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HEREDITARY PROPERTIES OF ALGEBRAS WITH RESPECT TO INDUCTIVE LIMITS

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Dedicated to Prof. Miroslav Laitoch on his 60th birthday

In this paper, there is shown, that any property of any member of the inductive set of universal algebras characterized by fulfilling some identity is hereditary with respect to inductive limit of this set, i.e. that the mentioned identity is also for the inductive limit true. The inductive system of algebras and the inductive limits of this system will be introduced in a natural way.

Theorem I. Let $E = \varinjlim (E_\alpha, f_{\beta\alpha})$ be the inductive limit of an inductive system $(E_\alpha, f_{\beta\alpha})$ of algebras of the type Δ and let

$$t_1 = t_2 \qquad (*)$$

be any identity of the type Δ. If $\forall\, \alpha \in I$ the identity (*) holds in E_α, then it holds also in algebra E.

Proof: Let $X = \{x_1, \ldots, x_n\}$ be a set of all variables which occur in one of the terms t_1, t_2 at least. Let us put $t_1 = t_1(y_1, \ldots, y_r)$, $t_2 = t_2(z_1, \ldots, z_s)$, where y_i, z_j $(i = 1, 2, \ldots, r;\ j = 1, 2, \ldots, s;\ r, s \leqq n)$ are variables from X.

Let us choose any interpretation H in E, e.g. let $H(y_1) = \bar{a}_1; \ldots; H(y_r) = \bar{a}_r$ and $H(z_1) = \bar{b}_1; \ldots; H(z_s) = \bar{b}_s$, where $\langle \bar{a}_1, \ldots, \bar{a}_r \rangle \in \dot{E}^r$, $\langle \bar{b}_1, \ldots, \bar{b}_s \rangle \in \dot{E}^s$ and $\dot{E}$ is the support of algebra E.

Let us denote

$$t_1' = H(t_1) = t_1(\bar{a}_1, \ldots, \bar{a}_r),$$
$$t_2' = H(t_2) = t_2(\bar{b}_1, \ldots, \bar{b}_s).$$

Let $\forall\, i = 1, 2, \ldots, r$; $\forall\, j = 1, 2, \ldots, s$ be a_i' and b_j' respectively, any elements of $\bar{a}_i$ and $\bar{b}_j$, and let $a_i' \in \dot{E}_{\alpha_i}$, $b_j' \in \dot{E}_{\beta_j}$, $\alpha_i, \beta_j \in I$, i.e. $f_{\alpha_i}(a_i') = \bar{a}_i$, $f_{\beta_j}(b_j') = \bar{b}_j$.

Then (from the properties of an inductive system of sets) $\exists\, \gamma_r \in I : \gamma_r \geqq \alpha_i$ and

$f_{\gamma_r\alpha_i}(a'_i) \in \dot{E}_{\gamma_r}$, $i = 1, 2, \ldots, r$, $\exists\, \delta_s \in I : \delta_s \geqq \beta_j$ and $f_{\delta_s\beta_j}(b'_j) \in \dot{E}_{\delta_s}$, $j = 1, 2, \ldots, s$ and $\exists\, \gamma \in I : \gamma \geqq \gamma_r, \gamma \geqq \delta_s$ and

$$f_{\gamma\alpha_i}(a'_i) = a_i \in E_\gamma,\ i = 1, 2, \ldots, r,$$
$$f_{\gamma\beta_j}(b'_j) = b_j \in E_\gamma,\ j = 1, 2, \ldots, s.$$

Then

$$f_\gamma(a_i) = f_\gamma(f_{\gamma\alpha_i}(a'_i)) = f_{\alpha_i}(a'_i) = \bar{a}_i,$$
$$f_\gamma(b_j) = f_\gamma(f_{\gamma\beta_j}(b'_j)) = f_{\beta_j}(b'_j) = \bar{b}_j$$

$\forall\, i = 1, 2, \ldots, r$ and $\forall\, j = 1, 2, \ldots, s$ and

$$t'_1 = t_1(f_\gamma(a_1), \ldots, f_\gamma(a_r)),$$
$$t'_2 = t_2(f_\gamma(b_1), \ldots, f_\gamma(b_s)).$$

Since f_γ is homomorphism,

$$t'_1 = f_\gamma(t_1(a_1, \ldots, a_r)),$$
$$t'_2 = f_\gamma(t_2(b_1, \ldots, b_s)).$$

Since in E_γ

$$t_1(a_1, \ldots, a_r) = t_2(b_1, \ldots, b_s)$$

and f_γ is a mapping, then necessarily

$$f_\gamma(t_1(a_1, \ldots, a_r)) = f_\gamma(t_2(b_1, \ldots, b_s)),$$

i.e. $H(t_1) = H(t_2)$ and since the interpretation H has been chosen arbitrabily in E, the identity (*) holds in E.

Corollary: If $\forall\, \alpha \in I$ are E_α algebras of the variety N, then $E = \varinjlim (E_\alpha, f_{\beta\alpha})$ is an algebra of the same variety N.

Note: Hereditary properties may be even those which are expressed by the validity of some formulas, e.g. the inductive limit of an inductive system of domains of integrity the domain of integrity ([6]).

Theorem II. Let $(E_\alpha, f_{\beta\alpha})$ be an inductive system of algebras of the type Δ. For every $\alpha \in I$ let f'_α be a homomorphism of algebra E_α in to algebra E' of the type Δ such that $f'_\alpha = f'_\beta \circ f_{\beta\alpha}$ for $\alpha \leqq \beta$.

Then $\exists!$ a homomorphism f' algebra $E = \varinjlim (E_\alpha, f_{\beta\alpha}) \to E'$ that $\forall\, \alpha \in I : f'_\alpha = f' \circ f_\alpha$.

Proof: The existence and unicity of such a mapping f' follows from the properties of an inductive system of sets. Let us prove that f' is homomorphism.

Let us choose any operation F of the type Δ, any $\langle \bar{a}_1, \ldots, \bar{a}_n \rangle \in \dot{E}$ and let $\bar{a}_i = f_{\alpha_i}(a'_i)$ where $a'_i \in E_{\alpha_i}$, $\alpha_i \in I$, $i = 1, 2, \ldots, n$. Then $\exists\, \gamma \in I$, $\gamma \geqq \alpha_i : f_{\gamma\alpha_i}(a'_i) = a_i \in E_\gamma$, $i = 1, 2, \ldots, n$, i.e. $F_{E_\gamma}(a_1, \ldots, a_n) \in E_\gamma$, $f_\gamma(a_i) = \bar{a}_i$, $i = 1, 2, \ldots, n$ and $f_\gamma(F_{E_\gamma}(a_1, \ldots, a_n) = F_E(f_\gamma(a_1), \ldots, f_\gamma(a_n)) = F_E(\bar{a}_1, \ldots, \bar{a}_n)$. It follows then $f'(F_E(\bar{a}_1, \ldots, \bar{a}_n)) = f'(f_\gamma(F_{E_\gamma}(a_1, \ldots, a_n))) = f'_\gamma(F_{E_\gamma}(a_1, \ldots, a_n)) = F_{E'}(f'_\gamma(a_1), \ldots, f'_\gamma(a_n)) = F_{E'}(f'(f_\gamma(a_1)), \ldots, f'(f_\gamma(a_n))) = F_{E'}(f'(\bar{a}_1), \ldots, f'(\bar{a}_n))$.

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Souhrn

DĚDIČNÉ VLASTNOSTI ALGEBER VZHLEDEM K INDUKTIVNÍM LIMITÁM

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V článku jsou studovány některé dědičné vlastnosti algeber téhož typu vzhledem k induktivním limitám, přičemž dědičnými jsou nazývány ty vlastnosti algeber, které se při přechodu k induktivní limitě zachovají. Induktivní systém algeber a induktivní limita tohoto systému jsou zavedeny přirozeným způsobem. Nejdůležitějším výsledkem je následující tvrzení:

Nechť $E = \varinjlim (E_\alpha, f_{\beta\alpha})$ je induktivní limita induktivního systému $(E_\alpha, f_{\beta\alpha})$ algeber typu Δ a nechť

$$t_1 = t_2 \tag{*}$$

je libovolná identita typu Δ. Jestliže $\forall\, \alpha \in I$ platí v E_α identita (*), pak platí i v algebře E.

Резюме

НАСЛЕДСТВЕННЫЕ СВОЙСТВА АЛГЕБР В ОТНОШЕНИИ К ИНДУКТИВНЫМ ПРЕДЕЛАМ

ИОСИФ МОЛНАР

В статье рассматриваются некоторые свойства индуктивных пределов индуктивных систем алгебр. Самый полезный результат выражается следующей теоремой:

Пусть $E = \varinjlim (E_\alpha, f_{\beta\alpha})$ — индуктивный предел индуктивной системы $(E_\alpha, f_{\beta\alpha})$ алгебр типа Δ и пусть

$$t_1 = t_2 \tag{*}$$

— любое тождество типа Δ. Если $\forall \alpha \in I$ формула (*) справедлива на E_α, то она справедлива на алгебре E.