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HOMOMORPHISMS OF PROJECTIVE PLANES OVER QUASIFIELDS AND NEARFIELDS

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It is well known, that there is a correspondence between framed projective planes and planar ternary rings (PTR), called sometimes ternary rings only (see [6], chap. 9.). Further, every homomorphism of a framed projective plane to another framed projective plane induces a place (Stelle, T-homomorphism) of corresponding ternary rings and conversely, every ternary rings' place induces a homomorphism of corresponding projective planes. Consequently, homomorphisms of projective planes can be investigated as places of ternary rings. The first definition of a place of PTR is due to Skornjakov ([5], 285). If PTR is linear, the definition of a place can be expressed by means of addition and multiplication defined in PTR.[1]) If such a linear PTR is one of the known algebraic structures, coordinatizing special types of projective planes, the definition can be simplified. Cartesian groups were investigated by J. André in [1]. In the case of semifields (see [5]), alternative rings ([2]), skew-fields ([1], [4]) and fields (e.g. [3]), the corresponding necessary and sufficient conditions are known. Moreover, a place of fields in our sence is identical with a notion of place (točka) used in algebraic geometry.

In the following text, we shall establish characteristic properties of places of quasifields and nearfields.

Definition.

The algebraic system $(\boldsymbol{T}, +, .)$ is called *a planar ternary ring*, if the following conditions are satisfied (see [6], p. 276):

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[1]) Let t be a ternary operation in a ternary ring $\boldsymbol{T}$. The condition of linearity can be expressed as so:

$$t(a, b, c) = a \,.\, b + c,$$

where

$$a + b := t(1, a, b)$$

and

$$a \,.\, b := t(a, b, 0)$$

for $a, b, c \in \boldsymbol{T}$.

(i) $(T, +)$ and $(T \dot{-} \{0\}, .)$ are loops with natural elements 0 and 1 respectively,

(ii) for all $a \in T$, $a . 0 = 0 . a = 0$,

(iii) $\forall\, a, b, c, d \in T$, $a \neq c$, there exists a unique $x \in T$ such that $x . a + b = x . c + d$,

(iv) $\forall\, a, b, c \in T$ there exists a unique $x \in T$ such that $a . b + x = c$,

(v) $\forall\, a, b, c, d \in T$, $a \neq c$, there exists a unique pair $(x, y) \in T \times T$ such that $a . x + y = b$ and $c . x + y = d$.

Definition.

A mapping Θ from a planar ternary ring $(T, +, .)$ to a planar ternary ring $(T', +', .')$ is called *a place*, if it satisfies:

(P1) if $a^\Theta \neq \infty$, $b^\Theta \neq \infty$, then $(a + b)^\Theta = a^\Theta +' b^\Theta$ and $(a . b)^\Theta = a^\Theta .' b^\Theta$,

(P2) if $a^\Theta \neq 0'$, $b^\Theta = \infty$, then $(a . b)^\Theta = (b . a)^\Theta = \infty$,

(P3) if $a^\Theta \neq \infty$, $b^\Theta = \infty$, then $(a + b)^\Theta = (b + a)^\Theta = \infty$,

(P4) if $x^\Theta = y^\Theta = \infty$, $b^\Theta \neq \infty$, where $y = a . x + b = a^* . x$, then $a^\Theta = a^{*\Theta}$,

(P5) if $a^\Theta = b^\Theta = \infty$, $(a . x + b)^\Theta \neq \infty$ and $a . x^* + b = 0$, then $x^\Theta = x^{*\Theta}$,

(P6) if $y = a . x + b = a^* . x$, $a . x^* + b = 0$ and $a^\Theta = b^\Theta = x^\Theta = y^\Theta = \infty$, then either $a^{*\Theta} = \infty$ or $x^{*\Theta} = \infty$,

(P7) the image T^Θ of T under Θ has at least two elements.

Our notation $x^\Theta \neq \infty$ (or $x^\Theta = \infty$) means, that an element x belongs (or does not belong) to the domain of Θ. Hence to those elements that have no image under Θ we give a common image, the symbol $\infty \notin T'$, and we can shortly write $\Theta : T \to T' \cup \{\infty\}$ to express that Θ is a place of $(T, +, .)$ to $(T', +', .')$.

Proposition 1.

Let $\Theta : T \to T' \cup \{\infty\}$ *be a place of PTRs. Then* $0^\Theta = 0'$ *and* $1^\Theta = 1'$.

Proof. Let $m' \in T^\Theta$ and let us choose $m \in T$ such that $m^\Theta = m' \neq \infty$. Then $m^\Theta = (m + 0)^\Theta = \infty$ by (P3), a contradiction. Thus $0^\Theta \neq \infty$ and we have $m^\Theta = m^\Theta + 0^\Theta$ by (P1). But an equation $m^\Theta + x = m^\Theta$ has a unique solution $x = 0'$, so it must be $0^\Theta = 0'$.

Suppose that $1^\Theta = \infty$. Then for all $x \in T$, x^Θ is either $0'$ or ∞, in contrary to (P7). Really, if $x^\Theta \neq 0'$, then $x^\Theta = (1 . x)^\Theta = \infty$ by (P2). Thus $1^\Theta \neq \infty$. Let m', m are chosen as above. Then $m' = m^\Theta = (m . 1)^\Theta = m^\Theta . 1^\Theta = m' . 1^\Theta$. An equation $m' . x = m'$ is uniquely soluble, thus $1^\Theta = 1'$.

It can be verified that an image T^Θ of a planar ternary ring T under a place Θ forms a planar ternary ring under operations $+', .'$ defined on T'. Thus those elements of T', which are not images, can be omitted and we can suppose that a place is surjective.

Definition.

A ternary ring $(T, +, .)$ with the properties

(i) $(T, +)$ is a group (i.e. T satisfies the associative law of addition),

(ii) $\forall\, a, b, c \in T$, $a . (b + c) = a . b + a . c$ (i.e. the right distributivity law holds) is called *a right quasifield.*

In a similar way, a left quasifield can be defined. It suffices to investigate right quasifields only, since by means of a new operation $x \circ y := y \,.\, x$, from a right quasifield can be obtained a left one and conversely. In the following text, under a quasifield we shall always understand a right one.

It can be proved that in a quasifield, $a \,.\, (-b) = -a \,.\, b$ and $a + b = b + a$. Thus the additive group of a quasifield is Abelian. It can be easily shown the following:

Proposition 2.

Let $\Theta : \mathbf{T} \to \mathbf{T}' \cup \{\infty\}$ *be a place of ternary rings and let* $(\mathbf{T}, +, .)$ *be a quasifield. Then* $(\mathbf{T}', +', .')$ *is also a quasifield.*

Theorem 1.

Let $(\mathbf{T}, +, .)$, $(\mathbf{T}', +', .)$ *are quasifields. A mapping* $\Theta : \mathbf{T} \to \mathbf{T}' \cup \{\infty\}$ *is a place, if and only if it satisfies*

(Q1) if $a^\Theta \neq \infty$, $b^\Theta \neq \infty$, then $(a - b)^\Theta = a^\Theta -' b^\Theta$ and $(a \,.\, b)^\Theta = a \,.' b^\Theta$,

(Q2) if $a^\Theta \neq 0'$, $b^\Theta = \infty$, then $(a \,.\, b)^\Theta = (b \,.\, a)^\Theta = \infty$,

(Q3) if $x^\Theta = \infty$ and $(-a \,.\, x + a^* \,.\, x)^\Theta \neq \infty$, then $a^\Theta = a^{*\Theta}$,

(Q4) if $a^* \,.\, x = a \,.\, x - a \,.\, x^*$, $a^\Theta = x^\Theta = \infty$ and $a^{*\Theta} \neq \infty$,

then $x^{*\Theta} = \infty$.

To prove this theorem, we first establish several propositions.

Proposition 3.

A place $\Theta : \mathbf{T} \to \mathbf{T}' \cup \{\infty\}$ *of quasifields satisfies*:

(i) $(-b)^\Theta = \infty \Leftrightarrow b^\Theta = \infty$,

(ii) $b^\Theta \neq \infty \Leftrightarrow (-b)^\Theta = -' b^\Theta$,

(iii) if $a^\Theta \neq \infty$, $b^\Theta \neq \infty$, then $(a - b)^\Theta = a^\Theta -' b^\Theta$.

Proof. If $b^\Theta = \infty$, $(-b)^\Theta \neq \infty$ (or $b^\Theta \neq \infty$, $(-b)^\Theta = \infty$), we conclude according to (P3) and Prop. 1, that $0' = 0^\Theta = (b + (-b))^\Theta = \infty$, which is a contradiction. This proves (i). Let $b^\Theta \neq \infty$. Then $(-b)^\Theta \neq \infty$ and $0' = 0^\Theta = b^\Theta + (-b)^\Theta$ by (P1). Thus (ii) is true. The property (iii) is an immediate consequence of (P1) and (ii).

Proposition 4.

Let $\mathbf{T}, \mathbf{T}'$ *be quasifields and* $\Theta : \mathbf{T} \to \mathbf{T}' \cup \{\infty\}$ *be a mapping with a property*

(*) If $a^\Theta \neq \infty$, $b^\Theta \neq \infty$ then $(a - b)^\Theta = a^\Theta -' b^\Theta$, $(a \,.\, b)^\Theta = a^\Theta \,.' b^\Theta$.

Then Θ *satisfies:*

(i) $0^\Theta = 0'$,

(ii) $b^\Theta = \infty \Leftrightarrow (-b)^\Theta = \infty$,

(iii) if $b^\Theta \neq \infty$ then $(-b)^\Theta = -' b^\Theta$,

(iv) if $a^\Theta \neq \infty$, $b^\Theta \neq \infty$, then $(a + b)^\Theta = a^\Theta +' b^\Theta$,

(v) if $a^\Theta \neq \infty$, $b^\Theta = \infty$, then $(a + b)^\Theta = (b + a)^\Theta = \infty$.

The proof is not difficult. We can now return to our theorem.

Proof of Theorem 1.

Let Θ be a place of quasifields. Then Θ possesses the properties (P1)–(P6). (Q1) follows from (P1) and Prop. 3. (iii). (Q2) is identical with (P2). (Q3) can be proved by means of (P4) and (P2). Really, suppose that $-a \,.\, x + a^* \,.\, x = b$, $b^\Theta \neq \infty$, $x^\Theta = \infty$. Let $y = a^* \,.\, x = a \,.\, x + b$. If $y^\Theta = \infty$, we use (P4) to obtain $a^\Theta = a^{*\Theta}$. If $y^\Theta \neq \infty$, i.e. $(a^* \,.\, x)^\Theta \neq \infty$, then $a^{*\Theta} = 0'$, by (P2). Since $a \,.\, x = y - b$, it holds $(a \,.\, x)^\Theta = y^\Theta -' b^\Theta \neq \infty$ and therefore $a^\Theta = 0'$. To prove (Q4), we suppose $b = -a \,.\, x^*$, $y = a \,.\, x + b = a^* \,.\, x$. Let all assumptions of (Q4) are satisfied. Now suppose $x^{*\Theta} \neq \infty$. Then $(a^* \,.\, x)^\Theta = (a \,.\, x^*)^\Theta = \infty$. Let us prove it. Suppose $(a^* \,.\, x)^\Theta \neq \infty$. Then $(a \,.\, x - a \,.\, x^*)^\Theta = (a \,.\, x + b)^\Theta \neq \infty$, where $a \,.\, x^* + b = 0$ and $x^\Theta = a^\Theta = \infty$. If $b^\Theta = \infty$, we use (P5) to obtain the identity $x^\Theta = x^{*\Theta} = \infty$, in contrary to our assumption. If $b^\Theta \neq \infty$, then $(-b)^\Theta = (a \,.\, x^*)^\Theta \neq \infty$, according to Proposition 3. (ii). By (P2), $x^{*\Theta} = 0'$, which is also a contradiction. Hence $(a^* \,.\, x)^\Theta = \infty$. Now suppose $(a \,.\, x^*)^\Theta \neq \infty$. Then $(-a \,.\, x + a^* \,.\, x)^\Theta = (-a \,.\, x^*)^\Theta \neq \infty$ and $a^\Theta = a^{*\Theta}$ by (Q3), in contrary to the assumptions of (Q4). Thus $(a \,.\, x^*)^\Theta = \infty$. Hence all assumptions of (P6) are satisfied and since $a^{*\Theta} \neq \infty$, we conclude $x^{*\Theta} = \infty$. This contradiction establishes (Q4).

Conversely, let Θ be a mapping of quasifields with the properties (Q1)–(Q4). Since (P1), (P3) follow from Prop. 4. and (P2), (Q3) are identical, it remains to show, that (P4)–(P6) are true. So let the assumptions of (P4) are satisfied. Then $b^\Theta = (-a \,.\, x + a^* \,.\, x)^\Theta \neq \infty$ and $a^\Theta = a^{*\Theta}$ by (Q3). This proves (P4). The assumptions of (P5) imply, that $b = -a \,.\, x^*$ and $(a \,.\, x - a \,.\, x^*)^\Theta \neq \infty$. Thus $(a \,.\, x + a \,.\, (-x^*))^\Theta = (a \,.\, (x - x^*))^\Theta \neq \infty$ and we conclude $(x - x^*)^\Theta = 0'$, according to (Q2). If $x^\Theta = \infty$, then $x^{*\Theta} = \infty$, too. If $x^\Theta \neq \infty$, then $x^{*\Theta} \neq \infty$ and $x^\Theta -' x^{*\Theta} = 0'$ by (Q1). In any case, $x^\Theta = x^{*\Theta}$. Suppose now, that the assumptions of (P6) are satisfied. Then either $a^{*\Theta} = \infty$, or $x^{*\Theta} = \infty$, according to (Q4).

Note that (Q3) can be substitued by a weaker condition

(Q3)′ if $(-a \,.\, x + a^* \,.\, x)^\Theta \neq \infty$, $x^\Theta = \infty$, $a^\Theta \neq \infty$, then $a^\Theta = a^{*\Theta}$.

Let $a^\Theta = \infty$ and suppose that $a^{*\Theta} \neq \infty$. Let $-c = -a \,.\, x + a^* \,.\, x$. Then $c^\Theta = (-(-a \,.\, x) - a^* \,.\, x)^\Theta = (a \,.\, x - a^* \,.\, x)^\Theta \neq \infty$. Since $a \neq 0$, the equation $a \,.\, z = c$ has a unique solution. Let us note it x^*. Then $a \,.\, x^* = a \,.\, x - a^* \,.\, x$, $a^* \,.\, x = -a \,.\, x^* + a \,.\, x = a \,.\, x - a \,.\, x^*$. This implies $x^{*\Theta} = \infty$ by (Q4) and since $a^\Theta \neq 0'$, we conclude $c^\Theta = (a \,.\, x^*)^\Theta = \infty$, a contradiction. Thus $a^{*\Theta} = \infty$ and the equality $a^\Theta = a^{*\Theta}$ holds.

Definition.

A nearfield (more precisely, *a right planar nearfield*) $(\boldsymbol{T}, +, .)$ is a quasifield with associative multiplication, i.e. $(\boldsymbol{T} - \{0\}, .)$ is a group.

In a nearfield, $(-a) \,.\, b = -a \,.\, b$. It can be verified that $(\boldsymbol{T}, +, .)$ is a right planar nearfield if and only if $(\boldsymbol{T}, +)$ and $(\boldsymbol{T} - \{0\}, .)$ are groups, $a \,.\, 0 = 0 \,.\, a = 0$ for all

$a \in \boldsymbol{T}$, $(a + b) . c = a . c + b . c$ for all $a, b, c \in \boldsymbol{T}$ and $-x . a + x . b = c$ has a unique solution x for given $a, b, c \in \boldsymbol{T}$, $a \neq b$.

An image of a nearfield under a place is again a nearfield. Note, that $a^\Theta = \infty \Leftrightarrow \Leftrightarrow (a^{-1})^\Theta = 0'$.

Theorem 2.

Let $(\boldsymbol{T}, +, .)$, $(\boldsymbol{T}', +', .')$ *be nearfields. A mapping* $\Theta : \boldsymbol{T} \to \boldsymbol{T}' \cup \{\infty\}$ *is a place, if and only if it satisfies conditions* (Q1), (Q2) *and* (Q3)$'$.

Proof. One implication is trivial. To prove the other, we must show that (Q4) follows from (Q1), (Q2), (Q3)$'$. Suppose that the assumptions of (Q4) are satisfied. Then $a^* . x = a . x + a . (-x^*) = a . (x - x^*)$. Since $a^\Theta = \infty$, we conclude $a \neq 0$ and $(a^{-1})^\Theta = 0'$. Now we shall express x^*. From the previous equality, $a^{-1} . (a^* . x) = = x - x^*$ and $x^* = x - a^{-1} . (a^* . x)$. Suppose $x^{*\Theta} \neq \infty$. Then $(x - x^*)^\Theta = \infty$ and $(a^* . x)^\Theta = (a . (x - x^*))^\Theta = \infty$. This implies $a^* \neq 0$. Now $x = a^{*-1} . (a^* . x)$ and after a substitution, $x^{*\Theta} = (-a^{-1} . (a^* . x) + a^{*-1} . (a^* . x))^\Theta \neq \infty$. Substituting a^{-1}, a^{*-1}, $a^* . x$ for a, a^* and x in (Q3)$'$, we obtain $(a^{*-1})^\Theta = (a^{-1})^\Theta$. Thus $(a^{*-1})^\Theta = 0'$, i.e. $a^{*\Theta} = \infty$. This is a contradiction. Hence $x^{*\Theta} = \infty$ and (Q4) holds.

For completeness, let us mention other structures related to projective planes. By a similar way as above, it can be checked that *a mapping* Θ *of Cartesian groups is a place, if and only if it satisfies* (Q1)$-$(Q3) and (C1), (C2), where

(C1) $(a . x - a . x^*)^\Theta \neq \infty$, $a^\Theta = \infty \to x^\Theta = x^{*\Theta}$,

(C2) if $a^* . x + a . x^* = a . x$, $a^\Theta = x^\Theta = (a^* . x)^\Theta = \infty$, $(a . x^*)^\Theta = \infty$ then either $a^{*\Theta} = \infty$ or $x^{*\Theta} = \infty$.

A semifields' place is characterized by the properties (Q1), (Q2) and (S): if $a . x^* = = (a - a^*) . x$, $a^\Theta = x^\Theta = \infty$, $a^{*\Theta} \neq \infty$ then $x^{*\Theta} = \infty$.

In the case of *alternative rings, skew-fields and fields,* (Q1) and (Q2) appear to be necessary and sufficient conditions for a mapping to be a place. Let us prove it for alternative rings. In the other cases, the proof is trivial.

It suffices to show that (Q1), (Q2) imply (S). Let $a . x^* = (a - a^*) . x$, i.e. $a . x - - a . x^* = a^* . x$ ... (A), $a^\Theta = x^\Theta = \infty$ and $a^{*\Theta} \neq \infty$. Then $(a - a^*)^\Theta = \infty$ and $((a - a^*) . x)^\Theta = \infty$. It can be easily seen that $x \neq 0$, $x^* \neq 0$. Thus there exists x^{-1}, x^{*-1} and the above formula (A) can be rewritten to the form $(a . x - a . x^*) . . x^{-1} = a^*$, $(a . x) . x^{-1} - (a . x^*) . x^{-1} = a^*$. According to the right inverse property, $a - (a . x^*) . x^{-1} = a^*$ and further, $(a . x^*) . x^{*-1} - (a . x^*) . x^{-1} = a^*$. Thus $(a . x^*) . (x^{*-1} - x^{-1}) = a^*$. This implies $(x^{*-1} - x^{-1})^\Theta = 0'$. Here $(x^{-1})^\Theta = = 0'$, since $x^\Theta = \infty$, and thus $(x^{*-1})^\Theta = 0'$. Hence $x^{*\Theta} = \infty$.

It can be verified that an analogy of Proposition 2. is true in remaining cases. Of course, an image under a place can have additional properties. For examples see e.g. [1].

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SHRNUTÍ

HOMOMORFISMY PROJEKTIVNÍCH ROVIN NAD KVAZITĚLESY A SKOROTĚLESY

ALENA VANŽUROVÁ

V článku jsou nalezeny charakteristické vlastnosti umístění (T-homomorfismů) pravých kvazitěles a skorotěles. V závěru je podán přehled podmínek charakterizujících umístění některých dalších algebraických struktur, s nimiž se setkáváme při souřadnicování projektivních rovin, totiž kartézských grup, semitěles, alternativních těles, nekomutativních a komutativních těles.

РЕЗЮМЕ

ГОМОМОРФИЗМЫ ПРОЕКТИВНЫХ ПЛОСКОСТЕЙ НАД КВАЗИТЕЛАМИ И ПОЧТИТЕЛАМИ

АЛЕНА ВАНЖУРОВА

В статье установлены характеристические свойства Т-гомоморфизмов правых квазител и почтител. В заключении указаны условия, характеризирующие Т-гомоморфизмы некоторых других структур, встречающихся при координатызации проективных плоскостей, а именно картезских групп, семител, альтернативных тел, тел и полей.