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ON THE PROPERTIES OF THE FUNDAMENTAL DISPERSIONS OF THE EQUATION $y'' = \lambda q(t) y$

SVATOSLAV STANĚK

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Dedicated to Academician O. Borůvka on his 80th birthday

1. Introduction

In this paper we consider a differential equation

$$y'' = \lambda q(t) y \qquad (\lambda q)$$

with $q \in C^0(j), j = \langle a, b) \ (a < b \leqq \infty)$ where λ is a real parameter. The object of our study is to investigate the zero distribution of solutions and the zero distribution of the derivative of solutions of (λq), described as functions $\varphi(t, \lambda)$, $\psi(t, \lambda)$, $\chi(t, \lambda)$ and $\omega(t, \lambda)$. On applying the "generalized Wronskian" $w := y_0 y_1' - y_0' y_1$, where y_0 and y_1 are respectively the solutions of the equations (λ_0q) and (λ_1q), we prove in analogy with [3] some results on the monotony of the functions φ, ψ, χ and ω with respect to the variable λ, which are well known in case of $q(t) \neq 0$ on j.

2. Basic definitions, relations and notation

Let $q \in C^0(j)$ and let λ be a (real) number. Throughout our discussion we exclude the trivial solution of (λq). Suppose that $x \in j$ and u, v are solutions of (λq) satisfying the condition $u(x) = 0$, $v'(x) = 0$. Denote by $\varphi(x, \lambda)$ ($\chi(x, \lambda)$; $\omega(x, \lambda)$) the first zero (if any) of the function u (u'; v) lying to the right of the point x. The function φ is called the 1st kind fundamental dispersion of (λq) and in case of $q(t) \neq 0$ the functions χ and ω are called respectively the 3rd and 4th kind fundamental dispersion of (λq). (Cf. [1, 2]). The functions χ and ω are introduced in analogy with [5].

Say, a function $p \in C^0(j)$ possesses the *property* H if there is not a cluster point of zeros of p lying on j. If the function p possesses the property H and $\lambda \neq 0$, then λp possesses this property, too.

Lemma 1. *Let a function p possess the property* H *and let u be a solution of $y'' = p(t)\,y$. Then the zeros of u' have no cluster point on j.*

Proof. Suppose that the function p possesses the property H and there exists a nontrivial solution u of $y'' = p(t)\,y$ together with a sequence $\{t_n\}$, $t_n \in j$, $t_n \neq c \in j$, $\lim_{n\to\infty} t_n = c$ where $u'(t_n) = 0$. Then $u'(c) = 0$, $u''(c) = 0$ and because of $u(c) \neq 0$ we have $p(c) = 0$. According to the assumption, p possesses the property H and therefore there exists a number $\varepsilon > 0$ such that $u(t) \neq 0$ for $t \in (c - \varepsilon, c + \varepsilon)$ and $p(t) \neq 0$ for $t \in (c - \varepsilon, c + \varepsilon) - \{c\}$. Then $u'(t_n) - u'(c) = \int_c^{t_n} p(t)\,u(t)\,\mathrm{d}t \neq 0$ holds for all n for which $t_n \in (c - \varepsilon, c + \varepsilon)$, which is a contradiction.

Suppose that $x \in j$ and let q possess the property H. Let v be a solution of (λq) satisfying the condition $v'(x) = 0$. Denote by $\psi(x, \lambda)$ the first zero (if any) of v' lying to the right of the point x. If $q(t) \neq 0$, then the function ψ is called the 2nd kind fundamental dispersion of (λq). (See [1, 2]).

Every equation (λq) may be associated with the functions $\varphi(t, \lambda)$, $\chi(t, \lambda)$, $\omega(t, \lambda)$, and even with the function $\psi(t, \lambda)$ if q possesses the property H. Thereby it follows from the definition of these functions that they need not be defined for every $t \in j$. On the assumption that the equation (λq) is oscillatory, i.e. the point b is the cluster point of zeros of a (and then of every) solution of (λq), these functions are defined for every $t \in j$.

Lemma 2. *Let* (λ_0q) *be an oscillatory equation. Then $\chi(t_1, \lambda_0) < \chi(t_0, \lambda_0)$ for $t_1 < t_0$, $t_1 \in j$, $t_0 \in j$.*

Proof. We may assume without any loss of generality that $a \leqq t_1 < t_0 < \chi(t_1, \lambda_0)$. Suppose that u, v are solutions of (λ_0q), $u(t_1) = v(t_0) = 0$, $u'(t_1) = v'(t_0) = 1$. Let $\chi(t_0, \lambda_0) \leqq \chi(t_1, \lambda_0)$. Then $u'(t) > 0$ for $t \in (t_1, \chi(t_0, \lambda_0))$. We put $w(t) := u(t)\,v'(t) - u'(t)\,v(t)$, $t \in j$. Then $w(t) = k$ (= a constant $\neq 0$) and next $k = u(t_0)$, $k = -u'(\chi(t_0, \lambda_0))\,v(\chi(t_0, \lambda_0))$. Because of $u(t_0) > 0$ we have $k > 0$ and since $v(\chi(t_0, \lambda_0)) > 0$, we have $u'(\chi(t_0, \lambda_0)) < 0$, i.e. a contradiction.

Convention. In so far as a function at x_0 passing to an infinite expression of the type "0/0" occurs in our consideration, the value of such a function at x_0 will be defined as its limit (if any).

In closing this section let us remark the following observation: If there exists an interval $(c, d) \subset j$ with $q(t) < 0$, then every solution of (λq) possesses at least two zeros on (c, d) for a sufficiently large λ.

3. Main results

Theorem 1. *Assume* (λ_0q) *to be oscillatory. If*:

a) $\lambda_0 > 0$, *then* (λq) *is oscillatory also for every $\lambda \geqq \lambda_0$ and $\varphi(t, \lambda_1) > \varphi(t, \lambda_2)$ for $\lambda_0 \leqq \lambda_1 < \lambda_2$, $t \in j$;*

b) $\lambda_0 < 0$, *then* (λq) *is oscillatory also for every* $\lambda \leqq \lambda_0$ *and* $\varphi(t, \lambda_1) > \varphi(t, \lambda_2)$ *for* $\lambda_2 < \lambda_1 \leqq \lambda_0$, $t \in j$.

Proof. Suppose (λ_0q) to be oscillatory and $\dfrac{\lambda_0}{\lambda - \lambda_0} > 0$, which means that either $0 < \lambda_0 < \lambda$ or $0 > \lambda_0 > \lambda$. Let $x \in j$ and let y_0 and y_1 be solutions of (λ_0q) and (λq), respectively, with $y_0(x) = y_1(x) = 0$, $y_0'(x) = y_1'(x) = 1$. Then $y_0(\varphi(x, \lambda_0)) = 0$ and $y_0(t) > 0$ for $t \in (x, \varphi(x, \lambda_0))$. Assume $\varphi(x, \lambda_0) \leqq \varphi(x, \lambda)$, consequently $y_1(t) > 0$ for $t \in (x, \varphi(x, \lambda_0))$. We set $w(t) := y_0(t) y_1'(t) - y_0'(t) y_1(t)$, $t \in j$. Then $w' = (\lambda - \lambda_0) q y_0 y_1$ and $w(x) = 0$. This gives

$$0 < \int_x^{\varphi(x, \lambda_0)} y_0'^2(t)\,dt = y_0(t)\, y_0'(t)\,\Big|_x^{\varphi(x, \lambda_0)} - \lambda_0 \int_x^{\varphi(x, \lambda_0)} q(t)\, y_0^2(t)\,dt =$$

$$= -\frac{\lambda_0}{\lambda - \lambda_0} \int_x^{\varphi(x, \lambda_0)} \frac{y_0(t)\, w'(t)}{y_1(t)}\,dt =$$

$$= -\frac{\lambda_0}{\lambda - \lambda_0} \left[\frac{y_0(t)\, w(t)}{y_1(t)} \Bigg|_x^{\varphi(x, \lambda_0)} + \int_x^{\varphi(x, \lambda_0)} \left(\frac{w(t)}{y_1(t)} \right)^2 dt = \right.$$

$$= -\frac{\lambda_0}{\lambda - \lambda_0} \int_x^{\varphi(x, \lambda_0)} \left(\frac{w(t)}{y_1(t)} \right)^2 dt,$$

which, however, contradicts the assumption $\dfrac{\lambda_0}{\lambda - \lambda_0} > 0$. Consequently $\varphi(t, \lambda) < \varphi(t, \lambda_0)$ for $t \in j$ and (λq) is oscillatory for every λ where $\dfrac{\lambda_0}{\lambda - \lambda_0} > 0$. The rest of this proof is carried out writing λ_1 and λ_2 for λ_0 and λ into the above part of the proof.

Remark 1. Suppose (λ_0q) to be oscillatory. Then the statement of Theorem 1 on the oscillation of (λq), where $\lambda_0 \leqq \lambda$ and $\lambda \leqq \lambda_0$ are respectively $\lambda_0 > 0$ and $\lambda_0 < 0$, follows also from Theorem 2. 60 [7, p. 105] or from Lemma 3 [4].

Corollary 1. *Let* $\lambda_0 > 0$ *and let* (λ_0q) *be an oscillatory equation. Then*

$$\lim_{\lambda \to \infty} \varphi(t, \lambda) = \Phi_q(t), \qquad t \in j,$$

where

$$\Phi_q(t) = \begin{cases} t & \text{when } q(t) < 0, \\ \inf \{x;\, x \in j,\, t < x,\, q(x) < 0\} & \text{when } q(t) \geqq 0. \end{cases}$$

Proof. Let $x \in j$. By Theorem 1 $\varphi(x, \lambda)$ is a decreasing function on the interval $\langle \lambda_0, \infty)$. There exists therefore $\lim_{\lambda \to \infty} \varphi(x, \lambda)$ whose value we denote as c; $\lim_{\lambda \to \infty} \varphi(x, \lambda) = c$. Let $q(x) < 0$. Then there exists $\varepsilon > 0$ with $q(t) < 0$ for $t \in \langle x, x + \varepsilon\rangle$ and hence necessarily $c = x = \Phi_q(x)$. Let $q(x) \geqq 0$. Then $q(t) \geqq 0$ for $t \in \langle x, \Phi_q(x)\rangle$

(the case of $x = \Phi_q(x)$ is not excluded) and there exists on every interval $\langle \Phi_q(x), \Phi_q(x) + \varepsilon \rangle$, $\varepsilon > 0$, a subinterval $(\mu_\varepsilon, \nu_\varepsilon) \subset \langle \Phi_q(x), \Phi_q(x) + \varepsilon \rangle$, where $q(t) < 0$. Then, of course, every solution of (λq) has at least two zeros on the interval $(\mu_\varepsilon, \nu_\varepsilon)$ for a sufficiently large λ; hence $\Phi_q(x) \leqq \varphi(x, \lambda) \leqq \varphi(\Phi_q(x), \lambda) < \Phi_q(x) + \varepsilon$ holds for such λ and therefrom $c = \Phi_q(x)$.

Corollary 2. *Suppose* $\lambda_0 > 0$ *and let* (λ_0q) *be oscillatory with* $\Phi_q(t)$ *being the function defined in terms of Corollary* 1. *Then* $\lim_{\lambda \to \infty} \varphi(t, \lambda) = \Phi_q(t)$ *uniformly on every compact subinterval of* j *exactly if* $\Phi_q(t) = t$ *for* $t \in \langle \Phi_q(a), b) := j_1$, *i.e. iff* $q(t) \leqq 0$ *for* $t \in j_1$ *and* $q(t)$ *does not vanish in any interval* ($\subset j_1$).

Proof. Suppose $\lim_{\lambda \to \infty} \varphi(t, \lambda)$ to be uniformly converging on every compact subinterval of j. Then $\Phi_q(t) = \lim_{\lambda \to \infty} \varphi(t, \lambda)$ is a continuous function on j.

According to Theorem 1, the function $\varphi(t, \lambda)$ is a decreasing one in the variable λ on the interval $\langle \lambda_0, \infty)$ and since $\varphi(t, \lambda)$ is a continuous function for every $\lambda \in \in \langle \lambda_0, \infty)$ on j, then by the generalized wellknown Dini's theorem $\lim_{\lambda \to \infty} \varphi(t, \lambda) = \Phi_q(t)$ uniformly on every compact subinterval of j. It is evident from the definition of $\Phi_q(t)$ that this function is continuous on j exactly if $\Phi_q(t) = t$ for $t \in j_1$ $(= \langle \Phi_q(a), b))$ which occours precisely in case of $q(t) \leqq 0$ for $t \in j_1$ and $q(t)$ nonvanishing on every interval ($\subset j_1$).

Remark 2. If $\lambda_0 = 0$, then (λ_0q) is a nonoscillatory equation and it is easy to verify that the domain of the function $\varphi(t, \lambda_0)$ is an empty set. There is, however, such a function q to be found where $\varphi(t, \lambda)$ is defined on the set $j \times \mathbf{R}_0$ with $j = \langle a, \infty)$, $\mathbf{R}_0 = (-\infty, \infty) - \{0\}$. From [6] that say q may be replaced by any function $q \in C^0(j)$, $q(t) \not\equiv 0$, $q(t + \pi) = q(t)$ for $t \in j$ and $\int_{x_0}^{x_0 + \pi} q(t) \, dt = 0$ $(x_0 \in j)$.

Theorem 2. *Suppose that* (λ_0q) *is oscillatory. If*:

a) $\lambda_0 > 0$, *then the function* $\chi(t, \lambda)$ *is defined at every point* $(t, \lambda) \in j \times \langle \lambda_0, \infty)$ *and* $\chi(t, \lambda_1) > \chi(t, \lambda_2)$ *for* $\lambda_0 \leqq \lambda_1 < \lambda_2$, $t \in j$,

b) $\lambda_0 < 0$, *then the function* $\chi(t, \lambda)$ *is defined at every point* $(t, \lambda) \in j \times (-\infty, \lambda_0\rangle$ *and* $\chi(t, \lambda_1) > \chi(t, \lambda_2)$ *for* $\lambda_2 < \lambda_1 \leqq \lambda_0$, $t \in j$.

Proof. Let $x \in j$ and $\dfrac{\lambda_0}{\lambda - \lambda_0} > 0$. Let next y_0 and y_1 be solutions of (λ_0q) and (λq), respectively, with $y_0(x) = y_1(x) = 0$, $y_0'(x) = y_1'(x) = 1$. Then $y_0'(\chi(x, \lambda_0)) = 0$ and $y_0'(t) > 0$ for $t \in (x, \chi(x, \lambda_0))$. Assume that $y_1'(t) > 0$ for $t \in (x, \chi(x, \lambda_0))$ and therefore $\chi(x, \lambda_0) \leqq \chi(x, \lambda)$. We set $w(t) := y_0(t)\, y_1'(t) - y_0'(t)\, y_1(t)$, $t \in j$ and get $w' = (\lambda - \lambda_0)\, q y_0 y_1$, $w(x) = 0$. This gives

$$0 < \int_x^{\chi(x,\lambda_0)} y_0'^2(t)\,dt = y_0(t)\,y_0'(t)\,|_x^{\chi(x,\lambda_0)} - \lambda_0 \int_x^{\chi(x,\lambda_0)} q(t)\,y_0^2(t)\,dt =$$

$$= -\frac{\lambda_0}{\lambda-\lambda_0}\int_x^{\chi(x,\lambda_0)} \frac{y_0(t)\,w'(t)}{y_1(t)}\,dt =$$

$$= -\frac{\lambda_0}{\lambda-\lambda_0}\left[\frac{y_0(t)\,w(t)}{y_1(t)}\,|_x^{\chi(x,\lambda_0)} + \int_x^{\chi(x,\lambda_0)}\left(\frac{w(t)}{y_1(t)}\right)^2 dt\right] =$$

$$= -\frac{\lambda_0}{\lambda-\lambda_0}\left[\frac{y_0^2(\chi(x,\lambda_0))\,y_1'(\chi(x,\lambda_0))}{y_1(\chi(x,\lambda_0))} + \int_x^{\chi(x,\lambda_0)}\left(\frac{w(t)}{y_1(t)}\right)^2 dt\right]$$

which yields a contradiction since $\frac{y_0^2(\chi(x,\lambda_0))\,y_1'(\chi(x,\lambda_0))}{y_1(\chi(x,\lambda_0))} + \int_x^{\chi(x,\lambda_0)}\left(\frac{w(t)}{y_1(t)}\right)^2 dt > 0$ and $\frac{\lambda_0}{\lambda-\lambda_0} > 0$. Consequently $\chi(t,\lambda) < \chi(t,\lambda_0)$ and thus the function $\chi(t,\lambda)$ is defined at the points (t,λ), where $t \in j$ and $\frac{\lambda_0}{\lambda-\lambda_0} > 0$. Writing λ_1 and λ_2 in the above part of the proof for λ_0 and λ satisfying the assumptions of the Theorem, we prove so the remaining part of its statement.

Theorem 3. *Suppose that* $(\lambda_0 q)$ *is oscillatory. If*:

a) $\lambda_0 > 0$, *then the function* $\omega(t,\lambda)$ *is defined at every point* $(t,\lambda) \in j \times \langle\lambda_0, \infty)$ *and* $\omega(t,\lambda_1) > \omega(t,\lambda_2)$ *for* $\lambda_0 \leqq \lambda_1 < \lambda_2$, $t \in j$,

b) $\lambda_0 < 0$, *then the function* $\omega(t,\lambda)$ *is defined at every point* $(t,\lambda) \in j \times (-\infty, \lambda_0\rangle$ *and* $\omega(t,\lambda_1) > \omega(t,\lambda_2)$ *for* $\lambda_2 < \lambda_1 \leqq \lambda_0$, $t \in j$.

Proof. Let $x \in j$ and $\frac{\lambda_0}{\lambda-\lambda_0} > 0$. Let next y_0 and y_1 be solutions of $(\lambda_0 q)$ and (λq), respectively, with $y_0(x) = y_1(x) = 1$, $y_0'(x) = y_1'(x) = 0$. Then $y_0(\omega(x,\lambda_0)) = 0$ and $y_0(t) > 0$ for $t \in (x, \omega(x,\lambda_0))$. Assume that $y_1(t) > 0$ for $t \in (x, \omega(x,\lambda_0))$ and therefore $\omega(x,\lambda_0) \leqq \omega(x,\lambda)$. We set $w(t) := y_0(t)\,y_1'(t) - y_0'(t)\,y_1(t)$, $t \in j$ and get $w' = (\lambda - \lambda_1)\,q y_0 y_1$, $w(x) = 0$. Then

$$0 < \int_x^{\omega(x,\lambda_0)} y_0'^2(t)\,dt = y_0(t)\,y_0'(t)\,|_x^{\omega(x,\lambda_0)} - \lambda_0 \int_x^{\omega(x,\lambda_0)} q(t)\,y_0^2(t)\,dt =$$

$$= -\frac{\lambda_0}{\lambda-\lambda_0}\int_x^{\omega(x,\lambda_0)} \frac{y_0(t)\,w'(t)}{y_1(t)}\,dt =$$

$$= -\frac{\lambda_0}{\lambda-\lambda_0}\left[\frac{y_0(t)\,w(t)}{y_1(t)}\,|_x^{\omega(x,\lambda_0)} + \int_x^{\omega(x,\lambda_0)}\left(\frac{w(t)}{y_1(t)}\right)^2 dt\right] =$$

$$= -\frac{\lambda_0}{\lambda-\lambda_0}\int_x^{\omega(x,\lambda_0)}\left(\frac{w(t)}{y_1(t)}\right)^2 dt,$$

which is a contradiction. Therefore $\omega(t,\lambda) < \omega(t,\lambda_0)$, $t \in j$ and thus the function

$\omega(t, \lambda)$ is defined at every point (t, λ), where $t \in j$ and $\frac{\lambda_0}{\lambda - \lambda_0} > 0$. If we replace γ_0 and λ in the above part of the proof by λ_1 and λ_2 satisfying the assumptions of the Theorem, we prove so the remaining part of its statement.

Lemma 3. *Let $x \in j$ and q possess the property* H. *Further let* $(\lambda_0 q)$ *be oscillatory and* $\psi(x, \lambda_0) > \omega(x, \lambda_0)$. *If*:

a) $\lambda_0 > 0$, *then* $\psi(x, \lambda) > \omega(x, \lambda)$ *for* $\lambda > \lambda_0$,

b) $\lambda_0 < 0$, *then* $\psi(x, \lambda) > \omega(x, \lambda)$ for $\lambda < \lambda_0$.

Proof. Let $x \in j$, $\psi(x, \lambda_0) > \omega(x, \lambda_0)$. Let $\psi(x, \lambda_1) < \omega(x, \lambda_1)$ for a number λ_1 satisfying the inequality $\frac{\lambda_0}{\lambda_1 - \lambda_0} > 0$ and thus also the inequality $\frac{\lambda_1}{\lambda_1 - \lambda_0} > 0$. Let y_0 and y_1 be solutions of $(\lambda_0 q)$ and $(\lambda_1 q)$, respectively, $y_0(x) = y_1(x) = 1$, $y_0'(x) = y_1'(x) = 0$. Then $y_1'(\psi(x, \lambda_1)) = 0$, $y_1(t) > 0$, $y_1'(t) < 0$ for $t \in (x, \psi(x, \lambda_1))$ and $y_0'(t) < 0$ for $t \in (x, \psi(x, \lambda_1))$, since by Theorem 3 we have $\omega(x, \lambda_0) > \omega(x, \lambda_1)$. Setting $w(t) := y_0(t)\, y_1'(t) - y_0'(t)\, y_1(t)$, $t \in j$ gives $w' = (\lambda_1 - \lambda_0)\, q y_0 y_1$ and $w(x) = 0$. From this

$$0 < \int_x^{\psi(x,\lambda_1)} y_0'(t)\, y_1'(t)\, \mathrm{d}t = y_1'(t)\, y_0(t)\, |_x^{\psi(x,\lambda_1)} - \lambda_1 \int_x^{\psi(x,\lambda_1)} q(t)\, y_0(t)\, y_1(t)\, \mathrm{d}t =$$

$$= -\frac{\lambda_1}{\lambda_1 - \lambda_0} \int_x^{\psi(x,\lambda_1)} w'(t)\, \mathrm{d}t = -\frac{\lambda_1}{\lambda_1 - \lambda_0} w(\psi(x, \lambda_1)) =$$

$$= \frac{\lambda_1}{\lambda_1 - \lambda_0} y_0'(\psi(x, \lambda_1))\, y_1(\psi(x, \lambda_1))$$

i.e. a contradiction to the fact that $y_0'(\psi(x, \lambda_1))\, y_1(\psi(x, \lambda_1)) < 0$.

Theorem 4. *Let $x \in j$ and q be possessing the property* H. *Let* $(\lambda_0 q)$ *be oscillatory with* $\psi(x, \lambda_0) > \omega(x, \lambda_0)$. *If*:

a) $\lambda_0 > 0$, *then* $\psi(x, \lambda_1) > \psi(x, \lambda_2)$ *for* $\lambda_0 \leqq \lambda_1 < \lambda_2$,

b) $\lambda_0 < 0$, *then* $\psi(x, \lambda_1) > \psi(x, \lambda_2)$ *for* $\lambda_2 < \lambda_1 \leqq \lambda_0$.

Proof. Suppose that $x \in j$ and $\psi(x, \lambda_0) > \omega(x, \lambda_0)$. Let $0 < \lambda_0 \leqq \lambda_1 < \lambda_2$. Then, from Lemma 3, we obtain $\psi(x, \lambda_1) > \omega(x, \lambda_1)$, $\psi(x, \lambda_2) > \omega(x, \lambda_2)$ and consequently $\psi(x, \lambda_1) = \chi(\omega(x, \lambda_1), \lambda_1)$, $\psi(x, \lambda_2) = \chi(\omega(x, \lambda_2), \lambda_2)$. Theorem 2 and Lemma 3 imply $\psi(x, \lambda_1) = \chi(\omega(x, \lambda_1), \lambda_1) > \chi(\omega(x, \lambda_2), \lambda_1) > \chi(\omega(x, \lambda_2), \lambda_2) = \psi(x, \lambda_2)$, hence $\psi(x, \lambda_1) > \psi(x, \lambda_2)$. We proceed similarly even in case of $0 > \lambda_0 \geqq \lambda_1 > \lambda_2$.

Theorem 5. *Let $x \in j$ and q be possessing the property* H. *Let* $(\lambda_0 q)$ *be oscillatory with* $\lambda_0 q(x) > 0$. *If*:

a) $\lambda_0 > 0$, *then* $\psi(x, \lambda_1) > \psi(x, \lambda_2)$ *for* $\lambda_0 \leqq \lambda_1 < \lambda_2$,

b) $\lambda_1 < 0$, *then* $\psi(x, \lambda_1) > \psi(x, \lambda_2)$ *for* $\lambda_2 < \lambda_1 \leqq \lambda_0$.

Proof. Let $x \in j$, $\lambda_0 q(x) > 0$ and $0 < \lambda_0 \lneqq \lambda_1 < \lambda_2$. Let y_1 and y_2 be solutions of (λ_1q) and (λ_2q), respectively, $y_1(x) = y_2(x) = 1$, $y_1'(x) = y_2'(x) = 0$ and $\psi(x, \lambda_1) \leqq \psi(x, \lambda_2)$. According to the assumption $\lambda_1 q(x) > 0$, $\lambda_2 q(x) > 0$ and therefore $y_1'(t) > 0, y_2'(t) > 0$ for $t \in (x, \psi(x, \lambda_1))$; $y_1(t) > 0, y_2(t) > 0$ for $t \in \langle x, \psi(x, \lambda_1)\rangle$. Setting $w(t) := y_1(t)\, y_2'(t) - y_1'(t)\, y_2(t)$, $t \in j$, gives $w' = (\lambda_2 - \lambda_1)\, q y_1 y_2$, $w(x) = 0$. From this it follows that

$$0 < \int_x^{\psi(x,\lambda_1)} y_1'^2(t)\,\mathrm{d}t = y_1(t)\, y_1'(t)\,|_x^{\psi(x,\lambda_1)} - \lambda_1 \int_x^{\psi(x,\lambda_1)} q(t)\, y_1^2(t)\,\mathrm{d}t =$$

$$= -\frac{\lambda_1}{\lambda_2 - \lambda_1} \int_x^{\psi(x,\lambda_1)} \frac{y_1(t)\, w'(t)}{y_2(t)}\,\mathrm{d}t =$$

$$= -\frac{\lambda_1}{\lambda_2 - \lambda_1}\left[\frac{y_1(t)\, w(t)}{y_2(t)}\,|_x^{\psi(x,\lambda_1)} + \int_x^{\psi(x,\lambda_1)} \left(\frac{w(t)}{y_2(t)}\right)^2 \mathrm{d}t =\right.$$

$$= -\frac{\lambda_1}{\lambda_2 - \lambda_1}\left[\frac{y_1^2(\psi(x,\lambda_1))\, y_2'(\psi(x,\lambda_1))}{y_2(\psi(x,\lambda_1))} + \int_x^{\psi(x,\lambda_1)} \left(\frac{w(t)}{y_2(t)}\right)^2 \mathrm{d}t\right.$$

contrary to $\dfrac{\lambda_1}{\lambda_2 - \lambda_1} > 0$ and $\dfrac{y_1^2(\psi(x,\lambda_1))\, y_2'(\psi(x,\lambda_1))}{y_2(\psi(x,\lambda_1))} \geqq 0$. In an analogous fashion we proceed in case of $0 > \lambda_0 \geqq \lambda_1 > \lambda_2$.

Remark 3. It becomes apparent from the proof of Theorem 5 that the assumption $\lambda_0 q(x) > 0$ may be replaced by a weaker one: $\lambda_0 q(x) \geqq 0$ and $\lambda_0 q(t) > 0$ in a right neighbourhood of the point x.

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Souhrn

VLASTNOSTI ZÁKLADNÍCH DISPERSÍ ROVNICE $y'' = \lambda q(t) y$

SVATOSLAV STANĚK

V práci je vyšetřováno rozložení nulových bodů řešení a nulových bodů derivace řešení rovnice

$$y'' = \lambda q(t) y, \qquad q \in C^0(j), \tag{λq}$$

kde $j = \langle a, b) \ (a < b \leqq \infty)$, které je popsáno pomocí základní centrální disperse 1. druhu $\varphi(t, \lambda)$ rovnice (λq) a pomocí jistých funkcí $\psi(t, \lambda)$, $\chi(t, \lambda)$ a $\omega(t, \lambda)$, které v případě $q(t) \neq 0 \ (t \in j)$ odpovídají postupně základním centrálním dispersím 2., 3. a 4. druhu rovnice (λq). Užitím ,,zobecněného wronskiánu" $w := y_0 y_1' - y_0' y_1$, kde y_0 a y_1 jsou řešení rovnic (λ_0q) a (λ_1q), je dokázána monotonnost funkcí φ, ψ, χ a ω vzhledem k proměnné λ.

Резюме

СВОЙСТВА ОСНОВНЫХ ДИСПЕРСИЙ УРАВНЕНИЯ $y'' = \lambda q(t) y$

СВАТОСЛАВ СТАНЕК

В работе исследовано расположение корней решений и корней их производных для уравнения

$$y'' = \lambda q(t) y, \ q \in C^0(j), \tag{λq}$$

где $j = \langle a, b) \ (a < b \leqq \infty)$. Их расположение описано при помощи основной дисперсии 1-го рода $\varphi(t, \lambda)$ уравнения (λq) и некоторых функций $\psi(t, \lambda)$, $\chi(t, \lambda)$ и $\omega(t, \lambda)$, которые в случае $q(t) \neq 0$ для $t \in j$ соответствуют постепенно основным дисперсиям 2-го, 3-го и 4-го родов уравнения (λq). С помощью ,,обобщенного вронскиана" $w: = y_0 y_1' - y_0' y_1$, где y_0 и y_1 решения уравнении (λ_0q) и (λ_1q), доказана монотонность функций φ, ψ, χ и ω относительно переменного λ.