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THE CHARACTERISTIC MULTIPLIERS OF A BLOCK AND OF AN INVERSE BLOCK OF SECOND-ORDER LINEAR DIFFERENTIAL EQUATIONS WITH π-PERIODIC COEFFICIENTS

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§ 1. INTRODUCTION

O. Borůvka in [2]–[5] and F. Neuman in [6] investigated characteristic (or Floquet's) multipliers of a differential equation (q): $y'' = q(t)y$ with π-periodic function q, $q \in C_R^0$, $\boldsymbol{R} = (-\infty, \infty)$, oscillatory on $\boldsymbol{R}$ by means of the dispersion of (q) in case of real characteristic multipliers or by means of a (first) phase of (q) in case of complex characteristic multipliers. In [3] and [5] it has been proved that all the equations from a block [q] have the same characteristic multipliers called the characteristic multipliers of [q].

This paper presents necessary and sufficient conditions for the π-periodicity of the carriers of equations in the block [q] and in the inverse block $[q]^{-1}$ and the relations between the characteristic multipliers of both blocks. The main results are in § 4.

§ 2. DEFINITIONS, NOTATION AND BASIC PROPERTIES

We consider differential equations of the type

$$y'' = q(t)\,y, \qquad q \in C_{\boldsymbol{R}}^0, \tag{q}$$

oscillatory on $\boldsymbol{R}$ (i.e. every nontrivial solution of (q) has an infinite number of zeros to the right and to the left of t_0, $t_0 \in \boldsymbol{R}$). The function q is occasionally called the *carrier of* (q).

A function α, is called the *(first) phase of* (q) if there exist independent solutions u and v of (q) such that

$$\operatorname{tg} \alpha(t) = \frac{u(t)}{v(t)} \qquad \text{for all } t \in \{t \in \boldsymbol{R},\ v(t) \neq 0\}.$$

For every phase of (q) we have:

$$\alpha \in C^3_{\boldsymbol{R}}, \qquad \alpha'(t) \neq 0,$$

$q(t) = -\{\alpha, t\} - \alpha'^2(t)$, where $\{\alpha, t\} = \dfrac{\alpha'''(t)}{2\alpha'(t)} - \dfrac{3}{4}\left(\dfrac{\alpha''(t)}{\alpha'(t)}\right)^2$. The set of all phases of $y'' = -y$ together with the composition rule form the group $\mathfrak{E}$ called the *fundamental group*. The function $t + a$ is an element of $\mathfrak{E}$ for every a, $a \in \boldsymbol{R}$ and $\varepsilon \in \mathfrak{E}$ if and only if there exist numbers a_{ij}, $i, j = 1, 2$, $\det a_{ij} \neq 0$ such that $\operatorname{tg} \varepsilon(t) = \dfrac{a_{11} \operatorname{tg} t + a_{12}}{a_{21} \operatorname{tg} t + a_{22}}$ holds for every $t \in \boldsymbol{R}$, where the righthand side of the formula is meaningful. Another group is the *group of elementary phases* $\mathfrak{H}$ formed by elementary phases, that is, by those phases α where $\alpha(t + \pi) = \alpha(t) + \pi \,.\, \operatorname{sign} \alpha'$; $\mathfrak{E} \subset \mathfrak{H}$.

Let $t_0 \in \boldsymbol{R}$, n be a positive integer and y be a nontrivial solution of (q) such that $y(t_0) = 0$. Denote $\varphi_n(t_0)$ ($\varphi_{-n}(t_0)$) the n^{th} zero of solution y lying to the right (to the left) of t_0. Then the function $\varphi_n(\varphi_{-n})$ defined on $\boldsymbol{R}$ is called the 1^{st} *kind central dispersion with index n (with index −n) of* (q). In what follows we briefly say the *dispersion of* (q) in place of the 1^{st} kind central dispersion with index 1 of (q) and instead of φ_1 we sometimes write only φ. The dispersion φ satisfies:

$$\varphi \in C^3_{\boldsymbol{R}}, \qquad \varphi(t) > t, \qquad \varphi'(t) > 0, \qquad \varphi \circ \varphi_{-1}(t) = \varphi_{-1} \circ \varphi(t) = t,$$
$$\varphi_n(t) = \underbrace{\varphi \circ \ldots \circ \varphi}_{n}(t), \qquad t \in \boldsymbol{R}.$$

Between every phase α and the dispersion φ of (q) there holds the Abel's relation

$$\alpha \circ \varphi(t) = \alpha(t) + \pi \,.\, \operatorname{sign} \alpha'.$$

For more details see [1].

We say that (q) *and* (q*) *are associated* and we write (q) ~ (q*), if there exist a phase α of (q) and $\varepsilon \in \mathfrak{E}$ with α^*, $\alpha^* := \alpha \circ \varepsilon$ being a phase of (q*). The associativity relation of equations is reflexive, symetric and transitive. Consequently it defines a decomposition on the set of all equations of type (q) oscillatory on $\boldsymbol{R}$. The elements of the decomposition are called *blocks* (see [2], [3], [5]). The block containing (q) will be denoted by [q]; (q) ∈ [q]. If α is a phase of (q), then $\mathfrak{E}\alpha\mathfrak{E} = \{\varepsilon_1 \circ \alpha \circ \varepsilon_2;\ \varepsilon_1 \in \mathfrak{E}, \varepsilon_2 \in \mathfrak{E}\}$ are the phases of all equations from [q]. Herefrom it follows

$$[\mathrm{q}] = \{(\mathrm{q}^*);\ q^*(t) = -1 + (1 + q \circ \varepsilon(t))\, \varepsilon'^2(t), \varepsilon \in \mathfrak{E}\}$$

(see [2], [3]).

We say that ($\bar{\mathrm{q}}$) *is inverse to* (q) if there exists such a phase α of (q) that the function α^{-1} is a phase of ($\bar{\mathrm{q}}$) (see [2], [3], [5]). If ($\bar{\mathrm{q}}$) is an inverse equation to (q), then (q) is an inverse equation to ($\bar{\mathrm{q}}$). Generally there exists an infinite number of inverse equations to (q). There are exactly those equations whose phases form the set $\mathfrak{E}\alpha^{-1}\mathfrak{E}$ i.e. a block of differential equations. This block is denoted by $[\mathrm{q}]^{-1}$ and is called the *inverse block of the block* [q]. The blocks [q] and $[\mathrm{q}]^{-1}$ have the following characteristic property: Every equation from [q] is inverse to all equations from $[q]^{-1}$ and vice versa every equation from $[q]^{-1}$ is inverse to all equations from [q]. Consequently, [q] is the inverse block to $[q]^{-1}$. If α is a phase of (q), then$[q]^{-1} = \{(\bar{q}^*); \bar{q}^*(t) = -1 - (1 + q \circ \alpha^{-1} \circ \varepsilon(t))(\alpha^{-1} \circ \varepsilon(t))'^2, \varepsilon \in \mathfrak{C}\}$.

Notation. The function f^{-1} denotes the inverse to f. For an integer n, $n \neq 0$ $f^{[n]}$ denotes the function $\underbrace{f \circ f \circ \ldots \circ f}_{u}$ or $\underbrace{f^{-1} \circ f^{-1} \circ \ldots \circ f^{-1}}_{-n}$ according as $n > 0$ or $n < 0$.

§ 3. CHARACTERISTIC MULTIPLIERS OF Π-PERIODIC DIFFERENTIAL EQUATION (q)

In this and the following paragraph we investigate only differential equations of type (q) whose carries are π-periodic functions on $\boldsymbol{R}$.

Lemma 1 ([2], [3]). *If the carrier of* (q) *is π-periodic, then the carriers of all equations from* [q] *are π-periodic, too.*

Lemma 2 ([2], [3]). *Let α be a phase of* (q). *Then $q(t + \pi) = q(t)$ for $t \in \boldsymbol{R}$ if and only if*

$$\alpha(t + \pi) = \varepsilon \circ \alpha(t),$$

where $\varepsilon \in \mathfrak{C}$.

There is associated an algebraic equation $s^2 - As + 1 = 0$ to every equation (q) with a π-periodic carrier in the Floquet theory. The constant A is given by: $A = \bar{u}(x + \pi) + \bar{v}'(x + \pi)$, where $x \in \boldsymbol{R}$ denotes an arbitrary number and $\bar{u}, \bar{v}$ are the solutions of (q) satisfying the initial conditions $\bar{u}(x) = 1$, $\bar{u}'(x) = 0$, $\bar{v}(x) = 0$, $\bar{v}'(x) = 1$. We denote the roots of the algebraic equation, the so-called *characteristic multipliers of* (q), by ϱ_1, ϱ_{-1}. Evidently $\varrho_1 \cdot \varrho_{-1} = 1$. Next (q) admits independent solutions u and v satisfying either

$$u(t + \pi) = \varrho_1 \cdot u(t), \qquad v(t + \pi) = \varrho_{-1} \cdot v(t), \tag{1}$$

or

$$u(t + \pi) = \varrho_1 \cdot u(t) + v(t), \qquad v(t + \pi) = \varrho_1 \cdot v(t), \qquad \varrho_1^2 = 1. \tag{2}$$

The characteristic multipliers ϱ_1, ϱ_{-1} of (q) can be calculated by means of a phase or by the dispersion of (q) as shown in the next two lemmas below

Lemma 3 ([2], [3], [5]). *Let φ be the dispersion of* (q). *Then* (q) *possesses the real characteristic multipliers ϱ_1, ϱ_{-1} if and only if there exist $x, x \in \boldsymbol{R}$ and a positive integer n:*

$$\varphi_n(x) = x + \pi.$$

In this case

$$\varrho_\sigma = (-1)^n (\varphi_n'(x))^{\frac{1}{2}\sigma}, \qquad \sigma = \pm 1.$$

The number x in Lemma 3 is called the 1^{st} *kind determining number of type n of* (q).

Lemma 4 ([3], [5], [6]). *Equation* (q) *possesses the complex characteristic multipliers* $e^{\pm ia\pi}$, $0 < a < 1$ *if and only if there exists a phase* α *of* (q) *with*

$$\alpha(t + \pi) = \alpha(t) + (a + 2n)\pi$$

where n is an integer.

The number a $(0 < a < 1)$ in Lemma 4 is called the *2nd kind determining number of type n of* (q). The equation (q) is said to be of the *category* (i, n) $(i = 1, 2;\ n$ an integer), if the i^{st} kind determining number of type n of (q) occurs.

Remark. If (q) possesses a phase α: $\alpha(t + \pi) = \alpha(t) + (a + 2n)\pi$ where n is an integer and $0 < a < 1$ then it follows by formula $q(t) = \{\alpha, t\} - \alpha'^2(t)$ that q is π-periodic. This fact will be particularly utilized in proving Theorems 4 and 5.

In the theory of blocks of differential equations there plays a basic role the result given in

Lemma 5 ([3], [5], "the law of inertia of characteristic multipliers"). *All equations* (q) *with* π*-periodic carriers which are contained in the same block, are of the same category and have the same characteristic multipliers. All such equations have or have not all solutions* π*-periodic or* π*-halfperiodic.*

From the above lemma it follows that we are justified to the following definitions: We say that the *block* [q] *has the characteristic multipliers* ϱ_1, ϱ_{-1} if ϱ_1, ϱ_{-1} are the characteristic multipliers of an (and then of every) equation from [q]. We say that [q] *is of category* (i, n) $(i = 1, 2;\ n$-integer), if (i, n) is the category of an (and then of every) equation from [q].

§ 4. CHARACTERISTIC MULTIPLIERS OF BLOCKS [q] AND $[q]^{-1}$

If q is a π-periodic function then it follows from Lemma 1 that all equations from [q] have π-periodic carriers as well. However, it doesn't generally follow that an (and by Lemma 1 every) equation from the inverse block $[q]^{-1}$ has a π-periodic carrier. Since we investigate the characteristic multipliers of blocks [q] and $[q]^{-1}$ we must also assume the carriers of equations from [q] and $[q]^{-1}$ to be π-periodic. If α is a phase of an equation from [q] then the above assumptions are with respect to Lemma 2 satisfied exactly if

$$\alpha(t + \pi) = \varepsilon_1 \bigcirc \alpha(t), \qquad \alpha^{-1}(t + \pi) = \varepsilon \bigcirc \alpha(t), \qquad t \in \boldsymbol{R},$$

where $\varepsilon \in \mathfrak{E}$, $\varepsilon_1 \in \mathfrak{E}$.

In the next five theorems we now give necessary and sufficient conditions for the carriers of equations from [q] and $[q]^{-1}$ to be π-periodic. Further we will investigate the relations between the characteristic multipliers of [q] and $[q]^{-1}$. Theorems 1, 2 and 3 are dealing with the characteristic multipliers of [q] being real, while Theorems 4 and 5 are concerned with the characteristic multipliers of [q] being complex.

Theorem 1. *Let the carriers of equations from* [q] *and* $[q]^{-1}$ *be* π*-periodic and let* $(1, n)$ *and* $(1, m)$ *be the categories of* [q] *and* $[q]^{-1}$, *respectively. Then* $n = m = 1$ *and both blocks have the same characteristic multipliers. If further all solutions of equations of at least one from* [q] *and* $[q]^{-1}$ *are* π*-halfperiodic, then this applies to all solutions of equations from the second block, too.*

Proof: Let the carriers of equations from [q] and $[q]^{-1}$ be π-periodic and $(1, n)$ and $(1, m)$ be the categories of [q] and $[q]^{-1}$, respectively. Let α be a phase of (q), sign $\alpha' = 1$ and let α^{-1} be a phase of $(\bar{q})$; $(\bar{q}) \in [\bar{q}]^{-1}$. Then

$$\alpha(t + \pi) = \varepsilon_1 \circ \alpha(t), \qquad \alpha^{-1}(t + \pi) = \varepsilon \circ \alpha^{-1}(t); \qquad \varepsilon, \varepsilon_1 \in \mathfrak{E}$$

and

$$\alpha \circ \varepsilon(t) = \alpha(t) + \pi, \qquad \alpha^{-1} \circ \varepsilon_1(t) = \alpha^{-1}(t) + \pi.$$

Consequently ε and ε_1 are the dispersions of (q) and $(\bar{q})$, respectively. By assumption the equations $(\bar{q})$ and (q) have real characteristic multipliers ϱ_σ and $\bar{\varrho}_\sigma$ $(\sigma = \pm 1)$, respectively, and therefore by Lemma 3 there exist such numbers x and x_1 that

$$\varepsilon^{[n]}(x) = x + \pi, \qquad \varepsilon_1^{[m]}(x_1) = x_1 + \pi$$

and

$$\varrho_\sigma = (-1)^n \,.\, (\varepsilon^{[n]\prime}(x))^{\sigma/2},$$
$$\bar{\varrho}_\sigma = (-1)^m \,.\, (\varepsilon_1^{[m]\prime}(x_1))^{\sigma/2}, \qquad \sigma = \pm 1.$$

From $\alpha(x + \pi) = \alpha \circ \varepsilon^{[n]}(x) = \alpha(x) + n\pi$ we get for $\bar{x} := \alpha(x)$ $(x = \alpha^{-1}(\bar{x}))$ $\alpha(x + \pi) = \alpha(\alpha^{-1}(\bar{x}) + \pi) = \alpha(x) + n\pi = \bar{x} + n\pi$ and from this $\alpha^{-1}(\bar{x} + n\pi) = \alpha^{-1}(\bar{x}) + \pi$. Further for any $t \in \boldsymbol{R}$ we have $\alpha^{-1} \circ \varepsilon_1(t) = \alpha^{-1}(t) + \pi$, hence especially for $t = \bar{x}$ we get $\alpha^{-1} \circ \varepsilon_1(\bar{x}) = \alpha^{-1}(\bar{x}) + \pi$ which together with $\alpha^{-1}(\bar{x} + n\pi) = \alpha^{-1}(\bar{x}) + \pi$ gives $\varepsilon_1(\bar{x}) = \bar{x} + n\pi$. Let $n \geqq 2$. Then $\varepsilon_1(t) > t + \pi$, thus also $\varepsilon_1^{[m]}(t) > t + \pi$ contrary to $\varepsilon_1^{[m]}(x_1) = x_1 + \pi$. Therefore $n = 1$ and $\varepsilon_1(\bar{x}) = \bar{x} + \pi$. From this follows $m = 1$ and $\bar{x}$ is the 1st kind determining number of type 1 of (q). From the equalities $\alpha \circ \varepsilon(t) = \alpha(t) + \pi$, $\alpha^{-1} \circ \varepsilon_1(t) = \alpha^{-1}(t) + \pi$, $\varepsilon(x) = x + \pi$, $\varepsilon_1(\bar{x}) = \bar{x} + \pi$, $\bar{x} = \alpha(x)$ we obtain

$$\varepsilon'(x) = \frac{\alpha'(x)}{\alpha' \circ \varepsilon(x)} = \frac{\alpha'(x)}{\alpha'(x + \pi)},$$

$$\varepsilon_1'(\bar{x}) = \frac{\alpha^{-1\prime}(\bar{x})}{\alpha^{-1\prime} \circ \varepsilon_1(\bar{x})} = \frac{\alpha' \circ \alpha^{-1} \circ \varepsilon_1(\bar{x})}{\alpha' \circ \alpha^{-1}(\bar{x})} = \frac{\alpha'(\alpha^{-1}(\bar{x}) + \pi)}{\alpha'(x)} = \frac{\alpha'(x + \pi)}{\alpha'(x)} = \frac{1}{\varepsilon'(x)}.$$

Hence $\varrho_\sigma = \bar{\varrho}_{-\sigma}$, $\sigma = \pm 1$. The blocks [q] and $[q]^{-1}$ have the same characteristic multipliers and the same category $(= (1, 1))$.

Let all solutions of equations at least of one from the blocks [q] and $[q]^{-1}$ be π-halfperiodic (because of $n = m = 1$ they cannot be π-periodic). For definiteness let this hold for all solutions of equations from [q]. Then all equations from [q] have the same dispersion equal to $t + \pi$. Consequently $\alpha(t + \pi) = \alpha(t) + \pi$ and thus also $\alpha^{-1}(t + \pi) = \alpha^{-1}(t) + \pi$. Then the equation ($\bar{q}$) has the dispersion equal to $t + \pi$, hence all its solutions are π-halfperiodic and it follows from Lemma 5 that even all solutions of equations from $[q]^{-1}$ are π-halfperiodic.

Theorem 2. *Let the carriers of equations from* [q] *be π-periodic,* [q] *have real characteristic multipliers and* (q) *admit independent solutions u, v satisfying* (1). *Then the carriers of equations from $[q]^{-1}$ are π-periodic and $[q]^{-1}$ has real characteristic multipliers if and only if there exists a phase α of* (q) *where $\alpha \circ \alpha$ is an elementary phase:* $\alpha \circ \alpha(t + \pi) = \alpha \circ \alpha(t) + \pi$.

Proof: Let the carriers of equations from [q] be π-periodic, characteristic multipliers of [q] be real and (q) admits independent solutions u, v satisfying (1).

a) Let α be such a phase of (q) that $\alpha \circ \alpha$ is an elementary phase. There exists $\varepsilon \in \mathfrak{E}$: $\alpha(t + \pi) = \varepsilon \circ \alpha(t)$. Let us put $\gamma(t) := \alpha \circ \alpha(t)$, $t \in \boldsymbol{R}$. Then $\alpha^{-1} = \alpha \circ \gamma^{-1}$ and because of $\gamma^{-1}(t + \pi) = \gamma^{-1}(t) + \pi$ we have

$$\alpha^{-1}(t + \pi) = \alpha \circ \gamma^{-1}(t + \pi) = \alpha(\gamma^{-1}(t) + \pi) = \varepsilon \circ \alpha \circ \gamma^{-1}(t) = \varepsilon \circ \alpha^{-1}(t)$$

hence $\alpha^{-1}(t + \pi) = \varepsilon \circ \alpha^{-1}(t)$ and by Lemmas 1 and 2 the carriers of equations from $[q]^{-1}$ are π-periodic. Let α^{-1} be a phase of ($\bar{q}$); $(\bar{q}) \in [q]^{-1}$. From the formulas $\alpha(t + \pi) = \varepsilon \circ \alpha(t)$, $\alpha^{-1}(t + \pi) = \varepsilon \circ \alpha^{-1}(t)$ it follows that ($\bar{q}$) and (q) possess the same 1st kind central dispersion with index k, $k = \operatorname{sign} \alpha'$, equal to ε implying that they have also the same dispersion. From Lemma 3 and from the assumption that [q] has real characteristic multipliers it follows that even the inverse block $[q]^{-1}$ has real characteristic multipliers, as well.

b) Suppose now the carriers of equations from $[q]^{-1}$ are π-periodic and $[q]^{-1}$ has real characteristic multipliers. Let α_1 be a phase of (q). By Theorem 1 the blocks [q] and $[q]^{-1}$ have the same characteristic multipliers, the same category (equal to (1, 1)) and all solutions of equations from [q] and $[q]^{-1}$ are or are not π-halfperiodic simultaneously. Thus, according to the Theorem from [7], there exist $\varepsilon \in \mathfrak{E}$, $\gamma \in \mathfrak{H}$: $\alpha_1^{-1} = \varepsilon \circ \alpha_1 \circ \gamma$. Herefrom $\alpha_1 = \gamma^{-1} \circ \alpha_1^{-1} \circ \varepsilon^{-1} = \gamma^{-1} \circ \varepsilon \circ \alpha_1 \circ \circ \gamma \circ \varepsilon^{-1}$, hence $\gamma \circ \alpha_1 \circ \varepsilon = \varepsilon \circ \alpha_1 \circ \gamma$. Consequently $\alpha_1^{-1} = \gamma \circ \alpha_1 \circ \varepsilon$, $\alpha_1^{-1} \circ \varepsilon^{-1} = = \gamma \circ \alpha_1 = \gamma \circ \varepsilon^{-1} \circ \varepsilon \circ \alpha_1$. Let $\alpha := \varepsilon \circ \alpha_1$. Then α is a phase of (q) and $\alpha^{-1} = = \gamma \circ \varepsilon^{-1} \circ \alpha$ thus $\alpha \circ \alpha = \varepsilon \circ \gamma^{-1} \in \mathfrak{H}$, because $\mathfrak{H}$ is a group and ε, γ^{-1} are its elements. This proves Theorem 2.

Theorem 3. *Let the carriers of equations from* [q] *be π-periodic and* $(1, n)$ *be the category of* [q]. *Then the carriers of equations from $[q]^{-1}$ are π-periodic and* $(2, m)$ *is the category of $[q]^{-1}$ if and only if $n \geqq 2$, $m = 0$ and $t + \dfrac{\pi}{n}$ is the dispersion of an equation from* [q].

Proof: Let the carriers of equations from [q] be π-periodic and $(1, n)$ be the category of [q].

a) Let the carriers of equations from $[q]^{-1}$ be π-periodic and $(2, m)$ be the category of $[q]^{-1}$. Let $e^{\pm a\pi i}$, $0 < a < 1$, be the characteristic multipliers of $[q]^{-1}$ and $(\bar{q}) \in [q]^{-1}$. Then by Lemma 4 there exists a phase α of $(\bar{q})$ such that $\alpha(t + \pi) = \alpha(t) + (a + 2m)\pi$. Now let α^{-1} be a phase of (q); $(q) \in [q]$. By Lemma 1 there exist $\varepsilon \in \mathfrak{C}$, $\varepsilon_1 \in \mathfrak{C}$: $\alpha(t + \pi) = \varepsilon \circ \alpha(t)$, $\alpha^{-1}(t + \pi) = \varepsilon_1 \circ \alpha^{-1}(t)$. It is clear that ε and ε_1 are the 1st kind central dispersions with index k, $k = \operatorname{sign} \alpha'$, of equations (q) and $(\bar{q})$, respectively, $\varepsilon(t) = t + (a + 2m)\pi$.

1. Let $\operatorname{sign} \alpha' = 1$. Then there exists a number x: $\varepsilon^{[n]}(x) = x + \pi$. Since $\varepsilon(t) = t + (a + 2m)\pi$ we get $\varepsilon^{[n]}(t) = t + n(a + 2m)\pi$, hence also

$$x + \pi = x + n(a + 2m)\pi \quad \text{and} \quad 1 = n(a + 2m).$$

From the last equality it follows: $m = 0$, $a = \dfrac{1}{n}$. Thus $n = \dfrac{1}{a} \geqq 2$.

2. Let $\operatorname{sign} \alpha' = -1$. Then there exists a number x_1: $\varepsilon^{[-n]}(x_1) = x_1 + \pi$. Since $\varepsilon(t) = t + (a + 2m)\pi$, so is $\varepsilon^{[-n]}(t) = t - n(a + 2m)\pi$, hence also $x_1 + \pi = x_1 - n(a + 2m)\pi$, $1 = -n(a + 2m)$. From the last equality it follows that $m = 0$. Then, however, $a = -\dfrac{1}{n} < 0$ which is contrary to $0 < a < 1$.

Thus we have proved that (q) has the dispersion $\varepsilon(t) = t + a\pi = t + \dfrac{\pi}{n}$, $n \geqq 2$, $m = 0$ and $[q]^{-1}$ has the characteristic multipliers $e^{\pm i\frac{\pi}{n}}$ and the category $(2, 0)$.

b) Let $n \geqq 2$ and $t + \dfrac{\pi}{n}$ be the dispersion of some equation from $[q]$. For definiteness let $t + \dfrac{\pi}{n}$ be the dispersion of (q). Further let α and α^{-1} be phases of (q) and $(\bar{q})$, respectively; $(q) \in [\bar{q}]^{-1}$. Then $\alpha\left(t + \dfrac{\pi}{n}\right) = \alpha(t) + \pi \,.\, \operatorname{sign} \alpha'$ which leads to $\alpha^{-1}(t + \pi) = \alpha^{-1}(t) + \dfrac{\pi}{n} \,.\, \operatorname{sign} \alpha'$. Then, of course, $e^{\pm i\frac{\pi}{n}}$ and $(2, 0)$ are the characteristic multipliers and the category of $[q]^{-1}$, respectively, as it follows from Lemmas 4 and 5.

Corollary 1. *Let the carriers of equations from* $[q]$ *and* $[q]^{-1}$ *be π-periodic, let the category of* $[q]$ *be* $(1, n)$, *and* $[q]^{-1}$ *having complex characteristic multipliers. Then* $n \geqq 2$, $(-1)^n$ *is the double characteristic multipliers of* $[q]$, $e^{\pm i\frac{\pi}{n}}$ *are the characteristic multipliers of* $[q]^{-1}$ *and* $(2, 0)$ *is its category.*

Proof: Let the carriers of equations from $[q]$ and $[q]^{-1}$ be π-periodic, let the. category of $[q]$ be $(1, n)$ and the characteristic multipliers of $[q]^{-1}$ being complex. Then by Theorem 3 yields: $n \geqq 2$, $(2, 0)$ is the category of $[q]^{-1}$ and $\varphi(t) = t + \dfrac{\pi}{n}$ is the dispersion of an equation from $[q]$; for definiteness let φ be the dispersion of (q). Then $\varphi_n(t) = t + \pi$ and from Lemma 3 it follows that the characteristic multiplier of (q), and therefore that of $[q]$ too, is double and equal to $(-1)^n$. In the proof of

Theorem 3 there has been even shown that $e^{\pm i \frac{\pi}{n}}$ are the characteristic multipliers of $[q]^{-1}$.

Theorem 4. *Let a, b be rational numbers, $0 < a < 1$, $0 < b < 1$. The carriers of equations from $[q]$ and $[q]^{-1}$ are π-periodic and the characteristic multipliers of $[q]$ and $[q]^{-1}$ are equal to $e^{\pm ia\pi}$ and $e^{\pm ib\pi}$, respectively, if and only if at least one of the following two conditions is satisfied:*

(i) *$a = \frac{\varkappa}{b} - 2n$, where $\varkappa = \pm 1$, $n \neq 0$ is an integer and there exists an elementary phase γ such that $\gamma(t + b\pi) = \gamma(t) + b\pi$ and $\frac{1}{b} \cdot \gamma(t)$ is a phase of an equation from $[q]$.*

(ii) *$b = \frac{\varkappa}{a} - 2m$, where $\varkappa = \pm 1$, $m \neq 0$ is an integer and there exists an elementary phase ϱ such that $\varrho\left(t + \frac{\pi}{a}\right) = \varrho(t) + \frac{\pi}{a}$ and $a \cdot \varrho(t)$ is a phase of an equation from $[q]$.*

If the condition (i) is satisfied, the categories of $[q]$ and $[q]^{-1}$ are $(2, n)$ and $(2, 0)$, respectively; if the condition (ii) is satisfied, the categories of $[q]$ and $[q]^{-1}$ are $(2, 0)$ and $(2, m)$, respectively.

Proof: 1. Let a, b be rational numbers, $0 < a < 1$, $0 < b < 1$, $\varkappa = \pm 1$ and $n \neq 0$ an integer such that $a = \frac{\varkappa}{b} - 2n$. Suppose next that there exists an elementary phase γ, $\gamma(t + b\pi) = \gamma(t) + b\pi$ and $\frac{1}{b} \cdot \gamma(t)$ is a phase of an equation from $[q]$, for definiteness let it be a phase of (q). Then also $\frac{\varkappa}{b} \cdot \gamma(t)$ is a phase of (q) and $\frac{\varkappa}{b} \cdot \gamma(t + \pi) = \frac{\varkappa}{b} \cdot (\gamma(t) + \pi) = \frac{\varkappa}{b} \cdot \gamma(t) + \frac{\varkappa}{b}\pi = \frac{\varkappa}{b} \cdot \gamma(t) + (a + 2n)\pi$. Therefore by Lemmas 4 and 5 $e^{\pm ia\pi}$ are characteristic multipliers of $[q]$ and $(2, n)$ is its category. Let the function $\gamma^{-1}(bt)$, which is inverse to $\frac{1}{b} \cdot \gamma(t)$, be a phase of $(\bar{q})$; $(\bar{q}) \in [q]^{-1}$. By assumption $\gamma(t + b\pi) = \gamma(t) + b\pi$, where from $\gamma^{-1}(t + b\pi) = \gamma^{-1}(t) + b\pi$. Therefore $(\bar{q})$ and $[q]^{-1}$ have the characteristic multipliers $e^{\pm ib\pi}$ and the category $(2, 0)$.

Let the condition (ii) be fulfilled. Completely analogous to the condition (i) we prove that $[q]$ has the characteristic multipliers $e^{\pm ia\pi}$ and the category $(2, 0)$ and $[q]^{-1}$ has the characteristic multipliers $e^{\pm ib\pi}$ and the category $(2, m)$.

2. Let the carriers of equations from $[q]$ and $[q]^{-1}$ be π-periodic and let them have the characteristic multipliers $e^{\pm ia\pi}$ and $e^{\pm ib\pi}$, respectively; a, b being rational numbers, $0 < a < 1$, $0 < b < 1$. Then $a = \frac{k}{l}$, $b = \frac{r}{s}$ with $0 < k < l$, $0 < r < s$ and k, l as well as r, s are comprime, positive integers. By Lemma 4 there exists a phase α of (q) and an integer n: $\alpha(t + \pi) = \alpha(t) + \left(\frac{k}{l} + 2n\right)\pi$. Let α^{-1} be a phase

of $(\bar{q})$; $(\bar{q}) \in [q]^{-1}$. From the structure of the phases of $(\bar{q})$ and from Lemma 4 then follows the existence of ε, $\varepsilon \in \mathfrak{E}$ and of an integer m with $\varepsilon \circ \alpha^{-1}(t + \pi) = \varepsilon \circ \alpha^{-1}(t) + \left(\frac{r}{s} + 2m\right)\pi$. So we have $\varepsilon \circ \alpha^{-1}(t + s\pi) = \varepsilon \circ \alpha^{-1}(t) + (r + 2ms)\,\pi = \varepsilon(\alpha^{-1}(t) + (r + 2ms)\,\pi \,.\, \text{sign}\,\varepsilon')$. Consequently

$$\alpha^{-1}(t + s\pi) = \alpha^{-1}(t) + (r + 2ms)\,\pi \,.\, \text{sign}\,\varepsilon'$$

and passing to the inverse functions we get to

$$\alpha(t) - s\pi = \alpha(t - (r + 2ms)\,\pi \,.\, \text{sign}\,\varepsilon'),$$

hence

$$\alpha(t + (r + 2ms)\,\pi \,.\, \text{sign}\,\varepsilon') = \alpha(t) + s\pi,$$

$$\alpha(t + (k + 2nl)(r + 2ms)\,\pi \,.\, \text{sign}\,\varepsilon') = \alpha(t) + s(k + 2nl)\,\pi. \tag{3}$$

Further $\alpha(t + \pi) = \alpha(t) + \left(\frac{k}{l} + 2n\right)\pi$ which yields

$$\alpha(t + sl\pi) = \alpha(t) + s(k + 2nl)\,\pi. \tag{4}$$

It then follows from (3) and (4) that

$$ls = (k + 2nl)(r + 2ms) \,.\, \text{sign}\,\varepsilon'$$

and further

$$1 = (a + 2n)(b + 2m) \,.\, \text{sign}\,\varepsilon'. \tag{5}$$

From (5) it immediately follows that $mn = 0$, $m^2 + n^2 > 0$.

a) Let $m = 0$. Then $n \neq 0$ and $a = \frac{1}{b} \,.\, \text{sign}\,\varepsilon' - 2n$. Let us put $\alpha_1(t) := \alpha \circ \varepsilon^{-1}(t)$, $t \in \boldsymbol{R}$. Then α_1 is a phase of an equation from $[q]$. From $\alpha_1(t + \pi) = \alpha \circ \varepsilon^{-1}(t+\pi) = \alpha(\varepsilon^{-1}(t) + \pi \,.\, \text{sign}\,\varepsilon') = \alpha \circ \varepsilon^{-1}(t) + \pi(a + 2n) \,.\, \text{sign}\,\varepsilon' = \alpha_1(t) + \frac{\pi}{b}$ we obtain $b \,.\, \alpha_1(t + \pi) = b \,.\, \alpha_1(t) + \pi$. So $b \,.\, \alpha_1(t)$ is an elementary phase written as γ, $\gamma(t) = b \,.\, \alpha_1(t)$. Further $\varepsilon \circ \alpha^{-1}(t + \pi) = \varepsilon \circ \alpha^{-1}(t) + b\pi$ which gives $\alpha \circ \varepsilon^{-1}(t + b\pi) = \alpha \circ \varepsilon^{-1}(t) + \pi$, $\alpha_1(t + b\pi) = \alpha_1(t) + \pi$ and $\gamma(t + b\pi) = b \,.\, \alpha_1(t + b\pi) = b \,.\, \alpha_1(t) + b\pi = \gamma(t) + b\pi$. This proves the existence of such an elementary phase $\gamma(t)$ with $\gamma(t + b\pi) = \gamma(t) + b\pi$ and $\frac{1}{b} \,.\, \gamma(t)$ is a phase of an equation from $[q]$. Evidently, $(2, n)$ and $(2, 0)$ are the categories of $[q]$ and $[q]^{-1}$, respectively.

b) Let $n = 0$. Then $m \neq 0$ and $b = \frac{1}{a} \,.\, \text{sign}\,\varepsilon' - 2m$. Let us put $\alpha_1(t) := \text{sign}\,\varepsilon' \,.\, \alpha \circ \varepsilon^{-1}(t)$, $t \in \boldsymbol{R}$. Then $\alpha_1(t)$ is a phase of an equation from $[q]$. From the equalities $\alpha_1(t + \pi) = \text{sign}\,\varepsilon' \,.\, \alpha \circ \varepsilon^{-1}(t + \pi) = \text{sign}\,\varepsilon' \,.\, \alpha(\varepsilon^{-1}(t) + \pi \,.\, \text{sign}\,\varepsilon') = \text{sign}\,\varepsilon' \,.\, (\alpha \circ \varepsilon^{-1}(t) + a\pi \,.\, \text{sign}\,\varepsilon') = \text{sign}\,\varepsilon' \,.\, \alpha \circ \varepsilon^{-1}(t) + a\pi = \alpha_1(t) + a\pi$ we obtain $\frac{1}{a}$.

$.\alpha_1(t + \pi) = \frac{1}{a} . \alpha_1(t) + \pi$. Consequently $\frac{1}{a} . \alpha_1(t)$ is an elementary phase written as ϱ, $\varrho(t) = \frac{1}{a} . \alpha_1(t)$. Further we have $\varepsilon \circ \alpha^{-1}(t + \pi) = \varepsilon \circ \alpha^{-1}(t) + (b + 2m)\pi = \varepsilon \circ \alpha^{-1}(t) + \frac{\pi}{a} . \text{sign } \varepsilon'$ which gives $\varepsilon \circ \alpha^{-1}(t + \pi) = \varepsilon \circ \alpha^{-1}(t) + \frac{\pi}{a} . \text{sign } \varepsilon'$ which in passing to the inverse functions gives $\text{sign } \varepsilon' . \alpha \circ \varepsilon^{-1}\left(t + \frac{\pi}{a}\right) = \text{sign } \varepsilon' . \alpha \circ \varepsilon^{-1}(t) + \pi$ equivalent to $\alpha_1\left(t + \frac{\pi}{a}\right) = \alpha_1(t) + \pi$. Herefrom $\varrho\left(t + \frac{\pi}{a}\right) = \frac{1}{a} . \alpha_1\left(t + \frac{\pi}{a}\right) = \frac{1}{a} . \alpha_1(t) + \frac{\pi}{a} = \varrho(t) + \frac{\pi}{a}$. This proves the existence of an elementary phase ϱ, such that $\varrho\left(t + \frac{\pi}{a}\right) = \varrho(t) + \frac{\pi}{a}$ and $a . \varrho(t)$ is a phase of an equation from [q]. Evidently, (2,0) and $(2, m)$ are the categories of [q] and $[q]^{-1}$, respectively.

Theorem 5. *Let at least one of the numbers a, b be irrational, $0 < a < 1$, $0 < b < 1$. The carriers of equations from [q] and $[q]^{-1}$ are π-periodic and have the characteristic multipliers $e^{\pm ia\pi}$ and $e^{\pm ib\pi}$, respectively, if and only if one of the two following conditions is satisfied:*

(i) $a = \frac{\varkappa}{b + 2m}$, *where $\varkappa = \pm 1$, m is an integer, and at is a phase of an equation from* [q].

(ii) $b = \frac{\varkappa}{a + 2n}$, *where $\varkappa = \pm 1$, n is an integer, and t/b is a phase of an equation from* [q].

If the condition (i) *is fulfilled, the categories of* [q] *and* $[q]^{-1}$ *are* $(2, 0)$ *and* $(2, m)$; *if the condition* (ii) *is fulfilled, the categories if* [q] *and* $[q]^{-1}$ *are* $(2, n)$ *and* $(2, 0)$, *respectively.*

Proof: 1. Let $a = \frac{\varkappa}{b + 2m}$ with $\varkappa = \pm 1$, m being an integer; let at least one of the numbers a, b be irrational, and let at be a phase of an equation from [q]. Then from Lemmas 4 and 5 and from $a . (t + \pi) = at + a\pi$ if follows that $e^{\pm ia\pi}$ are the characteristic multipliers of [q] and (2, 0) is its category. Further $\varkappa . \frac{t}{a}$ is the inverse function to $\varkappa at$. Thus from $\frac{\varkappa}{a} . (t + \pi) = \varkappa . \frac{t}{a} + \varkappa . \frac{\pi}{a} = \varkappa . \frac{t}{a} + (b + 2m)\pi$ we have: $e^{\pm ib\pi}$ being the characteristic multipliers of $[q]^{-1}$ and $(2, m)$ its category.

Let the condition (ii) be fulfilled. Completely analogous to the condition (*i*) we prove that [q] has the characteristic multipliers $e^{\pm ia\pi}$ and the category $(2, n)$ and $[q]^{-1}$ has the characteristic multipliers $e^{\pm ib\pi}$ and the category (2, 0).

2. Let the carriers of all equations from [q] and $[q]^{-1}$ be π-periodic. Let $e^{\pm ia\pi}$ and $e^{\pm ib\pi}$ be the characteristic multipliers of [q] and $[q]^{-1}$, respectively, with $0 <$

$< a < 1, 0 < b < 1$, and at least one of the numbers a, b be irrational (for definiteness let it be the number a). By Lemma 4 there exists a phase α of (q) $\in$ [q] $\alpha(t + \pi) = \alpha(t) + (a + 2n)\pi$, with n being an integer. Let α^{-1} be a phase of $(\bar{q})$; $(\bar{q}) \in [q]^{-1}$. From the equalities $\alpha^{-1}(t + (a + 2n)\pi) = \alpha^{-1}(t) + \pi$ and $\bar{q}(t) = -\frac{\alpha^{-1'''}(t)}{2 \cdot \alpha^{-1'}(t)} + \frac{3}{4}\left(\frac{\alpha^{-1''}(t)}{\alpha^{-1'}(t)}\right)^2 - \alpha^{-1'^2}(t)$ it follows that the continuous function $\bar{q}$ is periodic also with the period $a\pi$ and since a by our assumption is irrational, we have $q(t) = \text{const.}\ (= k < 0)$. Then, of course, $\sqrt{-k}t$ is a phase of $(\bar{q})$ and $\frac{t}{\sqrt{-k}}$ is a phase of an equation from [q]. Therefore there exist $\varepsilon \in \mathfrak{E}$, $\varepsilon_1 \in \mathfrak{E}$ and m being an integer such that $\varepsilon(\sqrt{-k}t + \sqrt{-k}\pi) = \varepsilon(\sqrt{-k}t) + (b + 2m)\pi$, $\varepsilon_1\left(\frac{t}{\sqrt{-k}} + \frac{\pi}{\sqrt{-k}}\right) = \varepsilon_1\left(\frac{t}{\sqrt{-k}}\right) + (a + 2n)\pi$ and also $\varepsilon(t + \sqrt{-k}\pi) = \varepsilon(t) + (b + 2m)\pi$, $\varepsilon_1\left(t + \frac{\pi}{\sqrt{-k}}\right) = \varepsilon_1(t) + (a + 2n)\pi$. We now show that from the last two equalities it follows $\varepsilon'(t) = \operatorname{sign} \varepsilon'$, $\varepsilon_1'(t) = \operatorname{sign} \varepsilon_1'$, $t \in \boldsymbol{R}$. From $\varepsilon(t + \sqrt{-k}\pi) = \varepsilon(t) + (b + 2m)\pi$ we have $\varepsilon'(t + \sqrt{-k}\pi) = \varepsilon'(t)$, hence $\varepsilon'(t)$ is a periodic function with the period $\sqrt{-k}\pi$. Hereby $\varepsilon'(t) = \frac{a_{11}a_{22} - a_{12}a_{21}}{(a_{11} \sin t + a_{12} \cos t)^2 + (a_{21} \sin t + a_{22} \cos t)^2}$ with a_{ij}, $(i, j = 1, 2)$ being appropriate numbers, $\det a_{ij} \neq 0$. Therefore, unless $\varepsilon'(t)$ is a constant function, $d\pi$, $d = \pm 1, \pm 2, \ldots$ are all periods of this function. So, if $\varepsilon'(t)$ is not a constant function, there exists a positive integer d_1: $\sqrt{-k} = d_1$. Then $(\bar{q})$ has the dispersion $\varphi(t) = t + \frac{\pi}{d_1}$ and it follows from the relation $\varphi_{d_1}(t) = t + \pi$ and from Lemma 3 that $(\bar{q})$ has the real characteristic multipliers contrary to our assumption. Therefore $\varepsilon'(t) = \text{const.}\ (= h)$ and from $\{\varepsilon, t\} - \varepsilon'^2(t) = -1$ we have $h = \operatorname{sign} \varepsilon'$. Completely analogous it can be shown that $\varepsilon_1'(t) = \operatorname{sign} \varepsilon_1'$, $t \in \boldsymbol{R}$. So, it holds $\sqrt{-k} \cdot \operatorname{sign} \varepsilon' = b + 2m$, $\frac{1}{\sqrt{-k}} \cdot \operatorname{sign} \varepsilon_1' = a + 2n$ and also $\operatorname{sign} \varepsilon' \cdot \operatorname{sign} \varepsilon_1' = (a + 2n)(b + 2m)$, hence $mn = 0$, $m^2 + n^2 > 0$. If $n = 0$, then $a = \frac{\operatorname{sign}(\varepsilon \circ \varepsilon_1)'}{b + 2m}$ and at is a phase of an equation from [q] and (2, 0) and (2, m) are the categories of [q] and $[q]^{-1}$, respectively. If $m = 0$, then $b = \frac{1}{a + 2n} \cdot \operatorname{sign}(\varepsilon \circ \varepsilon_1)'$ and $\frac{t}{b}$ is a phase of an equation from [q] and (2, n) and (2, 0) are the categories of [q] and $[q]^{-1}$, respectively. This completes the proof of Theorem 5.

Remark. If the carriers of equations from [q] and $[q]^{-1}$ are π-periodic and $e^{\pm ia\pi}$ and $e^{\pm ib\pi}$ are the characteristic multipliers of [q] and $[q]^{-1}$, respectively, wherein

$0 < a < 1$, $0 < b < 1$ and at least one of the numbers a, b is irrational, then follows from Theorem 5 that both numbers a, b are irrational.

From Theorem 5 immediately follows

Corollary 2. *Let a, b be irrational numbers, $0 < a < 1$, $0 < b < 1$. The carriers of equations from $[q]$ and $[q]^{-1}$ are π-periodic and $e^{\pm ia\pi}$ and $e^{\pm ib\pi}$ are the characteristic multipliers of $[q]$ and $[q]^{-1}$, respectively, if and only if $y'' = -a^2 y$ or $y'' = -\frac{1}{b^2} y$ belong to $[q]$ and $a = \frac{\varkappa}{b + 2m}$ or $b = \frac{\varkappa}{a + 2n}$ where $\varkappa = \pm 1$ and m, n are integers.*

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SOUHRN

CHARAKTERISTICKÉ KOŘENY BLOKU A INVERZNÍHO BLOKU LINEÁRNÍCH DIFERENCIÁLNÍCH ROVNIC DRUHÉHO ŘÁDU S π-PERIODICKÝMI KOEFICIENTY

SVATOSLAV STANĚK

V práci jsou uvedeny nutné a postačující podmínky pro π-periodičnost koeficientů diferenciálních rovnic typu (q): $y'' = q(t) y$, $q \in C_{\mathbf{R}}^{0}$, $\mathbf{R} = (-\infty, \infty)$, které jsou oscilatorické na $\mathbf{R}$ a leží v bloku $[q]$ a v inversním bloku $[q]^{-1}$. Za předpokladu, že rovnice v blocích $[q]$ a $[q]^{-1}$ mají π-periodické koeficienty, jsou dále vyšetřeny vztahy mezi charakteristickými kořeny obou bloků.

РЕЗЮМЕ

ХАРАКТЕРИСТИЧЕСКИЕ КОРНИ БЛОКА И ОБРАТНОГО БЛОКА ЛИНЕЙНЫХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ ВТОРОГО ПОРЯДКА С π-ПЕРИОДИЧЕСКИМИ КОЭФФИЦИЕНТАМИ

СВАТОСЛАВ СТАНЕК

В работе приводятся необходимые и достаточные условия для π-периодичности коэффициентов в блоке [q] и в обратном блоке $[q]^{-1}$ дифференциальных уравнений вида $y'' = q(t)\,y$, $q \in C_{\boldsymbol{R}}^{0}$, $\boldsymbol{R} = (-\infty, \infty)$ решения которых колеблются в $\boldsymbol{R}$. Иследуются соотношения между характеристическими корнями в блоках [q], $[q]^{-1}$ при условии что дифференциальные уравнения в обоих блоках имеют π-периодические коэффициенты.