1972 ACTA UNIVERSITATIS PALACKIANAE OLOMUCENSIS
FACULTAS RERUM NATURALIUM — TOM 37

NOTE ON THE PERIODIC SOLUTIONS OF SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

by

RAHMI IBRAHIM IBRAHIM ABDEL KARIM
Cairo University, Faculty of Science, Mathematical Department

(Received August 18th 1970)

§1. Preliminaries

In this paper we consider the system of differential equations

(a) $$\mathbf{x}' = A(t)\mathbf{x} + \mathbf{f}(t),$$

where $A(t)$ is an n . n continuous matrix of period p and $\mathbf{f}(t)$ is a continuous vector of n components of the same period p. It is well known [1] that a fundamental system of solutions $Y(t)$ of the corresponding homogeneous system of differential equations

(b) $$\mathbf{y}' = A(t)\mathbf{y}$$

can be obtained such that the constant matrix

(1) $$P = Y^{-1}(t)\, Y(t + p)$$

has the form $P = e^{Kp}$. Here the constant matrix K has the Jordan canonical normal form

(2) $$K = \begin{vmatrix} K_1 & & \\ & \ddots & \\ & & K_s \end{vmatrix} \quad \text{with} \quad K_\nu = \begin{vmatrix} \alpha_\nu^1 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \alpha_\nu \end{vmatrix},$$

where every submatrix K_ν is of order m_ν as well as the submatrix

(3) $$P_\nu = e^{K_\nu p}$$

of P. The fundamental system of solutions $Y(t)$ of (b) takes the form

(4) $$Y(t) = \Phi(t)\, e^{Kt}$$

(see [2]), where the matrix $\Phi(t)$ has the period p. The corresponding adjoint system of differential equations is

(c) $$\mathbf{z}' = -A^T(t)\, \mathbf{z}\,,$$

where A^T denotes the transposed of the matrix A.

We write $Y(t)$ in the form

(5) $$Y(t) = (Y_1(t), Y_2(t), \dots, Y_s(t))$$

with $Y_\nu(t) = (\mathbf{y}_{(\nu)}(t), \dots, \mathbf{y}_{[\nu]}(t))$,
where $Y_\nu(t)$ represents a rectangular matrix of type $n \cdot m_\nu$. The symbols (ν) and $[\nu]$ are defined by

(6) $$(\nu) = \sum_{\mu=1}^{\nu-1} m_\mu + 1, \qquad [\nu] = \sum_{\mu=1}^{\nu} m_\mu .$$

We subdivide the fundamental system of solutions $Z(t)$ of (c) and also $\Phi(t)$ in a similar way. From (1), (3) and (4), we get

(7) $$Y_\nu(t+p) = Y_\nu(t)P_\nu, \qquad Y_\nu(t) = \Phi_\nu(t)\, e^{K_\nu t} .$$

By means of the method of variation of parameters and referring to (5) and (7), the vector solution of (a) can be written as the sum of s vector components

(8) $$\mathbf{x}(t) = \sum_{1}^{s} {}^\nu\mathbf{x}(t)$$

with

(9) $${}^\nu\mathbf{x}(t) = \sum_{(\nu)}^{[\nu]} \mathbf{x}_\mu(t) = \sum_{(\nu)}^{[\nu]} \mathbf{y}_\mu(t)\, c_\mu(t) = Y_\nu(t)\,\mathbf{c}_\nu(t) = \Phi_\nu(t)\, e^{K_\nu t}\, \mathbf{c}(t) ,$$

where $\mathbf{c}_\nu(t)$ is the m_ν dimensional vector

(10) $$\mathbf{c}_\nu = \begin{bmatrix} c_{(\nu)} \\ c_{(\nu)+1} \\ \vdots \\ c_{[\nu]} \end{bmatrix} .$$

It is interesting to notice that the vector components ${}^\nu\mathbf{x}(t)$ of the solution $\mathbf{x}(t)$ of (a) satisfy the system of differential equations

(11) $${}^\nu\mathbf{x}' = A(t)\,{}^\nu\mathbf{x} + {}^\nu\mathbf{f}(t) \quad \text{with} \quad {}^\nu\mathbf{f}(t) = \Phi_\nu\, e^{K_\nu^T t}\, Z_\nu^T \mathbf{f} ,$$

which can be derived from the system (a) (see [3]).

In a previous paper [1] it was proved that to each submatrix K_ν of K vector solutions ${}^\nu\mathbf{x}(t)$ exist such that

(12) $${}^\nu\mathbf{x}(t) = \sum_{(\nu)}^{[\nu]} \mathbf{x}_\mu(t)$$

has the period p in either one of the following two cases:
(i) If the adjoint system (c) possesses the periodic solution $\mathbf{z}_{[\nu]}(t)$, i. e. if the eigenvalue α_ν of the corresponding submatrix K_ν equals zero and simultaneously

(13) $$\int_0^p \mathbf{z}_{[\nu]}^T(\tau)\, \mathbf{f}(\tau)\, \alpha\tau = 0 .$$

(This is the so-called exceptional case).
(ii) If the eigenvalue α_ν is not equal zero (principal case).

In this paper we investigate for both the preceding cases whether the sum (12), which satisfies the system of differential equations derived from (a), can be subdivided into periodic partical sums of the period p.

§ 2. Fundamental theorems

Let us subdivide the indices $(\nu), (\nu)+1, \ldots, (\nu)+m_\nu - 1 = [\nu]$ into two subsets, and let us differ between them by underlining the indices in the first subset or second subset once or twice respectively. Here it will be assumed that the first index (ν) belongs to the first subset, i. e. $(\nu) = \underline{(\nu)}$. Accordingly the sum ${}^\nu\mathbf{x} = \sum_{(\nu)}^{[\nu]} \mathbf{x}_\mu$ is subdivided into two partial sums

$$(14) \qquad {}^\nu\mathbf{x} = {}^\nu\underline{\mathbf{x}} + {}^\nu\underline{\underline{\mathbf{x}}}$$

with

$$(15) \qquad {}^\nu\underline{\mathbf{x}} = \sum \mathbf{x}_{\underline{\mu}} \quad \text{and} \quad {}^\nu\underline{\underline{\mathbf{x}}} = \sum \mathbf{x}_{\underline{\underline{\mu}}} \, .$$

Analogously the matrix Y_ν (see (5)) is subdivided into two matrices $\underline{Y}_\nu + \underline{\underline{Y}}_\nu$ and the wector $\mathbf{c}_\nu$ (see (10)) is subdivided into two vectors $\underline{\mathbf{c}}_\nu + \underline{\underline{\mathbf{c}}}_\nu$. Referring to (9), it follows that

$$(16) \qquad {}^\nu\underline{\mathbf{x}} = \underline{Y}_\nu \, . \, \underline{\mathbf{c}}_\nu(t) \quad \text{and} \quad {}^\nu\underline{\underline{\mathbf{x}}} = \underline{\underline{Y}}_\nu \, . \, \underline{\underline{\mathbf{c}}}_\nu \, .$$

We now assume that each of ${}^\nu\mathbf{x}(t)$ and ${}^\nu\underline{\mathbf{x}}(t)$ has the period p. This means the same as the assumption that the first partial sum ${}^\nu\underline{\mathbf{x}}(t)$ as well as the total sum ${}^\nu\mathbf{x}(t) = {}^\nu\underline{\mathbf{x}} + {}^\nu\underline{\underline{\mathbf{x}}}$ has the period p. Then holds the following theorem:

Theorem 1. *Let the vector* ${}^\nu\mathbf{x} = \sum_{(\nu)}^{[\nu]} \mathbf{x}_\mu$ *be subdivided into two partial sums* (15), *where the indices in each partial sum run over a subset of the indices* (ν), $(\nu)+1, \ldots, [\nu]$ *as it is described above, such that each of the partial sums has the period p. Then all components* $\mathbf{x}_{\underline{\underline{\mu}}}(t)$ *are identic zero for* $\mu \geq \operatorname{Min}(\underline{\underline{\mu}})$. *Consequently the sum* ${}^\nu\mathbf{x}(t)$ *is subdivided into two partial sums*

$$(17) \qquad {}^\nu\mathbf{x}(t) = \sum_{(\nu)}^{\gamma-1} \mathbf{x}_\mu(t) \quad \textit{and} \quad {}^\nu\underline{\underline{\mathbf{x}}}(t) = \sum_{\gamma}^{[\nu]} \mathbf{x}_\mu(t)$$

with $\gamma = \operatorname{Min}(\underline{\underline{\mu}})$, *in which the first sum has the period p and the second is identic zero.*

Proof. We have only to prove that all the components $\mathbf{x}_\mu(t)$ for $\mu \geq \operatorname{Min}(\underline{\underline{\mu}})$ vanish identically. Referring to the assumption, the sum

$$(18) \qquad {}^\nu\mathbf{x}(t) = \sum_{(\nu)}^{[\nu]} \mathbf{x}_\mu(t) = \sum_{(\nu)}^{[\nu]} \mathbf{y}_\mu \, c_\mu(t)$$

has the period p. Instead of (18), we can also write

$$(19) \qquad {}^\nu\mathbf{x}(t) = \sum_{(\nu)}^{[\nu]} \mathbf{y}_\mu \, \underline{c}_\mu(t) \, ,$$

where all functions $\underline{c}_\mu(t)$ for $\mu = \underline{\underline{\mu}}$ are identic zero. In particular

$$(20) \qquad \underline{c}_\gamma(t) = 0 \quad \text{for} \quad \gamma = \operatorname{Min}(\underline{\underline{\mu}}) \, .$$

By virtue of the periodicity of ${}^{\nu}\mathbf{x}(t)$ and referring to (9), (17), (3), we get

$$ {}^{\nu}\mathbf{x}(t+p) - {}^{\nu}\mathbf{x}(t) = \Phi_\nu(t)\, e^{K_\nu t}\,(P_\nu\, \mathbf{c}_\nu(t+p) - \mathbf{c}_\nu(t)) \,. $$

Consequently, it follows that

$$ P_\nu\, \mathbf{c}_\nu(t+p) = \mathbf{c}_\nu(t) \tag{21} $$

with P_ν from (3). Denoting $c_\mu(t+p)$ by $\tilde{c}_\mu(t)$, we get at once by virtue of the assumption (20)

$$ c_\gamma(t) = 0 = \tilde{c}_\gamma(t) \,. $$

The system (21) takes then the scalar form

$$ \left\{ \begin{aligned} c_{(\nu)}(t) &= e^{\alpha_\nu p} \sum_{\mu=0}^{[\nu]-(\nu)} \frac{p^\mu}{\mu!}\, \tilde{c}_{(\nu)+\mu}(t) \\ &\;\vdots \\ c_{\gamma-1}(t) &= e^{\alpha_\nu p} \sum_{\mu=0}^{[\nu]-\gamma+1} \frac{p^\mu}{\mu!}\, \tilde{c}_{\gamma+\mu-1}(t) \end{aligned} \right. \tag{22} $$

$$ c_\gamma(t) = e^{\alpha_\nu p} \sum_{\mu=0}^{[\nu]-\gamma} \frac{p^\mu}{\mu!}\, \tilde{c}_{\gamma+\mu}(t) \tag{$\overline{22}$} $$

$$ \left\{ \begin{aligned} c_{\gamma+1}(t) &= e^{\alpha_\nu p} \sum_{\mu=0}^{[\nu]-\gamma-1} \frac{p^\mu}{\mu!}\, \tilde{c}_{\gamma+\mu+1}s(t) \\ &\;\vdots \\ c_{[\nu]}(t) &= e^{\alpha_\nu p}\, \tilde{c}_{[\nu]}(t) \,. \end{aligned} \right. \tag{$\overline{\overline{22}}$} $$

In order to write $(\overline{22})$ and $(\overline{\overline{22}})$ in convenient forms, we denote the $([\nu]-\gamma)$ — dimensional row vector $\left(p, \frac{p^2}{2!}, \ldots, \frac{p^{[\nu]-\gamma}}{([\nu]-\gamma)!}\right)$ by $\mathbf{q}_*^T$ and the $([\nu]-\gamma)$ — dimensional column-vector

$$ \begin{bmatrix} c_{\gamma+1} \\ c_{\gamma+2} \\ \vdots \\ c_{[\nu]} \end{bmatrix} \quad \text{by } \mathbf{c}_* \,. \tag{23} $$

Further we denote the square matrix

$$ \begin{bmatrix} 1, & p, & \ldots, & \dfrac{p^{[\nu]-\gamma-1}}{([\nu]-\gamma-1)!} \\ & 1, & \ldots, & \dfrac{p^{[\nu]-\gamma-2}}{([\nu]-\gamma-2)!} \\ & & \ddots & \vdots \\ & & & 1 \end{bmatrix} $$

of order $[\nu] - \gamma$ by P_*. Hence the system of equations $(\bar{\bar{22}})$ and $(\bar{22})$ take the forms (see (20))

$$e^{\alpha_\nu p} P_* \bar{\mathbf{c}}_*(t) = \mathbf{c}_*(t) \tag{24}$$

and

$$\mathbf{q}_*^T \bar{\mathbf{c}}_*(t) = 0 = \mathbf{q}_*^T \mathbf{c}_*(t) \tag{25}$$

respectively. By virtue of (24) and (25), we obtain successively the following identities

$$\mathbf{q}_*^T P_*^\mu \bar{\mathbf{c}}_*(t) = 0 \quad \text{for} \quad \mu = 0, 1, \ldots, [\nu] - \gamma - 1 \,. \tag{26}$$

If namely (26) is valid for μ, then it will be also valid for $\mu + 1$ because

$$\mathbf{q}_*^T P_*^\mu \bar{\mathbf{c}}_*(t) = 0 = \mathbf{q}_*^T P_*^\mu \bar{\mathbf{c}}_*(t - p) = \mathbf{q}_*^T P_*^\mu \mathbf{c}_*(t) = e^{\alpha_\nu p} \mathbf{q}_*^T P_*^{\mu+1} \bar{\mathbf{c}}_*(t) \,.$$

By means of linear combinations of (26), we obtain

$$\mathbf{q}_*^T (P_* - I_*)^\mu \bar{\mathbf{c}}_*(t) = 0 \quad \text{for} \quad \mu = 0, 1, \ldots, [\nu] - \gamma - 1 \,, \tag{27}$$

where I_* is a unit matrix of the same order as P_*. Using further linear combinations of the equations (27), we get the system of equations

$$\mathbf{q}_*^T K_*^\mu \bar{\mathbf{c}}_*(t) = 0 \quad \text{for} \quad \mu = 0, 1, \ldots, [\nu] - \gamma - 1 \,, \tag{28}$$

where

$$K_* = \begin{vmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{vmatrix} = \frac{1}{p} \log P_*$$

(see (2) with $\alpha_\nu = 0$). W. r. t. convergence see e. g. [4] §8. Here we only need to expand K_* in the power series

$$K_* = \frac{1}{p}\Big[(P_* - I_*) - \frac{1}{2}(P_* - I_*)^2 + \frac{1}{3}(P_* - I_*)^3 \mp \ldots + (-1)^{[\nu]-\gamma} \frac{1}{[\nu] - \gamma - 1}(P_* - I_*)^{[\nu]-\gamma-1}\Big] \,.$$

(Notice that $(P_* - I_*)^r = 0$ for $r \geq [\nu] - \gamma$). From this we get K_*^μ for $\mu = 1, 2, \ldots, [\nu] - \gamma - 1$ by means of the power series. Clearly $K_*^0 = I_*$. Setting $\mathbf{q}_*^T$ in (28) and observing that

$$K_*^2 = \begin{vmatrix} 0 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & \ddots & 0 \\ & & & & 0 \end{vmatrix}, \ldots, K_*^{[\nu]-\gamma-1} = \begin{vmatrix} 0 & 0 & 0 & & 1 \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 0 \\ & & & \ddots & 0 \\ & & & & 0 \end{vmatrix}, K_*^{[\nu]-\gamma} = 0 \,,$$

we obtain the system of $([\nu]) - \gamma)$ equations

$$\begin{vmatrix} p, & \frac{p^2}{2!}, & \ldots, & \frac{p^{[\nu]-\gamma}}{([\nu] - \gamma)!} \\ & p, & \ldots, & \frac{p^{[\nu]-\gamma-1}}{([\nu] - \gamma - 1)!} \\ & & \ddots & \vdots \\ & & & p \end{vmatrix} \mathbf{c}_*^{(t+p)} = \mathbf{0} \,,$$

whose coefficient matrix is a regular triangular matrix. Referring to (20) and (23) it follows that $c_\mu(t) = 0$ for every index $\gamma \leqq \mu \leqq [\nu]$. Thus the theorem is proved

As an organization of theorem 1, we state the following

Theorem 2. Let the vector ${}^\nu\mathbf{x} = \sum_{(\nu)}^{[\nu]} \mathbf{x}_\mu(t)$ be subdivided into two partial sums

$$ {}^\nu\underline{\mathbf{x}} = \sum_{(\nu)}^{\gamma-1} \mathbf{x}_\mu(t) \quad \text{and} \quad {}^\nu\underline{\underline{\mathbf{x}}} = \sum_{\gamma}^{[\nu]} \mathbf{x}_\mu(t) $$

such that each of the two partial sums has the period p. Then it follows in the principal case that:

i) The sum ${}^\nu\underline{\underline{\mathbf{x}}}(t) = \sum_{\gamma}^{[\nu]} \mathbf{x}_\mu(t)$ is identic zero

ii) The vector $\boldsymbol{f}(t)$ satisfies the condition

$$ \mathbf{z}_\mu^T(t)\,\boldsymbol{f}(t) = 0 \quad \text{for} \quad \mu = \gamma, \gamma + 1, \ldots, [\nu]\,. $$

iii) ${}^\nu\underline{\mathbf{x}}(t)$ is uniquely determined by means of the vectors $\boldsymbol{f}(t)$. In the exceptional case, it follows that:

i) The sum ${}^\nu\underline{\underline{\mathbf{x}}}$ is again identic zero.

ii) The vector $\boldsymbol{f}(t)$ satisfies the condition

$$ \mathbf{z}_\mu^T(t)\,\boldsymbol{f}(t) = 0 \quad \text{for} \quad \mu = \gamma, \gamma + 1, \ldots, [\nu] $$

and the condition

ii′)
$$ \int_t^{t+p} \mathbf{z}_{\gamma-1}^T(\tau)\,\boldsymbol{f}(\tau)\,d\tau = 0\,. $$

iii) The sum ${}^\nu\underline{\mathbf{x}}(t)$ is uniquely determined up to an arbitrary additive multiple of $c_{(\nu)}(0) \cdot \mathbf{y}_{(\nu)}(t)$.

Proof. The statement i) in both cases is a direct consequence of theorem 1, because all the functions $c_\mu(t)$ for $\mu = \gamma, \gamma + 1, \ldots, [\nu]$ are identic zero. Moreover, since

$$ c_\mu(t) = \int_0^t \mathbf{z}_\mu^T(\tau)\,\boldsymbol{f}(\tau)\,d\tau + c_\mu(0) \tag{30} $$

(see [2], §3) then the integral on the R. S. of (30) must vanish identically in t. And consequently the integrand $\mathbf{z}_\mu^T(\tau)\,\boldsymbol{f}(\tau)$ must be identic zero for $\mu = \gamma, \gamma + 1, \ldots, [\nu]$.

In the exceptional case we have to prove also the validity of the statement ii′). Since $c_\mu(t)$ ($\mu = \gamma, \ldots, [\nu]$) vanish identicaly in t and $e^{\alpha_\nu p} = 1$ in the exceptional case, then the $(\gamma - (\nu))$-th equation in (22) becomes

$$ c_{\gamma-1}(t + p) = c_{\gamma-1}(t)\,. $$

And by virtue of (30), we immediately obtain ii′).

Whereas the statements i), ii) and ii′) necessarily follow from the assumption in the theorem, the statement iii) expresses that the conditions ii) and ii′) are sufficient for the existence of a periodic solution of the form ${}^\nu\underline{\mathbf{x}} = \sum_{(\nu)}^{\gamma-1} \mathbf{x}_\mu(t)$.

Even the condition $\int_0^p \mathbf{z}_{\gamma-1}^T(\tau)\, \boldsymbol{f}(\tau)\, d\tau = 0$ is sufficient instead of the condition ii').

The necessary and sufficient condition for the existence of a periodic solution ${}^{\nu}\mathbf{x}(t)$ for given $\boldsymbol{f}(t)$ can be easily obtained from (a), (7) and (3) (see also [1]) as:

$$ {}^{\nu}\mathbf{x}(t+p) - {}^{\nu}\mathbf{x}(t) = Y_{\nu}(t)\left[P_{\nu}\int_0^p Z_{\nu}^T(\tau)\, \boldsymbol{f}(\tau)\, d\tau - (I_{\nu} - P_{\nu})\mathbf{c}_{\nu}(0)\right] = \boldsymbol{0} $$

i. e.

$$ (P_{\nu}^{-1} - I_{\nu})\, \mathbf{c}_{\nu}(0) = \int_0^p Z_{\nu}^T(\tau)\, \boldsymbol{f}(\tau)\, d\tau\,. \tag{31} $$

where I_{ν} is the unit matrix of order m_{ν}. Or in components

$$ \begin{bmatrix} e^{-\alpha_{\nu}p}-1, & -p, & \dots, & \dfrac{(-p)^{m_{\nu}-1}}{(m_{\nu}-1)!} \\ & e^{-\alpha_{\nu}p}-1, & \dots, & \dfrac{(-p)^{m_{\nu}-2}}{(m_{\nu}-2)!} \\ & & \ddots & \vdots \\ & & & e^{-\alpha_{\nu}p}-1 \end{bmatrix} \cdot \begin{bmatrix} c_{(\nu)}(0) \\ c_{(\nu)+1}(0) \\ \vdots \\ c_{[\nu]}(0) \end{bmatrix} = \begin{bmatrix} \int_0^p \mathbf{z}_{(\nu)}^T(\tau)\, \boldsymbol{f}(\tau)\, d\tau \\ \int_0^p \mathbf{z}_{(\nu)+1}^T(\tau)\, \boldsymbol{f}(\tau)\, d\tau \\ \vdots \\ \int_0^p \mathbf{z}_{[\nu]}^T(\tau)\, \boldsymbol{f}(\tau)\, d\tau \end{bmatrix}. \tag{32} $$

If the condition ii) is satisfied, then the integrals $\int_0^p \mathbf{z}_{\mu}^T(\tau)\, \boldsymbol{f}(\tau)\, d\tau$ on the R. S. of (32) for $\mu = \gamma, \gamma + 1, \dots, [\nu]$ must vanish. Consequently, it follows from (32) in the principal case, that the constants $c_{\gamma}(0), c_{\gamma+1}(0), \dots, c_{[\gamma]}(0)$ are equal zero, while the other constants $c_{(\nu)}(0), \dots, c_{\gamma-1}(0)$ are uniquely determined because of the regularity of the coefficient matrix.

In the exceptional case, we consider the system (32) with $e^{\alpha_{\nu}p} = 1$ (exceptional condition). Referring to the condition ii), the integrals $\int_0^p \mathbf{z}_{\mu}^T(\tau)\, \boldsymbol{f}(\tau)\, d\tau$ for $\mu = \gamma$, $\gamma + 1, \dots, [\nu]$ must vanish and hence the integral $\int_0^p \mathbf{z}_{\gamma-1}(\tau)\, \boldsymbol{f}(\tau)\, d\tau$ must vanish also. Eliminating the first column and the last row of the coefficient matrix in (32), we obtain a system of equations in the constant $c_{(\gamma)+1}(0), \dots, c_{[\gamma]}(0)$ with a regular coefficient matrix. Therefore the constants $c_{\gamma}(0), \dots, c_{[\gamma]}(0)$ are zeros and the other constants $c_{(\gamma)+1}(0), \dots, c_{\gamma-1}(0)$ are uniquely determited. Clearly the constant $c_{(\nu)}(0)$ remains arbitrary. Since the functions $c_{\mu}(t)$ vanish in both cases, (see 30), then the statemetn i) follows from the formula $\mathbf{x}_{\mu}(t) = \mathbf{y}_{\mu}(t) \,.\, c_{\mu}(t)$

REFERENCES

[1] *R. I. I. Abdel Karim:* Studium des Resonanzfalles bei Systemen gewöhnlicher Differentialgleichungen. Arch. Rational Mech. Anal. 10, 229 – 241 (1962).

[2] *R. I. I. Abdel Karim:* Über den Resonanzfall bei gewöhnlichen linearen Differentialgleichungen n-ter Ordnung mit periodischen veränderlichen Koeffizienten. Archivum mathematicum (Brno), T. 3, 11 – 30 (1967).

[3] *R. I. I. Abdel Karim:* On the vector solution of the system of the differential equations $\mathbf{x}' = A(t)\,\mathbf{x} + \mathbf{f}(t)$. U. pub. in Acta F. R. N. (Bratislava).

[4] *W. Schmeidler:* Vorträge über Determinanten und Matrizen mit Anwendungen in Physik und Technik. Berlin: Akademic Verlag 1949.

Summary

NOTE ON THE PERIODIC SOLUTIONS OF SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

RAHMI IBRAHIM IBRAHIM ABDEL KARIM

In this paper we consider the system of equations (*) $\mathbf{x}' = A(t)\mathbf{x} + \mathbf{f}(t)$ with $A(t), \mathbf{f}(t)$ continuous and of period p. Let Y(t) be a fundamental system of solutions of the homogeneous system corresponding to (*) such that the constant matrix $P = Y^{-1}(t)\,Y(t + p)$ has the form $P = e^{Kp}$ and K has the Jordan canonical normal form with the submatrices K_ν (for $\nu = 1, \ldots, s$) of order m_ν. In a previous paper is proved that to each submatrix K_ν of K, there corresponds – in certain cases – vector solutions ${}^\nu\mathbf{x}(t)$ such that $(**)^\nu\ \mathbf{x}(t) = \sum_{(\nu)}^{[\nu]} \mathbf{x}_\mu(t)$ has the period p. Here $(\nu) = \sum_1^{\nu-1} m_\mu + 1$, $[\nu] = \sum_1^{\nu} m_\mu$ and $\mathbf{x}(t) = \sum_1^{s} {}^\nu\mathbf{x}(t)$. In this paper will be mainly investigated, whether the sum (**), which satisfies the system of differential equations derived from (*), can be subdivided into periodic partial sums of the period p.