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Jan Voráček

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Vedoucí katedry: Prof. RNDr. Miroslav Laitoch, CSc.

ON THE SOLUTION OF CERTAIN NON-LINEAR DIFFERENTIAL EQUATIONS OF THE THIRD ORDER

JAN VORÁČEK

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1. Let us at first consider the equation

$$x''' + f(x')\, x'' + g(x)\, x' + h(x) = e(t) \tag{1}$$

with $f(y), g(x), h(x), e(t)$ continuous for every real argument. With (1) deals our note [1]; here we will prove some more general results.

Let us pose $F(y) = \int_0^y f(s)\, ds$, $G(x) = \int_0^x g(s)\, ds$. In what follows we will also use some assumptions about f, g, h, e.

Assumption A_1: There exist positive numbers $g, \varepsilon, Y > 1, H, E$, such that

$$|\, g(x) \,| \leqq g \qquad \text{for every } x, \tag{2}$$

$$f(y) \geqq \varepsilon \qquad \text{for every } y, \qquad \frac{f(y)}{|\, y \,|} \geqq \varepsilon \qquad \text{for every } |\, y \,| \geqq Y, \tag{3}$$

$$|\, h(x) \,| \leqq H \qquad \text{for every } x, \tag{4}$$

$$|\, e(t) \,| \leqq E \qquad \text{for every } t. \tag{5}$$

Assumption A_2 : A_1 holds and

$$g(x) \geqq \varepsilon \qquad \text{for every } x, \tag{6}$$

$$\left|\, \int_0^t e(s)\, ds \,\right| \leqq E \qquad \text{for every } t. \tag{7}$$

Assumption A_3 : A_2 holds and there exists a positive number h, such that

$$h(x) \operatorname{sgn} x \geqq 0 \qquad \text{for every } |\, x \,| \geqq h. \tag{8}$$

Assumption A_4 : A_2 holds and there exist a real function $r(x)$ and a positive number ϱ, such that $r(x)$ is continuous for every $|\, x \,| \geqq \varrho$ and

$$\liminf_{|x| \to +\infty} r(x) \operatorname{sgn} x > 0. \tag{9}$$

Simultaneously, $h(x)$ satisfies the relation

$$\liminf_{|x| \to +\infty} r(x)\, h(x) > 0. \tag{10}$$

Remark 1 : $r(x)$ from A_4 may be f.i. $e^{n|x|} \operatorname{sgn} x$.

Assumption A_5 : A_2 holds and we have

$$\limsup_{|x| \to +\infty} G(x)\, h(x) < -H(D_1' + \max_{|y| \leqq D_1'} F(y) + E), \tag{11}$$

where the constant D_1' is defined in (25).

Theorem 1: If A_3 holds, then each solution $x(t)$ of (1) exists on the interval $I = \langle t_0, +\infty)$ (t_0 stands for a real number) and is bounded on I. There also exists a constant D' such, that every $x(t)$ fulfils the relations

$$\limsup_{t \to +\infty} |x'(t)| \leqq D', \qquad \limsup_{t \to +\infty} |x''(t)| \leqq D'. \tag{12}$$

The proof of theorem 1 will be divided in several steps.

Lemma 1: If A_1 holds, then every $x(t)$ exists on I and the relation

$$\limsup_{t \to +\infty} |x''(t)| \leqq \frac{1}{\varepsilon}(gY + H + E) + 1 = K \tag{13}$$

holds.

The proof of lemma 1 can be obtained with the same method as in the mentioned note [1]; a change is necessary only in the estimations for the function $\frac{1}{2}x''^2(t)$.

We get

$$\frac{d}{dt}\frac{1}{2}x''^2(t) = x''x''' = -f(x')x''^2 - g(x)x'x'' - h(x)x'' + e(t)x''$$

and hence, using (2), (3), (4) and (5)

$$\text{for } |x'| \leqq Y : \frac{d}{dt}\frac{1}{2}x''^2(t) \leqq -|x''|(\varepsilon|x''| - gY - H - E), \tag{14}$$

$$\text{for } |x'| \geqq Y ; \frac{d}{dt}\frac{1}{2}x''^2(t) \leqq -|x''|(\varepsilon|x'||x''| - g|x'| - H - E). \tag{15}$$

For $|x'| \geqq Y$ and $|x''| \geqq K$ we then obtain

$$\varepsilon|x'||x''| \geqq (gY + H + E + \varepsilon)|x'| \geqq g(x') + H + E + \varepsilon. \tag{16}$$

We have thus from (14), (15), (16)

$$\frac{d}{dt}\frac{1}{2}x''^2(t) \leqq -\varepsilon K \qquad \text{for every } |x''| \geqq K.$$

The remaining part of the proof equals that of [1].

Lemma 2: If A_2 holds, then there exists a constant D', such that for every $x(t)$ the relations (12) hold.

Proof: Let us prove at first that

$$\liminf_{t \to +\infty} |x'(t)| \leqq \frac{H}{\varepsilon} + 2. \tag{17}$$

We fix a $x(t)$. By lemma 1 there exists a $t_1 \geqq t_0$ such that $|x''(t)| \leqq K + 1$ for every $t \geqq t_1$. If, for a $t_2 > t_1$, should be $|x'(t)| \geqq \frac{H}{\varepsilon} + 2$ for every $t \geqq t_2$, we should get from (6)

$$|G(x)(t)) - G(x(t_2))| = \int_{t_2}^{t} g(x(s))\, x'(s)\, ds \operatorname{sgn} x'(s) \geqq (H + \varepsilon)(t - t_2). \tag{18}$$

But (4) implies

$$\left| \int_{t_2}^{t} h(x(s))\, ds \right| \leqq H(t - t_2)\ (t \geqq t_2), \tag{19}$$

and we thus obtain from (18) and (19) for every $t \geqq t_2$

$$[G(x(t)) - G(x(t_2))] \operatorname{sgn} x'(t) - \left| \int_{t_2}^{t} h(x(s))\, ds \right| \geqq \varepsilon(t - t_2). \tag{20}$$

Integrating (1) from t_2 to $t \geqq t_2$ we have

$$F(x'(t)) + G(x(t)) - G(x(t_2)) + \int_{t_2}^{t} h(x(s))\, ds = x''(t_2) - x''(t) + $$
$$+ F(x(t_2)) + \int_{t_2}^{t} e(s)\, ds,$$

and hence, using (7) and multiplying with the constant sgn $x'(t)$:

$$F(x'(t)) \operatorname{sgn} x'(t) + [G(x(t)) - G(x(t_2))] \operatorname{sgn} x'(t) - \left| \int_{t_2}^{t} h(x(s))\, ds \right| \leqq$$
$$\leqq 2(K + 1 + E) + F(x'(t_2)). \tag{21}$$

By means of (20) we get from (21) the inequality

$$F(x'(t)) \operatorname{sgn} x'(t) \leqq 2(K + 1 + E) + F(x'(t_2)) - \varepsilon(t - t_2),$$

which is, for $t - t_2$ large enough in contradiction to the properties of the function F (cf. (3)). Thus, (17) is proved.

Let us suppose now that there exists an interval $\langle T_1, T_2 \rangle$ $(t_1 \leqq T_1 < T_2 < +\infty)$, such that $x'(T_1) = x'(T_2) = \frac{H}{\varepsilon} + 2$, $x'(t) > \frac{H}{\varepsilon} + 2$ on (T_1, T_2). Let $\Theta \in (T_1, T_2)$ be the number with the property $x'(\Theta) = \max x'(t)$ on $\langle T_1, T_2 \rangle$. Integrating (1) from T_1 to Θ we obtain $(x''(\Theta) = 0)$:

$$0 \leqq F(x'(\Theta)) = F\left(\frac{H}{\varepsilon} + 2\right) + x''(T_1) - [G(x(\Theta)) - G(x(T_1))] -$$

$$- \int_{T_1}^{\Theta} h(x(t))\,\mathrm{d}t + \int_{T_1}^{\Theta} e(t)\,\mathrm{d}t. \tag{22}$$

On $\langle T_1, \Theta\rangle$ one can easily prove an inequality analogous to (20) i.e. in a weaker form

$$- [G(x(\Theta)) - G(x(T_1))] - \int_{T_1}^{\Theta} h(x(t))\,\mathrm{d}t < 0,$$

and thus, we obtain from (22)

$$0 \leqq F(x'(\Theta)) \leqq F\left(\frac{H}{\varepsilon} + 2\right) + K + 1 + 2E = L_1. \tag{23}$$

The case $x'(t) \leqq -\frac{H}{\varepsilon} - 2$ on $\langle T_1, T_2\rangle$ leads to the inequality

$$0 \geqq F(x'(\Theta)) \geqq F\left(\frac{H}{\varepsilon} - 2\right) - K - 1 - 2E = L_2. \tag{24}$$

Herewith the lemma 2 is proved. We have seen also that D' must satisfy the inequality

$$D' \leqq \max\,(K, F_{-1}(L_1), -F_{-1}(L_2)) = D_1', \tag{25}$$

($F_{-1}(y)$ is the inverse function of $F(y)$).

Proof of theorem 1: We fix again a $x(t)$. There exists then by lemma 2 a $t_x \geqq t_0$ with the property that

$$|\, x'(t) \,| \leqq D' + 1, \qquad |\, x''(t) \,| \leqq D' + 1 \quad \text{for every } t \geqq t_x. \tag{26}$$

If on any interval $\langle t_1, t_2)$ $(t_x \leqq t_1 < t_2 \leqq +\infty)$ the inequality

$$|\, x(t) \,| \geqq h, \tag{27}$$

holds, then, integrating (1) from t_1 to $t \in (t_1, t_2)$, multiplying it by the constant $\operatorname{sgn} x(t)$ and using (27), (8), (26), (7) and (3) we get

$$\operatorname{sgn} x(t)[G(x(t)) - G(x(t_1))] \leqq |\, F(x'(t) - F(x'(t_1)) \,| + |\, x''(t) -$$

$$- x''(t_1) \,| - \int_{t_1}^{t} h(x(s))\,\mathrm{d}s \operatorname{sgn} x(t) + \left|\int_{t_1}^{t} e(s)\,\mathrm{d}s\right| \leqq 2[\max\,(F(D' + 1),$$

$$-F(-D' - 1)) + D' + 1 + \mathrm{E}] = \mathrm{P}.$$

Hence

$$|\, G(x(t)) \,| \leqq |\, G(x(t_1)) \,| + \mathrm{P}. \tag{28}$$

If (27) is valid on $\langle t_1, +\infty\rangle$, then our theorem easy follows from (28). Is $t_2 < +\infty$, then it is possible by the same method we have deduced (28) to prove the inequality

$$|\, G(x(t)) \,| \leqq \max\,(G(h), -G(-h)) + \mathrm{P},$$

for every $t \geqq t_2$. Theorem 1 is proved.

Remark 2: When A_2 holds and

$$xh(x) > 0 \qquad \text{for every } x \neq 0, \tag{29}$$

then every $x(t)$ is oscillatory or fulfils the relation

$$\lim_{t \to = +\infty} x(t) = 0.$$

The proof of this assertion can be obtained by the same method as f.i. the proof of theorem 5 in [2].

Theorem 2: If A_4 holds, is (1) dissipative.

Proof: From (10) we see, that a positive constant h_1 exists, such that sgn $r(x) =$ = sgn $h(x)$ for every $| x | \geqq h_1$. By (9) there also exists a positive constant r_1, such that $| x | \geqq r_1$ implies sgn $r(x) =$ sgn x. If we now pose $h = \max(h_1, r_1)$ it is clear that A_4 implies A_3 and thus the validity of theorem 1. With a fixed $x(t)$ we now define

$$\sup_{t \geqq t_0} | x(t) | = X, \qquad \sup_{t \geqq t_0} | x'(t) | = X', \qquad \sup_{t \geqq t_0} | x''(t) | = X''. \tag{30}$$

Let us further set $\liminf_{|x| \to +\infty} r(x)\, h(x) = 2\beta$. By (11) it is $\beta > 0$ and a positive number $r > \max(\varrho, r_1)$ may be find, such that for every $| x | \geqq r$ the inequality $r(x)\, h(x) \geqq \beta$ holds, i.e.

$$h(x) \operatorname{sgn} x \geqq \frac{\beta}{r(x) \operatorname{sgn} x} \qquad \text{for every } | x | \geqq r. \tag{31}$$

We assume $X > r$ and pose $R = \max r(x) \operatorname{sgn} x$ on $\langle r, X \rangle$. Then (30) and (31) yield

$$h(x(t)) \operatorname{sgn} x(t) \geqq \frac{\beta}{R} \qquad \text{for every } | x | \geqq r. \tag{32}$$

Hence, if there exists a t_0 such that $| x(t) | \geqq r$ for all $t > t_0$, we obtain from (32) and (8)

$$\lim_{t \to +\infty} \int_{t_0}^{t} h(x(s))\, \mathrm{d}s \operatorname{sgn} x(t) = \lim_{t \to +\infty} \left| \int_{t_0}^{t} h(x(s))\, \mathrm{d}s \right| = +\infty. \tag{33}$$

But, by integration of (1) from t_0 to $t \geqq t_0$ and by use of (7) it follows

$$\left| \int_{t_0}^{t} h(x(s))\, \mathrm{d}s \right| \leqq 2[X'' + \max(F(X), \quad -F(-X')) + \max(G(X), \quad -G(-X)) + E]$$

i.e. a contradiction to (33). Thus, the relation

$$\liminf_{t \to +\infty} | x(t) | \leqq r, \tag{34}$$

is proved. The proof can be achieved again with the method of [1].

Remark 3: If (1) fulfils a condition of unicity and $e(t)$ is periodical, then, if A_4 holds, (1) has a periodical solution.

Theorem 3: If A_5 holds, there exist solutions of (1), satisfying the relation

$$\lim_{t \to +\infty} | x(t) | = +\infty, \tag{35}$$

(and simultaneously the relations (12)).
This assertion can be proved by transforming (1) into the system

$$\frac{dx_1}{dt} = x_2, \quad \frac{dx_2}{dt} = x_3, \quad \frac{dx_3}{dt} = -f(x_2)\,x_3 - g(x_1)\,x_2 - h(x_1) + e(t)$$

and using the function

$$2U(x_1, x_2, x_3; t) = (x_3 + F(x_2) + G(x_1) - \int_0^t e(s)\, ds)^2$$

in the manner shown in the proof of an analogous assertion in [3].

2. Let us now, consider the equation

$$x''' + f(x')\,x'' + g(x') + h(x) = e(t), \tag{36}$$

with f, g, h, e continuous for every real value of their argument. For the purpose of studying this equation we recall the following theorem, proved in [4].

Theorem: Let us consider the differential equation

$$x^{(n)} = f(x, x, \ldots, x^{(n-1)}; t), \tag{37}$$

with $f(x_1, x_2, \ldots, x_n; t)$ continuous on $E_{n+1}(x_1, x_2 \ldots, x_n; t)$. Assume further that there exist functions $v_i(x_2, x_3, \ldots, x_n)$ $(i = 1, 2, 3)$, continuous on $E_{n-1}(x_2, x_3, \ldots, x_n)$ and a function $V(x_2, x_3, \ldots, x_n; t)$, with all partial derivatives continuous on $E_n(x_2, x_3, \ldots x_n; t)$. These functions may have following properties:

(i) There exists a positive number R such that for $\sum_{i=2}^{n} | x_i | \geqq R$ and for every t the inequality

$$v_1(x_2, x_3, \ldots, x_n) \geqq V(x_2, x_3, \ldots, x_n; t) \geqq v_2(x_2, x_3, \ldots, x_n) \text{ holds.} \tag{38}$$

(ii) We have

$$\lim v_2(x_2, x_3, \ldots, x_n) = +\infty \qquad \text{for } \sum_{i=2}^{n} | x_i | \to +\infty. \tag{39}$$

(iii) On the set $\sum_{i=2}^{n} | x_i | \geqq R$ the inequality $v_3(x_2, x_3, \ldots, x_n) > 0$ holds.

(iv) For every point from $E_{n+1}(x_1, x_2, \ldots, x_n; t)$, satisfying the inequalities $\sum_{i=2}^{n} | x_i | \geqq$ $\geqq R$, $-\infty < t < +\infty$ we have

$$\frac{\partial V}{\partial t} + \sum_{i=2}^{n-1} \frac{\partial V}{\partial x_i} x_{i+1} + \frac{\partial V}{\partial x_n} f(x_1, x_2, \ldots, x_n; t) \leqq -v_3(x_2, \ldots, x_n). \tag{40}$$

Then there exists each solution $x(t)$ of (37) on the interval $I = \langle t_0, +\infty)$ and satisfies the inequality

$$\limsup_{t \to +\infty} \sum_{i=1}^{n-1} | x^{(i)} | \leqq D' \tag{41}$$

with a common constant D'.

Next, we will use following assumptions:

Assumption A_6: There exist positive numbers ε, H, E, Y, such that (4), (5) hold and

$$f(y) \geqq 4\varepsilon \qquad \text{for every } y, \tag{42}$$

$$g(y) \operatorname{sgn} y \geqq E + H + \varepsilon \qquad \text{for every } | y | \geqq Y. \tag{43}$$

Assumption A_7 : A_6 and (7) hold and there exist positive numbers d, m, such that

$$| g(y) - \mathrm{d}y | \leqq m \qquad \text{for every } y. \tag{44}$$

Assumption A_8 : A_7 holds and there exist a positive number h, such that

$$h(x) \operatorname{sgn} x \geqq m \qquad \text{for every } | x | \geqq h. \tag{45}$$

Assumption A_9: A_7 holds and further

$$\liminf_{|x| \to +\infty} h(x) \operatorname{sgn} x > m. \tag{46}$$

Assumption A_{10}: A_7 holds and there exist two positive constants h, δ, such that

$$h(x) \operatorname{sgn} x \leqq -m - \delta \qquad \text{for every } | x | \geqq h. \tag{47}$$

Theorem 4: If A_8 holds, then each solution $x(t)$ of (36) exists on I and is bounded there.

Proof: At first, the following lemma will be proved:

Lemma 3: If A_6 holds, then every $x(t)$ exists on I and there exists a constant D', such that (12) holds.

Proof of lemma 3: Let us consider two functions (inspired by [5])

$$u(y) = \int_0^y g(s) + \varepsilon\left(1 - \frac{1}{1 + | s |}\right) \frac{s f(s)}{1 + a | s |} \, \mathrm{d}s$$

and

$$2w(y, z) = z^2 + 2\varepsilon z \left(1 - \frac{1}{1 + | y |}\right) \frac{y}{1 + a | y |},$$

where a stands for a positive constant satisfying the inequality

$$0 < a < 8\varepsilon^3 (H + E)^{-2}. \tag{48}$$

Using A_6, it is easy to show that u, w are lower bounded and that

$$\lim_{|y| \to +\infty} u(y) = \lim_{|z| \to +\infty} w(y, z) = +\infty.$$

Let us now consider the function

$$V(y, z) = u(y) + w(y, z), \tag{49}$$

with the property

$$\lim_{|y|+|z|\to+\infty} V(y, z) = +\infty. \tag{50}$$

The system equivalent to (36) is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = z, \quad \frac{\mathrm{d}z}{\mathrm{d}t} = -f(y)\,z - g(y) - h(x) + e(t)$$

and thus, the expression on the left side of (40) is

$$V' = z^2\left(\varepsilon\left(\frac{|y|}{(1 + a|y|)(1 + |y|)}2 + \frac{|y|}{(1 + |y|)(1 + a|y|)}2\right) - \right.$$
$$\left. - f(y) + \frac{1}{z}(e(t) - h(x))\right) + \frac{\varepsilon|y|\,y(e(t) - h(x) - g(y))}{(1 + a|y|)(1 + |y|)}. \tag{51}$$

Using (42), (4) and (5) we can estimate

$$V' \leqq z^2\left(-2\varepsilon + \frac{1}{|z|}(E + H)\right) + \frac{\varepsilon|y|\,y(e(t) - h(x) - g(y))}{(1 + a|y|)(1 + |y|)} \tag{52}$$

and hence for $|y| \leqq Y$ (with $G = \max|g(y)|$ for $|y| \leqq Y$)

$$V' \leqq z^2\left(-2\varepsilon + \frac{1}{|z|}(E + H)\right) + \varepsilon Y^2(E + H + G).$$

This leads us finally to the inequality

$$V' \leqq -\varepsilon < 0 \text{ for every } |y| \leqq Y \text{ and every } |z| \geqq M =$$
$$= \max\left(\frac{1}{\varepsilon}(E + H), (Y^2(E + H + G) + 1)^{1/2}\right). \tag{53}$$

For $|y| \geqq Y$ we have because of (43)

$$yg(y) = |y|\,g(y)\operatorname{sgn} y \geqq |y|\,(E + H + \varepsilon), \tag{54}$$

and thus also the following relation must be true (note that $M > 1$)

$$V' \leqq -\varepsilon z^2 + \frac{\varepsilon^2 y^2}{(1 + |y|)(1 + a|y|)} < -\varepsilon < 0 \quad \begin{array}{l}\text{for every } |y| \geqq Y\\ \text{and every } |z| \geqq M.\end{array} \tag{55}$$

Let us further consider the case $|z| \leqq M$. From (52) it follows also

$$V' \leqq \max_{|z|\leqq M}(-2\varepsilon z^2 + (E + H)|z|) + \frac{\varepsilon|y|\,y(e(t) - h(x) - g(y))}{(1 + a|y|)(1 - |y|)} =$$
$$= \frac{1}{8\varepsilon}(E + H)^2 + \frac{\varepsilon|y|\,y(e(t) - h(x) - g(y))}{(1 + a|y|)(1 + |y|)}.$$

Using (54) we see that it must hold

$$\limsup_{|y| \to +\infty} \frac{\varepsilon \mid y \mid y(e(t) - h(x) - g(y))}{(1 + a \mid y \mid)(+ \mid y \mid)} \leqq -\frac{\varepsilon^2}{a}$$

and thus, if a fulfils the inequality (48), there must exist a positive N, such that

$$V' \leqq -\frac{4\varepsilon^3}{(E + H)} 2 + \frac{a}{2} = -\eta < 0 \qquad \text{for every } |z| \leqq M \text{ and every } |y| \geqq N. \tag{56}$$

Resuming, we can write

$$V' \leqq \max(-\varepsilon, -\eta) < 0 \text{ for every } |y| + |z| \geqq M + N = R. \tag{57}$$

Thus, if we pose $v_1(y, z) = v_2(y, z) = V(y, z)$, $v_3(y, z) = \min(\varepsilon, \eta)$, we see that by the above mentioned theorem lemma 3 is proved.

Proof of theorem 4: We pose $g(y) = \mathrm{d}y + \psi(y)$; because of (44) is then $|\psi(y)| \leqq m$. The above lemma results in the existence of a $t_x \geqq t_0$, such that

$$|x'(t)| \leqq D' + 1, \; |x''(t)| \leqq D' + 1 \qquad \text{for every } t \geqq t_x. \tag{58}$$

Integrating (36) and multiplying by sgn $x(t)$ gives

$$\mathrm{d}\,|x(t)| = \mathrm{d}\,|x(t_x)| + \big(x''(t_x) - x''(t) + F(x'(t_x)) - F(x'(t))\big) \,.$$
$$. \operatorname{sgn} x(t) - \int_{t_x}^{t} \big(h(x(s)) + \psi(x'(s)) - e(s)\big)\, \mathrm{d}s \operatorname{sgn} x(t), \tag{59}$$

if we suppose sgn $x(s) =$ const. for $s \in \langle t_x, t \rangle$. Hence, for $|x(s)| \geqq h$ on $\langle t_x, t \rangle$ (note that from (41) and (42) it follows then $|h(x(t))| - |\psi(x(t))| \geqq 0$) we obtain

$$\mathrm{d}\,|x(t)| \leqq \mathrm{d}\,|x(t_x)| + 2(D' + 1 + \max_{|y| \leqq D'+1} F(y) + E).$$

Hereby is our theorem proved.

Theorem 5: If A_g holds, then (35) is dissipative.

Proof: Let us pose $\gamma = \liminf_{|x| \to +\infty} h(x) \operatorname{sgn} x$; there exists a h_1, such that $|h(x)| \geqq \frac{1}{2}(m + \gamma)$ for every $|x| \geqq h_1$. From (59) we get now

$$\mathrm{d}\,|x(t)| \leqq \mathrm{d}\,|x(t_x)| + 2(D' + 1 + \max_{|y| \leqq D'+1} F(y) + E) - \frac{1}{2}(m + \gamma)(t - t_x),$$

if only $|x(t)| \geqq h$ on $\langle t_x, t \rangle$. For $t - t_x$ large enough this leads to a contradiction and hence the relation

$$\liminf_{t \to +\infty} |x(t)| \leqq \mathrm{h},$$

must be valid. Now the proof can be achieved as f.i. the proof of theorem 2.

Remark 4: If (36) fulfils a condition of unicity and $e(t)$ is periodical, then, if A_9 holds, (36) has a periodical solution.

Theorem 6: If A_{10} holds, there exist $x(t)$, satisfying the relation (35) and simultaneously (12).

Theorem 6 can be proved like theorem 3 using the function

$$2U(x, y, z; t) = (z + F(y) + \mathrm{d}x - \int_0^t e(s)\,\mathrm{d}s)^2 + 2\int_0^y (s)\,\mathrm{d}s.$$

LITERATURE

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Shrnutí

O ŘEŠENÍCH JISTÝCH NELINEÁRNÍCH DIFERENCIÁLNÍCH ROVNIC TŘETÍHO ŘÁDU

JAN VORÁČEK

V první části práce jsou odvozeny postačující podmínky pro dissipativnost rovnice (1) (podmínka A_4), resp. pro existenci D' – divergentních řešení (podmínka A_5). Ve druhé části je uvedena postačující podmínka omezenosti řešení rovnice (36) (podmínka A_8), resp. její dissipativnosti (podmínka A_9). Je-li splněna podmínka A_{10}, pak má (36) D' – divergentní řešení.