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LEFT APP-PROPERTY OF FORMAL POWER SERIES RINGS

Liu Zhongkui and Yang Xiaoyan

Abstract. A ring R is called a left APP-ring if the left annihilator $l_R(Ra)$ is right s-unital as an ideal of R for any element $a \in R$. We consider left APP-property of the skew formal power series ring $R[[x;\alpha]]$ where α is a ring automorphism of R. It is shown that if R is a ring satisfying descending chain condition on right annihilators then $R[[x;\alpha]]$ is left APP if and only if for any sequence $(b_0, b_1, \dots)$ of elements of R the ideal $l_R\left(\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} R\alpha^k(b_j)\right)$ is right s-unital. As an application we give a sufficient condition under which the ring $R[[x]]$ over a left APP-ring R is left APP.

Throughout this paper, R denotes a ring with unity. Recall that R is *left principally quasi-Baer* if the left annihilator of every principal left ideal of R is generated by an idempotent. Similarly, right principally quasi-Baer rings can be defined. A ring is called *principally quasi-Baer* if it is both right and left principally quasi-Baer. Observe that biregular rings and quasi-Baer rings (i.e. the rings over which the left annihilator of every left ideal of R is generated by an idempotent of R) are principally quasi-Baer. For more details and examples of left principally quasi-Baer rings, see [3], [1], [2], [4], and [7]. A ring R is called a *right* (resp. *left*) *PP-ring* if the right (resp. left) annihilator of every element of R is generated by an idempotent. R is called a *PP-ring* if it is both right and left PP. As a generalization of left principally quasi-Baer rings and right PP-rings, the concept of left APP-rings was introduced in [9]. A ring R is called a *left APP-ring* if the left annihilator $l_R(Ra)$ is right s-unital as an ideal of R for any element $a \in R$. For more details and examples of left APP-rings, see [9] and [6].

There are a lot of results concerning left principal quasi-Baerness and right PP-property of polynomial extensions of a ring. It was proved in ([2], Theorem 2.1) that a ring R is left principally quasi-Baer if and only if $R[x]$ is left principally quasi-Baer. If all right semicentral idempotents of R are central, then it was shown in [7] that the ring $R[[x]]$ is left principally quasi-Baer if and only if R is left principally quasi-Baer and every countable family of idempotents in R has a generalized join in $I(R)$, the set of all idempotents of R. It was shown in [5] that R is a reduced PP-ring if and only if $R[[x]]$ is a reduced PP-ring. In [8] the PP-property of the rings of generalized power series over a ring R has been

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considered. For left APP-rings, It was noted in [9] that there exists a commutative von Neumann regular ring R (hence left APP), but the ring $R[[x]]$ is not APP. It was also shown in [9] that if R is a left APP-ring satisfying descending chain condition on left and right annihilators then $R[[x]]$ is left APP. In this note we consider left APP-property of skew formal power series rings. We will show that if R is a ring satisfying descending chain condition on right annihilators then $R[[x;\alpha]]$ is left APP if and only if for any sequence $(b_0, b_1, \dots)$ of elements of R the ideal $l_R\big(\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} R\alpha^k(b_j)\big)$ is right s-unital. As an application we give a sufficient condition under which the ring $R[[x]]$ over a left APP-ring R is left APP.

For a nonempty subset Y of R, $l_R(Y)$ and $r_R(Y)$ denote the left and right annihilator of Y in R, respectively.

An ideal I of R is said to be *right s-unital* if, for each $a \in I$ there exists an element $x \in I$ such that $ax = a$. It follows from ([11, Theorem 1]) that I is right s-unital if and only if for any finitely many elements $a_1, a_2, \dots, a_n \in I$ there exists an element $x \in I$ such that $a_i = a_i x$, $i = 1, 2, \dots, n$. A submodule N of a left R-module M is called a *pure submodule* if $L \otimes_R N \to L \otimes_R M$ is a monomorphism for every right R-module L. By ([10], Proposition 11.3.13), an ideal I is right s-unital if and only if R/I is flat as a left R-module if and only if I is pure as a left ideal of R.

Lemma 1. *Let $R[[x;\alpha]]$ be a left APP-ring and $b_0, b_1, \dots$ in R. If $a_0, a_1, \dots, a_n \in R$ are such that for any $r \in R$ and any $s = 0, 1, \dots,$*

$$\begin{aligned}
&a_0 r\alpha^s(b_0) = 0\\
&a_0 r\alpha^s(b_1) + a_1\alpha(r)\alpha^{1+s}(b_0) = 0\\
&\quad\vdots\\
&a_0 r\alpha^s(b_{n-1}) + a_1\alpha(r)\alpha^{1+s}(b_{n-2}) + \cdots + a_{n-1}\alpha^{n-1}(r)\alpha^{n-1+s}(b_0) = 0\\
&a_0 r\alpha^s(b_n) + a_1\alpha(r)\alpha^{1+s}(b_{n-1}) + \cdots + a_n\alpha^n(r)\alpha^{n+s}(b_0) = 0\,,
\end{aligned}$$

then for any s,

$$a_0 R\alpha^s(b_j) = 0\,, \quad j = 0, 1, \dots n\,.$$

Proof. We prove this result by induction on n.

Suppose that $n = 1$. For any $\phi(x) = c_0 + c_1x + c_2x^2 + \cdots \in R[[x;\alpha]]$, $a_0\phi(x)b_0 = a_0c_0b_0 + a_0c_1\alpha(b_0)x + a_0c_2\alpha^2(b_0)x^2 + \cdots = 0$ since $a_0R\alpha^s(b_0) = 0$ for any s. Thus $a_0R[[x;\alpha]]b_0 = 0$. Since $R[[x;\alpha]]$ is a left APP-ring, there exists $h(x) = h_0 + h_1x + h_2x^2 + \cdots \in l_{R[[x;\alpha]]}(R[[x;\alpha]]b_0)$ such that $a_0 = a_0h(x)$. Clearly $a_0 = a_0h_0$ and for any $r \in R$ and any s, $h(x)(rx^s)b_0 = 0$. Thus $h_0r\alpha^s(b_0) = 0$ for any s. Take $r = h_0r'$ in $a_0r\alpha^s(b_1) + a_1\alpha(r)\alpha^{1+s}(b_0) = 0$. Then $a_0r'\alpha^s(b_1) = a_0h_0r'\alpha^s(b_1) = a_0h_0r'\alpha^s(b_1) + a_1\alpha\big(h_0r'\alpha^s(b_0)\big) = a_0h_0r'\alpha^s(b_1) + a_1\alpha(h_0r')\alpha^{1+s}(b_0) = 0$. Thus $a_0R\alpha^s(b_1) = 0$.

Now suppose that $n \geq 2$. From the first n equations and the induction hypothesis, it follows that $a_0R\alpha^s(b_j) = 0$, $j = 0, 1, \dots n-1$. Thus for any $r \in R$ and any s, $a_0(rx^s)(b_0 + b_1x + \cdots + b_{n-1}x^{n-1}) = a_0r\alpha^s(b_0)x^s + a_0r\alpha^s(b_1)x^{s+1} + \cdots + a_0r\alpha^s(b_{n-1})x^{s+n-1} = 0$. Hence $a_0R[[x;\alpha]](b_0 + b_1x + \cdots + b_{n-1}x^{n-1}) = 0$.

Since $R[[x;\alpha]]$ is a left APP-ring, there exists $h(x) = h_0 + h_1x + h_2x^2 + \cdots \in l_{R[[x;\alpha]]}\big(R[[x;\alpha]](b_0+b_1x+\cdots+b_{n-1}x^{n-1})\big)$ such that $a_0 = a_0h(x)$. Thus $a_0 = a_0h_0$ and $h(x)(rx^s)(b_0 + b_1x + \cdots + b_{n-1}x^{n-1}) = 0$ for any $r \in R$ and any s. Now we have

$$\begin{aligned}
&h_0r\alpha^s(b_0) = 0\\
&h_0r\alpha^s(b_1) + h_1\alpha(r)\alpha^{1+s}(b_0) = 0\\
&\quad\vdots\\
&h_0r\alpha^s(b_{n-1}) + h_1\alpha(r)\alpha^{1+s}(b_{n-2}) + \cdots + h_{n-1}\alpha^{n-1}(r)\alpha^{n-1+s}(b_0) = 0\,.
\end{aligned}$$

By the induction hypothesis, it follows that $h_0R\alpha^s(b_j) = 0$, $j = 0, 1, \ldots n-1$. Thus, for any $r' \in R$, taking $r = h_0r'$ in the last equation yields

$$\begin{aligned}
0 &= a_0h_0r'\alpha^s(b_n) + a_1\alpha(h_0r')\alpha^{1+s}(b_{n-1}) + \cdots + a_n\alpha^n(h_0r')\alpha^{n+s}(b_0)\\
&= a_0r'\alpha^s(b_n) + a_1\alpha(h_0r'\alpha^s(b_{n-1})) + \cdots + a_n\alpha^n(h_0r'\alpha^s(b_0))\\
&= a_0r'\alpha^s(b_n)\,.
\end{aligned}$$

Hence $a_0R\alpha^s(b_n) = 0$. Now the result follows. □

Theorem 2. *Let R be a ring satisfying descending chain condition on right annihilators and α a ring automorphism of R. Then the following conditions are equivalent:*

(1) *$R[[x;\alpha]]$ is a left APP-ring.*

(2) *For any sequence $(b_0, b_1, \ldots)$ of elements of R, $l_R(\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} R\alpha^k(b_j))$ is right s-unital.*

Proof. (1)⇒(2). Suppose that $(b_0, b_1, \ldots)$ is a sequence of elements of R. Set $g(x) = b_0 + b_1x + b_2x^2 + \cdots \in R[[x;\alpha]]$. Let $a \in l_R\big(\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} R\alpha^k(b_j)\big)$. Then $aR[[x;\alpha]]g(x) = 0$. Since $R[[x;\alpha]]$ is a left APP-ring, there exists $h(x) = h_0 + h_1x + h_2x^2 + \cdots \in l_{R[[x;\alpha]]}(R[[x;\alpha]]g(x))$ such that $a = ah(x)$. Thus we have $a = ah_0$ and $h(x)(rx^s)g(x) = 0$. Hence

$$\sum_{i+j=n} h_i\alpha^i(r)\alpha^{i+s}(b_j) = 0\,, \quad \forall\ n\,.$$

By Lemma 1, $h_0R\alpha^s(b_j) = 0$ for any j. Thus $h_0 \in l_R\big(\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} R\alpha^k(b_j)\big)$.

(2)⇒(1). Suppose that $f(x) = a_0+a_1x+a_2x^2+\ldots$, $g(x) = b_0+b_1x+b_2x^2+\cdots \in R[[x;\alpha]]$ are such that $f(x)R[[x;\alpha]]g(x) = 0$. Then for any $r \in R$, $f(x)(rx^s)g(x) = 0$. It follows that

$$\sum_{i+j=k} a_i\alpha^i(r)\alpha^{i+s}(b_j) = 0\,, \qquad k = 0, 1, 2, \ldots\,, \tag{1}$$

where r is an arbitrary element of R. Thus, since $a_0r\alpha^s(b_0) = 0$ for any s, one has $a_0 \in l_R\big(\sum_{s=0}^{\infty} R\alpha^s(b_0)\big)$. By the hypothesis for the sequence $(b_0, 0, 0, \ldots)$ of elements of R, there exists $p_0 \in l_R\big(\sum_{k=0}^{\infty} R\alpha^k(b_0)\big)$ such that $a_0 = a_0p_0$.

Suppose that $c_0, c_1, \cdots \in R$ are such that $a_i = \alpha^i(c_i)$. Let $r' \in R$ and take $r = p_0r'$ in $a_1\alpha(r)\alpha^{1+s}(b_0)+a_0r\alpha^s(b_1) = 0$. Then $a_1\alpha(p_0r')\alpha^{1+s}(b_0)+a_0p_0r'\alpha^s(b_1) = 0$.

Since $p_0 \in l_R\big(\sum_{k=0}^{\infty} R\alpha^k(b_0)\big)$, we have $a_1\alpha(p_0r')\alpha^{1+s}(b_0) = a_1\alpha(p_0r'\alpha^s(b_0)) = 0$. Thus $a_0r'\alpha^s(b_1) = a_0p_0r'\alpha^s(b_1) = 0$ for any $s = 0, 1, \dots$, which implies that $a_0 \in l_R\big(\sum_{k=0}^{\infty} R\alpha^k(b_1)\big)$. Also $a_1\alpha(r)\alpha^{1+s}(b_0) = 0$ for any $r \in R$. Thus $\alpha(c_1r\alpha^s(b_0)) = 0$. Since α is an automorphism, it follows that $c_1r\alpha^s(b_0) = 0$ for any $s = 0, 1, \dots$. This means that $c_1 \in l_R\big(\sum_{k=0}^{\infty} R\alpha^k(b_0)\big)$.

Inductively, assume that $q \geq 1$ is such that

$$c_i \in l_R\Big(\sum_{k=0}^{\infty} R\alpha^k(b_j)\Big)\,, \qquad i+j = 0, 1, 2, \dots, q-1\,.$$

Note that $c_0 = a_0$.

Since $c_0, c_1, \dots, c_{q-1} \in l_R\big(\sum_{k=0}^{\infty} R\alpha^k(b_0)\big)$ and $l_R\big(\sum_{k=0}^{\infty} R\alpha^k(b_0)\big)$ is right s-unital, there exists $r_0 \in l_R\big(\sum_{k=0}^{\infty} R\alpha^k(b_0)\big)$ such that $c_i = c_ir_0$, $i = 0, 1, \dots, q-1$. Let $r' \in R$ and take $r = r_0r'$. Then by the equation of (1) for the case when $k = q$, we have

$$a_0r_0r'\alpha^s(b_q) + \cdots + a_{q-1}\alpha^{q-1}(r_0r')\alpha^{q-1+s}(b_1) + a_q\alpha^q(r_0r')\alpha^{q+s}(b_0) = 0\,.$$

For any i with $0 \leq i \leq q-1$, we have $a_i\alpha^i(r_0r')\alpha^{i+s}(b_{q-i}) = \alpha^i\big(c_ir_0r'\alpha^s(b_{q-i})\big) = \alpha^i\big(c_ir'\alpha^s(b_{q-i})\big) = a_i\alpha^i(r')\alpha^{i+s}(b_{q-i})$. Also $a_q\alpha^q(r_0r')\alpha^{q+s}(b_0) = a_q\alpha^q\big(r_0r'\alpha^s(b_0)\big) = 0$ since $r_0 \in l_R\big(\sum_{k=0}^{\infty} R\alpha^k(b_0)\big)$. Thus

$$(2) \qquad a_0r'\alpha^s(b_q) + a_1\alpha(r')\alpha^{1+s}(b_{q-1}) + \cdots + a_{q-1}\alpha^{q-1}(r')\alpha^{q-1+s}(b_1) = 0\,.$$

By (1) it follows that $a_q\alpha^q(r)\alpha^{q+s}(b_0) = 0$. Thus $\alpha^q\big(c_qr\alpha^s(b_0)\big) = 0$, which implies that $c_qr\alpha^s(b_0) = 0$ for any s and any $r \in R$. Hence $c_q \in l_R\big(\sum_{k=0}^{\infty} R\alpha^k(b_0)\big)$.

Since $c_0, c_1, \dots, c_{q-2} \in l_R\big(\sum_{k=0}^{\infty} R\alpha^k(b_0) + \sum_{k=0}^{\infty} R\alpha^k(b_1)\big)$, there exists $r_1 \in l_R\big(\sum_{k=0}^{\infty} R\alpha^k(b_0) + \sum_{k=0}^{\infty} R\alpha^k(b_1)\big)$ such that $c_i = c_ir_1$ for any i with $0 \leq i \leq q-2$. Thus $a_i\alpha^i(r_1r'')\alpha^{i+s}(b_{q-i}) = \alpha^i\big(c_ir_1r''\alpha^s(b_{q-i})\big) = \alpha^i\big(c_ir''\alpha^s(b_{q-i})\big) = a_i\alpha^i(r'')\alpha^{i+s}(b_{q-i})$ for any i with $0 \leq i \leq q-2$. Now setting $r' = r_1r''$ in (2) yields

$$a_0r''\alpha^s(b_q) + a_1\alpha(r'')\alpha^{1+s}(b_{q-1}) + \cdots + a_{q-2}\alpha^{q-2}(r'')\alpha^{q-2+s}(b_2) = 0$$

for any $r'' \in R$ since $a_{q-1}\alpha^{q-1}(r_1r'')\alpha^{q-1+s}(b_1) = a_{q-1}\alpha^{q-1}(r_1r''\alpha^s(b_1)) = 0$. Thus, by (2), $a_{q-1}\alpha^{q-1}(r')\alpha^{q-1+s}(b_1) = 0$. This means that $c_{q-1}r'\alpha^s(b_1) = 0$ since α is an automorphism. Hence $c_{q-1} \in l_R\big(\sum_{k=0}^{\infty} R\alpha^k(b_1)\big)$. Continuing this procedure yields $c_{q-2} \in l_R\big(\sum_{k=0}^{\infty} R\alpha^k(b_2)\big) \dots, c_1 \in l_R\big(\sum_{k=0}^{\infty} R\alpha^k(b_{q-1})\big), c_0 \in l_R\big(\sum_{k=0}^{\infty} R\alpha^k(b_q)\big)$.

Hence we have shown that for any i and j, $c_i \in l_R\big(\sum_{k=0}^{\infty} R\alpha^k(b_j)\big)$. Thus $c_i \in l_R\big(\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} R\alpha^k(b_j)\big)$. Consider the descending chain as following:

$$r_R(c_0) \supseteq r_R(c_0, c_1) \supseteq r_R(c_0, c_1, c_2) \supseteq \dots\,,$$

there exists n such that $r_R(c_0, c_1, \dots, c_n) = r_R(c_0, c_1, \dots, c_n, c_{n+1}) = \dots$. By the hypothesis, $l_R\big(\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} R\alpha^k(b_j)\big)$ is right s-unital by considering sequence $(b_0, b_1, \dots)$. Thus there exists $e \in l_R\big(\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} R\alpha^k(b_j)\big)$ such that $c_i = c_ie$, $i = 0, 1, \dots, n$. Clearly $1 - e \in r_R(c_0, c_1, \dots, c_n)$. Thus $c_k = c_ke$ for all $k = 0, 1, \dots$. Now $f(x) = a_0 + \alpha(c_1)x + \alpha^2(c_2)x^2 + \cdots = a_0e + \alpha(c_1e)x + \alpha^2(c_2e)x^2 + \cdots =$

$a_0e+a_1\alpha(e)x+a_2\alpha^2(e)x^2+\cdots=f(x)e$ and $e\in l_{R[[x;\alpha]]}\big(R[[x;\alpha]]g(x)\big)$. This means that $R[[x;\alpha]]$ is a left APP-ring. □

It was shown in [9] that if R is a left APP-ring satisfying descending chain condition on left and right annihilators then $R[[x]]$ is left APP. By Theorem 2 we have the following result.

Corollary 3. *Let R be a ring satisfying descending chain condition on right annihilators. Then the following conditions are equivalent:*

(1) *$R[[x]]$ is a left APP-ring.*

(2) *For any sequence $(b_0, b_1, \dots)$ of elements of R, $l_R(\sum_{j=0}^{\infty} Rb_j)$ is right s-unital.*

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Northwest Normal University
Department of Mathematics
Lanzhou 730070, Gansu, People's Republic of China
E-mail: `liuzk@nwnu.edu.cn`; `xiaoxiao800218@163.com`