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Closed embeddings into complements of Σ-products

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Abstract. In some sense, a dual property to that of Valdivia compact is considered, namely the property to be embedded as a closed subspace into a complement of a Σ-subproduct of a Tikhonov cube. All locally compact spaces are co-Valdivia spaces (and only those among metrizable spaces or spaces having countable type). There are paracompact non-locally compact co-Valdivia spaces. A possibly new type of ultrafilters lying in between P-ultrafilters and weak P-ultrafilters is introduced. Under Martin axiom and negation of CH, no countable nowhere dense space is a co-Valdivia space.

Keywords: Σ-product, Tikhonov cube, Valdivia compact, locally compact space

Classification: 54B10, 54C25, 54D35, 54D45

1. Introduction

All topological spaces are supposed to be Tikhonov (Hausdorff completely regular spaces).

A Valdivia compact space X can be embedded into a Tikhonov cube in such a way that its intersection P with a Σ-product is dense in X. Thus, X is a compactification of P (in fact, $X = \beta P$) and the remainder $X \setminus P$ is a closed subspace of the complement of a Σ-product in the Tikhonov cube. One may ask what spaces are remainders of those Σ-parts of Valdivia compacts. For a survey on Valdivia compacts see [3] or [4] for other results.

Another motivation for the investigation of closed embeddings into complements of Σ-products is to look for spaces having nice remainders in a compactification. Every remainder of a closed subspace of the complement of a Σ-product in a Tikhonov cube (in its closure in the cube) is a normal, Fréchet and ω-bounded (thus countably compact) space.

So, it may be interesting to know which topological spaces can be embedded into complements of Σ-products in Tikhonov cubes as closed subspaces. If those spaces are nowhere locally compact, then their closures in Tikhonov cubes are Valdivia compacts.

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The symbol $\mathbb{I}$ denotes the closed real interval [0,1]. By $\Sigma_a X^\kappa$ we denote the Σ-product in X^κ determined by a point $a = \{a_\alpha\}_\kappa \in X^\kappa$, i.e.,

$$\Sigma_a X^\kappa = \{\{x_\alpha\}_\kappa \in X^\kappa; |\{\alpha \in \kappa; x_\alpha \neq a_\alpha\}| \leq \omega\}.$$

We shall denote the complement $X^\kappa \setminus \Sigma_a X^\kappa$ simply by $X^\kappa \setminus \Sigma_a$. It is known that $\Sigma_a \mathbb{I}^\kappa$ is a Fréchet ([5]) collectionwise normal space ([2]) and is closed under closures (in $\mathbb{I}^\kappa$) of its countable subsets. Every continuous function defined on a closed subspace of $\Sigma_a \mathbb{I}^\kappa$ continuously extends onto the whole product $\mathbb{I}^\kappa$ (it depends on countably many coordinates — [2]).

It is not clear how homeomorphs of closed subspaces of complements of Σ-products should be called, also from the reason that it is not clear whether such spaces will be shown useful. For the purpose of the present paper we shall call them cV-spaces, which comes from co-Valdivia spaces.

Definition 1. A topological space X is said to be a *cV-space* (or to have property cV) if it can be embedded as a closed subspace into the complement of a Σ-product in a power of $\mathbb{I}$.

It is convenient to realize that a Σ-product $\Sigma_a \mathbb{I}^\kappa$ in the previous definition can be always chosen with $a = 0$ (having all its coordinates equal to 0). Indeed, for every $\alpha \in \kappa$ there is an embedding of $\mathbb{I}$ into $\mathbb{I} \times \mathbb{I}$ that maps the point $a_\alpha \neq 0$ into (0,0) (mapping homeomorphically, e.g., $[0, a_\alpha]$ onto $\{0\} \times \mathbb{I}$ and $[a_\alpha, 1]$ onto $\mathbb{I} \times \{0\}$). So, a space X can be embedded as a closed subspace into a complement of $\Sigma_a \mathbb{I}^\kappa$ in $\mathbb{I}^\kappa$ iff it can be embedded as a closed subspace into the complement of $\Sigma_0 \mathbb{I}^\kappa$ in $\mathbb{I}^\kappa$.

In the previous paragraphs we were speaking about embeddings into Tikhonov cubes. Valdivia compacts can also be defined by embeddings into powers of reals. That would give a different concept in our case, which follows from the following fact: every realcompact space can be embedded as a closed subspace into the complement of a Σ-product in a power of reals $\mathbb{R}$ — we shall see later that not all realcompact spaces have cV (e.g. the space of rational numbers has not cV — see Corollary 5). Indeed, $\mathbb{R}$ can be embedded into $\mathbb{R}^\mathbb{R} \setminus \Sigma$ as a closed subspace of $\mathbb{R}^\mathbb{R}$ (e.g. by the map $r \to \{s \to r + s\}$). It follows that every power $\mathbb{R}^\kappa$ embeds as a closed subspace into $\mathbb{R}^{2^\omega \cdot \kappa} \setminus \Sigma$.

Clearly, the converse is true: every cV space can be embedded as a closed subspace into the complement of a Σ-product in a power of reals $\mathbb{R}$. And if a space can be embedded as a closed subspace into the complement of a Σ-product in a Cantor space 2^κ, it is a cV space. In all the cases, the Σ-products may be considered determined by the point 0.

2. General results

If X has cV then its closure in the corresponding $\mathbb{I}^\kappa$ is a compactification of X with its remainder lying in $\Sigma_a \mathbb{I}^\kappa$. According to the previous paragraph, the remainder is a Fréchet ω-bounded space (i.e., every its countable subset has a compact closure in the remainder) that is C-embedded in X (in fact in $\mathbb{I}^\kappa$). We shall call that compactification a *cV-compactification* of X.

We shall now transform the definition of cV spaces into a form more convenient for applications.

Proposition 2. *A topological space X has cV iff it there exist families $\mathcal{G} = \{G_i\}_I \subset$ cozero(βX) and $\mathcal{Z} = \{Z_i\}_I \subset$ zero(βX) having the following properties:*

1. *for every $i \in I$, either $Z_i \subset G_i$ or $Z_i \cap G_i = \emptyset$;*
2. *$\{G_i \cap X; i \in I\}$ is an open subbase of X;*
3. *$\{\beta X \setminus Z_i; i \in I\}$ is point-countable on $\beta X \setminus X$;*
4. *$\{G_i \setminus Z_i; i \in I\}$ is point-uncountable at every $x \in X$.*

Proof: The conditions are necessary. Indeed, if X embeds as a closed subspace into $\mathbb{I}^\kappa \setminus \Sigma$ and γX is the closure of X in $\mathbb{I}^\kappa$ one may take for $\mathcal{G}$ all the preimages of a countable open base in $\mathbb{I}$, under the compositions of the natural map $\beta X \to \gamma X$ and all the projections $\gamma X \to \mathbb{I}$. The family $\mathcal{Z}$ is formed by preimages of 0 under the same maps.

Suppose now that the conditions are fulfilled for some families $\mathcal{G}$ and $\mathcal{Z}$. For every $G_i \in \mathcal{G}$ find cozero sets $G_{i,n}$ in βX with $G_{i,n} \subset \overline{G_{i,n}} \subset G_{i,n-1}$, $G = \bigcup G_{i,n}$. Then find continuous functions $f_{i,n} : \beta X \to \mathbb{I}$ such that

$$f_{i,n}(x) = \begin{cases} 0 & \text{for } x \in \beta X \setminus G_i \\ 1 & \text{for } x \in G_{i,n} \end{cases} \quad \text{if } Z_i \cap G_i = \emptyset,$$

$$f_{i,n}(x) = \begin{cases} 0 & \text{exactly for } x \in Z_i \\ 1 & \text{for } x \in \beta X \setminus G_i \end{cases} \quad \text{if } Z_i \subset G_i.$$

Denote by φ the mapping $\beta X \to \mathbb{I}^{I \cdot \omega}$ determined by all $f_{i,n}$. According to the second condition, φ is homeomorphic on X. The third condition gives the inclusion $\varphi(\beta X \setminus X) \subset \Sigma_0$ and the fourth condition gives the inclusion $\varphi(X) \subset \mathbb{I}^{\mathcal{G} \times \omega} \setminus \Sigma$. Since $\varphi(\beta X)$ is compact, $\varphi(X)$ is closed in $\mathbb{I}^{\mathcal{G} \times \omega} \setminus \Sigma$ and, consequently, X is a cV space. □

It follows directly from the definition that the class of cV spaces is closed hereditary. The previous characterization helps to show that the class of cV spaces is closed under disjoint sums. It will follow from Corollary 5 that cV spaces are not closed under countable products (for instance, the space of irrationals is not a cV space), and under quotients (fan with ω_2 spikes).

Although it seems that the previous characterization is too complicated to be useful, it gives several interesting consequences. The first one describes a big class of cV spaces.

Theorem 3. *Every locally compact space X is cV.*

Proof: Let X be a locally compact space. Take for $\mathcal{G}$ in Proposition 2 the open base of X composed of cozero sets having compact closures, repeating each of it uncountably many times. The family $\mathcal{Z}$ is composed of complements in βX of those corresponding cozero sets. □

Clearly, there are nowhere locally compact spaces having cV; for instance, the complements of Σ-products in Tikhonov cubes have cV. In the next section we shall describe some classes of non-locally compact spaces having cV.

Another consequence of Proposition 2 gives a necessary condition for a space to have cV.

Proposition 4. *If X is a cV space then for every compact set in X there exists a family $\{U_\alpha\}_{\omega_1}$ of its neighborhoods in βX such that*

$$\bigcap_S \overline{U_\alpha} \subset X \quad \textit{for every uncountable} \quad S \subset \omega_1 .$$

Especially, every compact set in X is contained in a compact set $K \subset X$ with $\chi(K, X) \leq \omega_1$.

Proof: Let X have cV and C be a compact set in X. Take families $\mathcal{G}$ and $\mathcal{Z}$ from Proposition 2. For every point $x \in C$ take some $G_{i_x} \in \mathcal{G}$ with $x \in G_{i_x} \setminus Z_{i_x}$. There is a finite set $F \subset I$ such that $\bigcup_F (G_{i_x} \setminus Z_{i_x}) \supset C$. Denote $W_0 = \bigcup_F (G_{i_x} \setminus Z_{i_x})$ and $I_0 = I \setminus F$. Suppose that we have already constructed open sets $W_\alpha \supset C$ and collections $I_\alpha \subset I$ for all $\alpha < \delta$ for some $0 < \delta < \omega_1$ such that

1. $I_\alpha \supset I_\beta$ and $|I_\alpha \setminus I_\beta| \leq \omega$ for $\alpha < \beta < \delta$;
2. W_α is a union of sets $G_i \setminus Z_i$ for a finite number of indices $i \in I \setminus I_\alpha$.

We shall construct W_δ and I_δ. For every $x \in C$ there is an $i_x \in \bigcap_{\alpha<\delta} I_\alpha$ with $x \in G_{i_x} \setminus Z_{i_x}$, because only at most countably many elements were removed from the uncountable family $G_i \setminus Z_i$ containing x. There is a finite set $F \subset \bigcap_{\alpha<\delta} I_\alpha$ such that $\bigcup_F (G_i \setminus Z_i) \supset C$. Denote $W_\delta = \bigcup_F (G_i \setminus Z_i)$ and $I_\delta = \bigcap_{\alpha<\delta} I_\alpha \setminus F$. Both conditions above are satisfied for $\{W_\alpha; \alpha \leq \delta\}$ and $\{I_\alpha; \alpha \leq \delta\}$.

The family $\{W_\alpha; \alpha < \omega_1\}$ is point-countable on $\beta X \setminus X$. Indeed, if $x \in W_\alpha$ then $x \in (G_i \setminus Z_i)$ for a finite number of indices i, and those finite sets of indices are disjoint (by our construction of W_α). Consequently, if x belongs to uncountably many of W_α's, it belongs to uncountably many of $G_i \setminus Z_i$'s and, thus, $x \in X$. So, $\bigcap_S W_\alpha \subset X$ for any uncountable set S of countable ordinals.

For any $\alpha \in \omega_1$ take cozero sets $U_{\alpha,n}$ with $C \subset V_{\alpha,n} \subset \overline{V_{\alpha,n}} \subset V_{\alpha,n-1} \subset W_\alpha$. Then the family $\{V_{\alpha,n}; \alpha \in \omega_1, n \in \mathbb{N}\}$ is the requested family and its intersection is the requested compact set $K \supset C$. □

The previous proposition has interesting consequences. We remind that a point x is said to be a *weak P-point* in a space if it does not belong to closures of countable sets not containing x.

Corollary 5. 1. *If X is a cV space then every its point x is a weak P-point in $\{x\} \cup (\beta X \setminus X)$.*

2. *If X is a cV space then every its compact set having countable character has a compact neighborhood.*
3. *If X is a cV space having a point with countable π-character then X has a point with a compact neighborhood.*
4. *If X is a cV space then every its open set is a union of compact G_{ω_1}-sets (i.e., compact G_{ω_1}-sets form a network of X).*

PROOF: 1. Let $x \in X$ be an accumulation point of a countable set $\{p_n\} \subset \beta X \setminus X$ not containing x. By Proposition 4, there is a family $\{G_\alpha\}_{\omega_1}$ of neighborhoods of x in βX such that every intersection of uncountably many of them is a part of X. Every G_α contains some p_n, thus there is a p_k belonging to uncountably many G_α's, which is impossible.

2. Let $C \subset X$ be a compact space having a countable base $\{U_n\}$ of neighborhoods. By Proposition 4, there is a family $\{G_\alpha\}_{\omega_1}$ of neighborhoods of C in βX such that every intersection of closures of uncountably many of them is a part of X. Since uncountably many of G_α's must contain some U_k, the closure $\overline{U_k}$ is a compact neighborhood of C in X.

3. Let $\{U_n\}$ be a countable π-base at some $x \in X$. Taking a family $\{G_\alpha\}$ as in the previous part, some U_k must belong to uncountably many of G_α's and so, every point of U_k has a compact neighborhood, namely $\overline{U_k}$.

4. Let H be an open subset of X containing a point x. Again, there is the above family $\{G_\alpha\}$ for x. It suffices to take open $U_{\alpha,n}$ such that $x \in U_{\alpha,n} \subset \overline{U_{\alpha,n}} \subset U_{\alpha,n-1} \subset H \cap G_\alpha$. Then $\bigcap\{\overline{U_{\alpha,n}}; \alpha < \omega_1, n \in \mathbb{N}\}$ is the requested compact G_{ω_1}-set. □

An easy consequence of Corollary 5 says that if ξ is not a weak P-point of $\beta X \setminus X$, then $X \cup \{\xi\}$ has not cV (regarded as a subspace of βX).

In Corollary 5, item 4, one can write G_{ω_1}-set instead of open set.

Because of its importance we shall state the next corollary as a theorem. Recall that a space is said to be of *countable type* if every point is contained in a compact set having countable character.

Theorem 6. *A space of countable type is cV iff it is locally compact.*

In particular, a metrizable space or a space with countable local character is a cV space iff it is locally compact.

3. Special spaces

We shall now look at two special classes of non-locally compact spaces, namely at those having a unique accumulation point, and at dense-in-itself spaces.

By the previous results, if in a space X its compact sets coincide with finite sets, then X has cV only if $\chi(X) \leq \omega_1$. One type of such spaces are non-locally compact spaces with one accumulation point. Denote those spaces as $X \oplus 1 = X \cup \{\xi\}$, where X is a discrete open subset of $X \oplus 1$ and ξ is its accumulation point. A necessary condition for non-locally compact $X \oplus 1$ to have cV is that $\chi(\xi) \leq \omega_1$. If $\chi(\xi) = \omega$ then $X \oplus 1$ has cV iff ξ has a compact neighborhood. So it remains to consider the cases when $\chi(\xi) = \omega_1$.

Theorem 7. *For a space $X \oplus 1$ with $\chi(\xi) = \omega_1$, each of the following conditions implies the next one:*

1. *ξ is a P-point;*
2. *ξ is a P-point in $\beta(X \oplus 1) \setminus X$;*
3. *$X \oplus 1$ has cV;*
4. *ξ is a weak P-point in $\beta(X \oplus 1) \setminus X$.*

Proof: The implication $1 \to 2$ follows from a general fact that if $A \subset Z$ is dense in Z and a point $z \in Z \setminus A$ is a P-point of $A \cup \{z\}$ then it is a P-point of $Z \setminus A$.

The implication $3 \to 4$ follows from Corollary 5. It remains to prove the implication $2 \to 3$. Let $\{U_\alpha\}_{\omega_1}$ be a basis of cozero neighborhoods of ξ in $\beta(X \oplus 1)$ such that, for every $\alpha < \omega_1$, $\overline{U_{\alpha+1}} \subset U_\alpha$ and $U_\alpha \setminus X \subset \bigcap_{\beta<\alpha} U_\beta$. Let $U_\alpha = f_\alpha^{-1}(0,1]$ for some continuous $f_\alpha : \beta X \to \mathbb{I}$. Define $\mathcal{G} = \{\{\{x\}_{\omega_1}; x \in X\} \cup \{U_\alpha\}_{\omega_1}$ and the corresponding zero sets $Z_i = \beta(X \oplus 1) \setminus G_i$ if $|G_i| = 1$ and $Z_i = f_\alpha^{-1}(0)$ if $G_i = U_\alpha$. It is easy to see that the families satisfy the conditions of Proposition 2. □

We may now apply Theorem 7 to several examples of spaces having exactly one accumulation point: the subspace of the ordered space of ordinals $\kappa + 1$, where κ is a regular cardinal, composed of isolated ordinals and of the largest element κ, or the subspace of the Čech-Stone compactification of a discrete infinite set D composed of the set D and of a one point ξ of the remainder. We shall denote the former space by $\kappa \oplus 1$ and the latter space by D_ξ.

The space $\kappa \oplus 1$ has character κ and so, only for $\kappa \leq \omega_1$ the space may have cV. The space $\omega \oplus 1$ is compact and it has cV. It remains to consider $\omega_1 \oplus 1$. By Theorem 7 we have:

Corollary 8. *The space $\kappa \oplus 1$ has cV iff $\kappa \leq \omega_1$.*

So, there exists a space that is not locally compact, has a unique accumulation point (and is thus paracompact) and has cV.

If a space D_ξ has cV, then $\chi(\xi) = \omega_1$ (it cannot have countable character). It implies that ξ belongs to a closure of a countable subset of D and, consequently, D_ξ may be considered as a disjoint sum of a discrete space and $\mathbb{N}_\xi$. So, it remains to consider $\mathbb{N}_\xi$.

Corollary 9. *Let ξ be a free ultrafilter on a discrete set D.*

1. *If D_ξ has cV then ξ is a weak P-ultrafilter containing a countable set and $\chi(\xi) = \omega_1$.*

2. *If ξ is a P-ultrafilter on $\mathbb{N}$ and $\chi(\xi) = \omega_1$ then N_ξ has cV.*

We do not know whether there is a weak P, non-P-ultrafilter ξ on $\mathbb{N}$ such that N_ξ has cV. Denote by (V) the following property of free ultrafilters on $\mathbb{N}$:

(V) the ultrafilter has a base $\mathcal{A}$ such that $\{\overline{A}^{\beta\mathbb{N}}\}_{\mathcal{A}}$ is point-countable on $\beta\mathbb{N} \setminus \mathbb{N}_\xi$.

So, N_ξ has cV iff ξ has character ω_1 and satisfies (V) and, thus, every P-ultrafilter on $\mathbb{N}$ having character ω_1 has (V). Clearly, every ultrafilter with (V) is a weak P-ultrafilter. We do not know any other relation among those concepts and may formulate the following question (the best situation is under CH, when the assumption on characters may be omitted).

Question 10. *Is it true that either every ultrafilter having (V) and character ω_1 is P-ultrafilter or that every weak P-ultrafilter has (V)?*

Other interesting spaces having a unique accumulation point are fans F_κ, $\kappa \geq \omega$ regular (quotients of disjoint union of κ many converging sequences sewed together at their limit points). The spaces F_κ are never locally compact and their character is bigger than κ. Therefore, only F_ω may be a cV space.

The fan F_ω has character $\mathfrak{d}$, the minimal cardinality of a cofinal set of functions $\mathbb{N} \to \mathbb{N}$ in the order $f \prec g$ if $f(n) \leq g(n)$ for almost all n (up to finitely many). It is known that $\omega_1 \leq \mathfrak{d} \leq 2^\omega$. If $\mathfrak{d} = \omega_1$ then a cofinal set $\{f_\alpha\}_{\omega_1}$ may be found to be a scale: if $\alpha < \beta$ then $f_\alpha \prec f_\beta$.

In our notation, $X = \mathbb{N} \times \mathbb{N}, F_\omega = X \oplus 1$ and the accumulation point ξ has basic neighborhoods $U_f = \{\xi\} \cup \{(n,k); k \geq f(n)\}$ determined by $f : \mathbb{N} \to \mathbb{N}$.

Theorem 11. *The fan F_ω is a cV space iff $\mathfrak{d} = \omega_1$.*

PROOF: The necessity follows from Proposition 4. For the sufficiency we shall show that ξ is a P-point of $\beta(X \oplus 1) \setminus X$ and use Theorem 7. Take a cofinal set $\{f_\alpha\}_{\omega_1}$ of functions $\mathbb{N} \to \mathbb{N}$ in $\prec$ being also a scale (see above). Denote the neighborhoods of ξ determined by f_α as U_α. Take neighborhoods $G_n, n \in \mathbb{N}$, of ξ in $\beta(X \oplus 1)$ with $\overline{G_n} \subset G_{n-1}$ for every n. It suffices to show that $\bigcap_n G_n \supset \overline{U_\alpha}^{\beta(X\oplus 1)} \setminus X$ for some α.

There are α_n such that $U_{\alpha_n} \subset G_n$. Take f_γ following each f_{α_n} in the order $\prec$. Thus $U_\gamma \subset U_{\alpha_n} \cup C_n$, where C_n is a compact subset of F_ω (a finite number of converging sequences). Consequently, $\overline{U_\gamma}^{\beta(X\oplus 1)} \setminus X \subset \overline{U_{\alpha_n}}^{\beta(X\oplus 1)} \subset \overline{G_n} \subset G_{n-1}$ for every n, which was to be proved. □

The procedure of the previous proof can be used for connected fans obtained by sewing together all points 0 in a disjoint union of intervals [0,1]. One gets the same characterization of those fans belonging to cV.

We shall now look at spaces having no isolated points. It follows from the previous results that they need not have cV even when they have small characters

— e.g. irrationals or rationals. Are there countable spaces with cV having no isolated points? No point of a countable dense-in-itself space has a compact neighborhood. So, to have cV, it cannot have a countable basis of neighborhoods at any point.

We can give a final solution under MA+¬CH only.

Proposition 12. *Under* [MA+¬CH], *no countable space without isolated points has cV.*

Proof: By a result proved by Šapirovskii ([6], [7]) and Tall ([8]), every point-countable collection of open sets in a Čech complete ccc space is countable, provided MA+¬CH holds.

Let X be a countable cV space without isolated points. No point of X has a compact neighborhood in X and, thus, X is a countable remainder of $\gamma X \setminus X$, where γX is the cV-compactification of X. Consequently, $\gamma X \setminus X$ is Čech complete and has ccc.

Take $x \in X$ and a family $\{U_\alpha\}_{\omega_1}$ of its open neighborhoods in βX such that $\bigcap_S \overline{U_\alpha} \subset X$ for any uncountable $S \subset \omega_1$. Then $\{U_\alpha\}$ is point-countable on $\beta X \setminus X$ and, thus, a countable collection by the above theorem of Šapirovskii and Tall. Consequently, there is an uncountable $S \subset \omega_1$ such that $U_\alpha \setminus X = U_\beta \setminus X$ for any $\alpha, \beta \in S$. That implies $\bigcap_S \overline{U_\alpha} \supset \overline{U_\alpha} \setminus X$ and, therefore, $\overline{U_\alpha} \setminus X = \emptyset$ for $\alpha \in S$. Hence, $\overline{U_\alpha}$ is a compact neighborhood of x in X, which is not possible. □

We do not know if Theorem 12 holds in ZFC (or, say, under CH).

Question 13. *Is it true that no countable dense-in-itself space X is a cV space?*

To answer the question, it may be useful to notice that the preceding proof shows we need less than countability of point-countable open collections on $\beta X \setminus X$.

Definition 14. A space X is said to have property (P) if every uncountable and point countable collection of open sets contains a countable subcollection with empty intersection.

Under MA+¬CH, every Čech complete ccc space has (P). If $\beta X \setminus X$ has (P) then $\gamma X \setminus X$ has (P) for any compactification γX of X.

Proposition 15. *If $\beta X \setminus X$ has property (P) then X is a cV space iff it is locally compact.*

Proof: Assume that X has (P). Take $x \in X$ and a family $\{U_\alpha\}_{\omega_1}$ as in the proof of Proposition 12. According to the property (P) there is a countable set $A \subset \omega_1$ such that $\bigcap_A U_\alpha \setminus X = \emptyset$. Thus $\bigcap_A \overline{U_\alpha}$ is a compact set. Taking open neighborhoods $V_{\alpha,n}$ with $V_{\alpha,n} \subset \overline{V_{\alpha,n}} \subset V_{\alpha,n-1} \subset U_\alpha$ and their intersection (for $\alpha \in A$, $n \in \mathbb{N}$), one gets a compact set in X having a countable local base. Consequently, some of its neighborhood must be compact (see Corollary 5). □

There are many other classes of spaces as candidates for cV spaces. It has been recently shown that topological groups have as remainders Lindelöf or pseudocompact spaces only [1]; so, the second possibility suggests they can be cV spaces. Nontrivial cases are non-locally compact groups having character ω_1 (they are nowhere locally compact and their cV compactification is then Valdivia compact).

Another possibility is to look at P-spaces. Every P-space has finite compact sets only. Thus, if it is a cV space, every its point is either isolated or has character equal to ω_1.

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