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## Products of partially ordered quasigroups

MILAN DEMKO

*Abstract.* We describe necessary and sufficient conditions for a direct product and a lexicographic product of partially ordered quasigroups to be a positive quasigroup. Analogous questions for Riesz quasigroups are studied.

*Keywords:* partially ordered quasigroup, positive quasigroup, Riesz quasigroup, direct product, lexicographic product

*Classification:* 20N05, 06F99

### 1. Preliminaries

The concept of an ordered loop was introduced by D. Zelinsky [10] who was the first to consider valuations of nonassociative algebras. Their values are in ordered loops and D. Zelinsky [11] determined all such ordered loops. Ordered loops and quasigroups were later studied by several other authors (e.g. [1], [2], [3], [4]), also in the connection with the ordered planar ternary rings ([5]). The previous research seems to indicate that the area is interesting and rich enough to justify a systematic study. In this paper we shall consider products of special types of ordered quasigroups — positive quasigroups and Riesz quasigroups.

The concept of a positive quasigroup was introduced by V.M. Tararin [7]. Further, properties of left-positive quasigroups and left-positive Riesz quasigroups were studied by V.A. Testov [8], [9].

Let  $(Q, \cdot)$  be a quasigroup. Let  $a \in Q$ . By  $e_a$  ( $f_a$ ) we denote the local left (right) unit element for  $a$ , i.e.,  $e_a, f_a$  are such elements that  $e_a a = a$  and  $a f_a = a$ . If  $(Q, \cdot)$  is a loop, we denote by 1 the unit element of  $(Q, \cdot)$ .

A nonempty set  $Q$  with a binary operation  $\cdot$  and a relation  $\leq$  is called a *partially ordered quasigroup* (*po-quasigroup*) if

- (i)  $(Q, \cdot)$  is a quasigroup;
- (ii)  $(Q, \leq)$  is a partially ordered set;
- (iii) for all  $x, y, a \in Q$ ,  $x \leq y \Leftrightarrow ax \leq ay \Leftrightarrow xa \leq ya$ .

A po-quasigroup  $Q$  is called a *partially ordered loop* (*po-loop*) if  $(Q, \cdot)$  is a loop.

We say that a po-quasigroup  $Q$  is *trivially ordered*, if any two different elements  $a, b \in Q$  are non-comparable (for non-comparable elements we will use the notation  $a \parallel b$ ).

Let  $Q$  be a po-quasigroup. An element  $p \in Q$  is called a *positive element*, if  $px \geq x$  and  $xp \geq x$  for all  $x \in Q$ . The set of all positive elements of  $Q$  will be denoted by  $P_Q$ . Obviously,  $P_Q = \{p \in Q : p \geq e_x \text{ and } p \geq f_x \text{ for all } x \in Q\}$ . A po-quasigroup  $Q$  is said to be a *positive quasigroup*, if for all  $x, y \in Q$ ,  $x < y$ , there exist positive elements  $p, q \in P_Q$  such that  $y = px$  and  $y = xq$ .

Clearly, a partially ordered loop  $Q$  (and obviously a partially ordered group, too) is a positive quasigroup with the set of all positive elements  $P_Q = \{p \in Q : p \geq 1\}$ . If  $Q$  is a trivially ordered quasigroup, then  $Q$  is a positive quasigroup. The following example shows that there exists a non-trivially ordered positive quasigroup which is not a loop.

**1.1 Example.** Let  $Q = \mathbb{R} \times \mathbb{R}$  ( $\mathbb{R}$  is the set of all real numbers). Take the operation

$$(x, y) \cdot (u, v) = (x + u, 2y + v)$$

and the relation  $\leq$ , where  $=$  is defined componentwise and  $<$  is defined by

$$(x, y) < (u, v) \Leftrightarrow x < u.$$

Then  $Q$  is a positive quasigroup with the set of all positive elements  $P_Q = \{(x, y) : x > 0\}$ . The set of all local unit elements of  $Q$  is  $E = \{(0, y) : y \in \mathbb{R}\}$ .

It is easy to verify that a direct product and a lexicographic product of partially ordered quasigroups is a partially ordered quasigroup. The situation is different in the case of positive quasigroups. In this note we are interested in conditions under which a direct product and a lexicographic product of partially ordered quasigroups is a positive quasigroup. Analogous questions for Riesz quasigroups are studied as well.

## 2. Direct products of positive quasigroups

Let  $I$  be a nonempty set and let  $\{Q_i : i \in I\}$  be a family of partially ordered quasigroups. By the *direct product* of po-quasigroups  $Q_i$ ,  $i \in I$ , we mean the Cartesian product of sets  $Q_i$  with the operation  $\cdot$  and the relation  $\leq$  defined componentwise, i.e.,

$$\begin{aligned} x \cdot y = z &\Leftrightarrow x(i) \cdot y(i) = z(i) \text{ for all } i \in I, \\ x \leq y &\Leftrightarrow x(i) \leq y(i) \text{ for all } i \in I, \end{aligned}$$

where  $x(i)$ ,  $y(i)$  and  $z(i)$  is the  $i$ th component of  $x$ ,  $y$  and  $z$ , respectively. The direct product of po-quasigroups  $Q_i$ ,  $i \in I$  will be denoted by  $\prod_{i \in I} Q_i$ .

The direct product of positive quasigroups need not be a positive quasigroup. For instance, take  $Q$  as in 1.1. Then the direct product  $\prod_{i \in I} Q_i$ , where  $I = \{1, 2\}$  and  $Q_i = Q$  for  $i = 1, 2$ , is not a positive quasigroup.

**2.1 Lemma.** *Let  $I$  be a nonempty set and let  $\{Q_i : i \in I\}$  be a family of partially ordered quasigroups. If  $Q = \prod_{i \in I} Q_i$  is a positive quasigroup, then each  $Q_i$  is a positive quasigroup, too.*

PROOF: Let  $Q = \prod_{i \in I} Q_i$  and let  $j \in I$ . If  $Q_j$  is a trivially ordered quasigroup, then it is a positive quasigroup. Assume that  $Q_j$  is non-trivially ordered. Let  $a, b \in Q_j$ ,  $a > b$ . There exist  $p, q \in Q_j$  such that  $a = qb = bp$ . We are going to show that  $p, q$  are positive elements. Let us define  $x, y \in Q$  in such a way that  $x(j) = a$ ,  $y(j) = b$  and  $x(i) = y(i)$  for each  $i \in I$ ,  $i \neq j$ . Clearly  $x > y$ . Since  $Q$  is a positive quasigroup, there exists  $u, v \in P_Q$  such that  $x = uy = yv$ . Evidently  $u(j) = q$ ,  $v(j) = p$ . Now, let  $c$  be any element of  $Q_j$ . Take  $z \in Q$  with  $z(j) = c$ . Since  $u \in P_Q$ ,  $uz \geq z$  and  $zu \geq z$ . Thus we have  $qc \geq c$  and  $cq \geq c$ . Analogously,  $pc \geq c$  and  $cp \geq c$ . Thus  $p, q \in P_{Q_j}$ .  $\square$

**2.2 Lemma.** *Let  $I$  be a nonempty set and let  $\{Q_i : i \in I\}$  be a family of partially ordered quasigroups such that there are at least two non-trivially ordered quasigroups. Then  $Q = \prod_{i \in I} Q_i$  is a positive quasigroup if and only if  $Q_i$  is a po-loop for each  $i \in I$ .*

PROOF: Let  $Q = \prod_{i \in I} Q_i$  be a positive quasigroup. Let  $e_a, e_b$  be the local left unit elements for  $a, b \in Q_j$ , respectively. We are going to show that  $e_a = e_b$ . By assumption there exists  $k \in I$ ,  $k \neq j$ , such that  $Q_k$  is a non-trivially ordered quasigroup. Let us take  $x, y \in Q$  such that  $x(j) = y(j) = b$ ,  $x(k) > y(k)$  and  $x(i) = y(i)$  for each  $i \in I - \{j, k\}$ . Since  $x > y$ , there is  $p \in P_Q$  such that  $x = py$ . Obviously,  $p(j) = e_b$ . Now, let  $z$  be an element of  $Q$  with  $z(j) = a$ . Since  $p \in P_Q$ ,  $pz \geq z$ . Hence  $p(j)z(j) \geq z(j)$ , i.e.,  $e_b a \geq a$ . This yields  $e_b \geq e_a$ . Analogously we can prove that  $e_a \geq e_b$ . Therefore  $e_a = e_b$ . By the similar way we obtain that any two local right unit elements from  $Q_j$  are equal. Thus we can conclude that  $Q_j$  is a po-loop.

Conversely, if  $Q_i$  is a partially ordered loop for each  $i \in I$ , then  $Q = \prod_{i \in I} Q_i$  is a partially ordered loop, too. Thus  $Q$  is a positive quasigroup.  $\square$

**2.3 Theorem.** *Let  $I$  be a nonempty set and  $\{Q_i : i \in I\}$  a family of partially ordered quasigroups. Then  $Q = \prod_{i \in I} Q_i$  is a positive quasigroup if and only if one of the following conditions is fulfilled:*

- (i)  $Q_i$  is a trivially ordered quasigroup for each  $i \in I$ ;
- (ii) there exists an index  $k \in I$  such that  $Q_k$  is a non-trivially ordered positive quasigroup and for all  $i \in I$ ,  $i \neq k$ ,  $Q_i$  is a trivially ordered loop;
- (iii) there exist at least two indices  $j, k \in I$  such that  $Q_j, Q_k$  are non-trivially ordered quasigroups and  $Q_i$  is a partially ordered loop for each  $i \in I$ .

PROOF: Let  $Q = \prod_{i \in I} Q_i$  be a positive quasigroup. By 2.1  $Q_i$  is a positive quasigroup for each  $i \in I$ . Suppose that there exists exactly one index  $k \in I$  such that  $Q_k$  is a non-trivially ordered positive quasigroup. By the same way as

in the proof of 2.2 we obtain that  $Q_j$  is a loop for each  $j \neq k$ . Obviously  $Q_j$  is a trivially ordered loop. If there exist two indices  $k, j \in I$  such that  $Q_k$  and  $Q_j$  are non-trivially ordered positive quasigroups, then, according to 2.2,  $Q_i$  is a partially ordered loop for each  $i \in I$ .

Conversely, from (i) it follows that  $Q = \prod_{i \in I} Q_i$  is a trivially ordered quasigroup and thus it is a positive quasigroup. Let (ii) hold and let  $x, y \in Q, x > y$ . There exist  $p, q \in Q$  such that  $x = py = qy$ . We are going to show that  $p, q \in P_Q$ . Obviously  $x(k) > y(k)$  and  $x(i) = y(i)$  for each  $i \neq k$ . By assumption  $Q_k$  is a positive quasigroup, therefore  $p(k), q(k) \in P_{Q_k}$ . Further, for each  $i \in I, i \neq k, p(i) = q(i) = 1$ . Thus  $pz \geq z$  and  $zq \geq z$  for each  $z \in Q$ . Hence  $p, q \in P_Q$  and  $Q$  is a positive quasigroup. Finally, in view of 2.2, (iii) implies that  $Q$  is a positive quasigroup.  $\square$

**2.4 Corollary.** *Let  $Q$  be a partially ordered quasigroup which can be expressed as a direct product of non-trivially ordered quasigroups. Then  $Q$  is positive if and only if  $Q$  is a po-loop.*

### 3. Lexicographic products of positive quasigroups

Let  $I$  be a well-ordered set and let  $\{Q_i : i \in I\}$  be a family of partially ordered quasigroups. By the *lexicographic product* of  $Q_i, i \in I$ , we mean the direct product of quasigroups  $Q_i$  with the relation  $\leq$  defined by

$$x \leq y \Leftrightarrow x = y \text{ or } x(i) < y(i) \text{ for the least } i \in I \text{ with } x(i) \neq y(i),$$

where  $x(i)$  and  $y(i)$  is the  $i$ th component of  $x$  and  $y$ , respectively. The lexicographic product of po-quasigroups  $Q_i, i \in I$ , will be denoted by  $\Gamma_{i \in I} Q_i$ .

The lexicographic product of positive quasigroups need not be a positive quasigroup. For instance, the lexicographic product  $\Gamma_{i \in I} Q_i$ , where  $I = \{1, 2\}$  and  $Q_i$  is a positive quasigroup from 1.1 for each  $i \in I$ , is not a positive quasigroup.

Using the similar methods to those in the proof of 2.1 the following lemma can be proved.

**3.1 Lemma.** *Let  $I$  be a well-ordered set and  $\{Q_i : i \in I\}$  a family of partially ordered quasigroups. If  $Q = \Gamma_{i \in I} Q_i$  is a positive quasigroup, then each  $Q_i$  is a positive quasigroup, too.*

**3.2 Lemma.** *Let  $I$  be a well-ordered set and  $\{Q_i : i \in I\}$  a family of partially ordered quasigroups. Let  $Q = \Gamma_{i \in I} Q_i$  be a positive quasigroup. If  $Q_k, k \in I$ , is a non-trivially ordered quasigroup, then  $Q_i$  is a po-loop for each  $i \in I, i < k$ .*

PROOF: Let  $j \in I, j < k$ . Let  $e_a, e_b$  be local left unit elements for  $a, b \in Q_j$ , respectively. We take  $x, y \in Q$  with  $x(j) = y(j) = b, x(k) > y(k)$  and  $x(i) = y(i)$  for each  $i \neq j, k$ . Clearly  $x > y$ . Thus there is  $p \in P_Q$  such that  $x = py$ . Evidently,  $p(i)$  is the local left unit for  $x(i)$  for each  $i \neq k$ . Especially, for  $j$  we

have  $p(j) = e_b$ . Now, let  $z$  be an element of  $Q$  with  $z(j) = a$  and  $z(i) = x(i)$  for each  $i \neq j$ . Since  $p \in P_Q$ ,  $z \leq pz$ . Therefore  $z(j) \leq p(j)z(j)$ , i.e.,  $a \leq e_b a$ . Hence  $e_a \leq e_b$ . Using analogous methods, it can be shown that  $e_b \leq e_a$ . Therefore  $e_a = e_b$ . Similarly we obtain that any two local right unit elements from  $Q_j$  are equal.  $\square$

**3.3 Theorem.** *Let  $I$  be a well-ordered set and  $\{Q_i : i \in I\}$  a family of partially ordered quasigroups. Let each  $Q_i$ ,  $i \in I$ , contain more than one element. Then  $Q = \Gamma_{i \in I} Q_i$  is a positive quasigroup if and only if  $Q_i$  is a positive quasigroup for each  $i \in I$  and*

- (i) if  $Q_k$ ,  $k \in I$ , is a non-trivially ordered quasigroup, then  $Q_j$  is a po-loop for each  $j \in I$ ,  $j < k$ ;
- (ii) if  $Q_k$ ,  $k \in I$ , is a non-trivially ordered quasigroup and  $Q_i$  is a trivially ordered quasigroup for each  $i > k$ , then every two different local unit elements from  $Q_k$  are non-comparable.

PROOF: Let  $Q = \Gamma_{i \in I} Q_i$  be a positive quasigroup. From 3.1 it follows that  $Q_i$  is a positive quasigroup for each  $i \in I$ . The assertion (i) follows from 3.2. To prove (ii) suppose that  $Q_k$  is a non-trivially ordered quasigroup and  $Q_i$  is a trivially ordered quasigroup for each  $i > k$ . First, assume that there are different local left unit elements  $e_a, e_b$  for  $a, b \in Q_k$ , respectively. We are going to show that  $e_a \parallel e_b$ . Assume that  $e_a > e_b$ . There exists  $c \in Q_k$  such that  $b = e_a c$ . Since  $e_a b > b$ , we have  $b > c$ . Let  $x, y$  be elements of  $Q$  with  $x(k) = b$ ,  $y(k) = c$ ,  $x(i) \parallel y(i)$  for  $i > k$  and  $x(i) = y(i)$  for  $i < k$ . Obviously,  $x > y$ , and thus  $x = py$ , where  $p \in P_Q$ . Clearly  $p(k) = e_a$  and, by (i),  $p(i) = 1$  for each  $i < k$ . Now, take  $z \in Q$  with  $z(k) = a$ ,  $z(i) = y(i)$  for  $i > k$  and  $z(i) = x(i)$  for  $i < k$ . Then  $p(i)z(i) = z(i)$  for  $i \leq k$ . Further, since  $p(i)z(i) = p(i)y(i) = x(i)$  for  $i > k$ , we can conclude that  $p(i)z(i) \parallel z(i)$  for each  $i \in I$ ,  $i > k$ . Thus  $pz \parallel z$ . But, on the other hand, since  $p \in P_Q$ , we have  $pz \geq z$ , which contradicts the relation above. Assuming  $e_b < e_a$  we again obtain a contradiction. Therefore  $e_a \parallel e_b$ . Analogously we can show that any two different local right units from  $Q_k$  are non-comparable. Finally, to end this direction of the proof, suppose that  $e_a$  is the local left unit for  $a \in Q_k$ ,  $f_b$  is the local right unit for  $b \in Q_k$  and  $e_a \neq f_b$ . Let  $e_a > f_b$ . Then  $be_a > b$  and since  $Q_k$  is a positive quasigroup, we have  $be_a = bp$ , where  $p \in P_{Q_k}$ . Therefore  $e_a \in P_{Q_k}$ , i.e.,  $e_a \geq e_x$  for each  $x \in Q_k$ . Now, using the fact that any two different local left unit elements from  $Q_k$  are non-comparable, we obtain  $e_a = e_x$  for each  $x \in Q_k$ . This yields  $e_a e_a = e_a$  and therefore  $e_a = f_{e_a}$ , where  $f_{e_a}$  is the local right unit for  $e_a$ . And since  $e_a > f_b$ , we have  $f_{e_a} > f_b$ , which contradicts the fact that any two different local right unit elements from  $Q_k$  are non-comparable. Analogously the case  $f_b > e_a$  cannot occur. Thus  $e_a \parallel f_b$ .

To prove the converse, let  $Q_i$  be a positive quasigroup for each  $i \in I$  and let (i), (ii) be valid. A quasigroup  $Q = \Gamma_{i \in I} Q_i$  is trivially ordered if and only if  $Q_i$  is a trivially ordered quasigroup for each  $i \in I$ . In this particular case,  $Q$  is a positive

quasigroup. In the next we assume that  $Q$  is a non-trivially ordered quasigroup.

Let  $x, y \in Q, x > y$ . There are  $p, q \in Q$  such that  $x = py = yq$ . We are going to show that  $p, q \in P_Q$ . Since  $x > y$ , there exists  $k \in I$  such that  $x(k) > y(k)$  and  $x(i) = y(i)$  for each  $i \in I, i < k$ . Obviously  $p(k) \in P_{Q_k}$  and, by (i),  $Q_i$  is a loop for each  $i < k$ . Moreover, for each  $i < k, p(i)$  is the unit of  $Q_i$ . Let  $z$  be any element of  $Q$ . Clearly,  $p(k)z(k) \geq z(k)$  and  $p(i)z(i) = z(i)$  for  $i < k$ . Suppose that there exists  $r \in I, r > k$ , such that  $Q_r$  is a non-trivially ordered quasigroup. Then, by (i),  $Q_k$  is a loop. Since  $x(k) = p(k)y(k)$  and  $x(k) > y(k)$ , we have  $p(k) \neq 1$ . Thus  $p(k)z(k) > z(k)$ , and therefore  $pz > z$ . Assume that  $Q_i$  is a trivially ordered quasigroup for each  $i > k$ . If  $Q_k$  is a loop, then again  $pz > z$ . Suppose that  $Q_k$  is not a loop. Since  $p(k) \in P_{Q_k}, p(k)$  is greater than or equal to any local unit from  $Q_k$ . But, according to (ii), any two different local unit elements from  $Q_k$  are non-comparable. This yields that  $p(k)$  is the local unit for none element of  $Q_k$ . Therefore  $p(k)z(k) > z(k)$  and hence  $pz > z$ . We have shown that  $p \in P_Q$ . Analogously we can prove that  $q \in P_Q$ . Thus  $Q$  is a positive quasigroup.  $\square$

**3.4 Corollary.** *Let  $I$  be a well-ordered set,  $\{Q_i : i \in I\}$  a family of non-trivially ordered quasigroups. Then  $Q = \Gamma_{i \in I} Q_i$  is a positive quasigroup if and only if  $Q_i$  is a positive quasigroup for each  $i \in I$  and for each couple  $(j, k)$  of elements of  $I$ , if  $k$  covers  $j$ , then  $Q_j$  is a po-loop.*

#### 4. Riesz quasigroups

Riesz groups were studied by L. Fuchs, G. Birkhoff and some other authors. The necessary and sufficient conditions for a lexicographic product of a family of partially ordered group to be a Riesz group were given by J. Lihová in [6]. In this section we deal with the lexicographic product of Riesz quasigroups. As for direct products of Riesz quasigroups, it is routine to verify that the direct product of po-quasigroups is a Riesz quasigroup if and only if each factor is a Riesz quasigroup.

A partially ordered quasigroup  $Q$  is said to be a *directed quasigroup* if  $Q$  is a directed set (i.e. for each  $a, b \in Q$  there exist  $c, d \in Q$  such that  $c \leq a, b$  and  $a, b \leq d$ ).

Let  $a, b \in Q$ . By  $U(a, b)$  ( $L(a, b)$ ) we denote the set of all upper (lower, respectively) bounds of the set  $\{a, b\}$ .

**4.1 Lemma.** *Let  $Q$  be not a directed po-quasigroup. Then there exist  $u, v, z \in Q$  such that  $U(z, u) = \emptyset$  and  $L(z, v) = \emptyset$ .*

PROOF: Suppose that  $Q$  is not a directed quasigroup. Then there are elements  $a, b \in Q$  such that  $U(a, b) = \emptyset$ . In fact, if we assume that  $U(x, y) \neq \emptyset$  for all  $x, y \in Q$ , then for each  $x, y \in Q$  there exists  $g \in Q$  such that  $R_x^{-1}x, R_y^{-1}x \leq g$ . Hence  $L_g^{-1}x \leq x, y$  and thus  $L(x, y) \neq \emptyset$  for all  $x, y \in Q$ . This yields that  $Q$  is

a directed quasigroup, which contradicts the assumption. Analogously, provided  $L(x, y) \neq \emptyset$  for all  $x, y \in Q$  we also arrive at contradiction. Therefore there exist  $a, b, c, d \in Q$  such that  $U(a, b) = \emptyset$  and  $L(c, d) = \emptyset$ . Then  $U(ac, bc) = \emptyset$  and  $L(ac, ad) = \emptyset$ . Now, setting  $z = ac$ ,  $u = bc$  and  $v = ad$  we obtain the required elements.  $\square$

**4.2 Definition.** A partially ordered quasigroup  $Q$  is called a Riesz quasigroup if it is directed and satisfies the following interpolation property

$$(IP) \quad \begin{aligned} &\text{for all } a_i, b_j \in Q \text{ with } a_i \leq b_j, i, j \in \{1, 2\}, \\ &\text{there exists } c \in Q \text{ such that } a_i \leq c \leq b_j. \end{aligned}$$

Evidently every Riesz group is a Riesz quasigroup (Riesz groups are exactly the associative Riesz quasigroups). To give an example of a Riesz quasigroup which is not a Riesz group consider  $Q = \mathbb{R}^2$  with operation  $(x, y) \cdot (u, v) = (x + u, \frac{1}{2}(y + v))$  and relation  $(x, y) < (u, v) \Leftrightarrow x < u$  (cf. [9]).

**4.3 Remark.** Let  $h$  be any element from a partially ordered quasigroup  $Q$ . To see that the condition (IP) holds, it is sufficient to show that for all elements  $x, y, z \in Q$  such that  $h, x$  are non-comparable,  $y, z$  are non-comparable,  $h < y, z$  and  $x < y, z$  there exists  $c \in Q$  such that  $h, x \leq c \leq y, z$ .

Using similar methods as in [6] (by 4.3, the group unit can be replaced by any  $h \in Q$ ) we can prove both following lemmas.

**4.4 Lemma** (cf. [6, Lemma 2.1]). *Let  $I$  be a well-ordered set,  $\{Q_i : i \in I\}$  a family of partially ordered quasigroups. If  $Q = \Gamma_{i \in I} Q_i$  satisfies (IP), then each  $Q_i$  satisfies (IP), too.*

**4.5 Lemma** (cf. [6, Lemma 2.2]). *Let  $I$  be a well-ordered set,  $\{Q_i : i \in I\}$  a family of partially ordered quasigroups. Let  $Q = \Gamma_{i \in I} Q_i$ . If  $h \leq u, v$ ,  $a \leq u, v$ ,  $a \parallel h$ ,  $u \parallel v$  for some  $h, a, u, v \in Q$ , then there exists an index  $i \in I$  such that  $a(i) < u(i), v(i)$ ,  $h(i) < u(i), v(i)$  and  $h(j) = a(j) = u(j) = v(j)$  for all  $j \in I$ ,  $j < i$ .*

By an *antilattice* we mean such a po-quasigroup, in which only pairs of comparable elements may have a greatest lower and a least upper bound. Choose any element  $h \in Q$ . To verify that a partially ordered quasigroup  $Q$  is an antilattice, it is sufficient to show that if  $a \in Q$ ,  $a \parallel h$ , then  $a, h$  do not have a least upper bound.

A partially ordered quasigroup  $Q$  will be said to be *dense* if, whenever  $a, b \in Q$ ,  $a < b$ , there exists  $c \in Q$  with  $a < c < b$ . And again, to see that  $Q$  is dense, it is sufficient to show that for any chosen (fixed) element  $h \in Q$  and all  $b \in Q$  such that  $h < b$ , there exists  $c \in Q$  with  $h < c < b$ .

The following theorem generalizes Theorem 2.3 in [6] which was formulated for Riesz groups.



**4.6 Theorem.** *Let  $I$  be a well-ordered set and  $\{Q_i : i \in I\}$  a family of partially ordered quasigroups such that each  $Q_i$  contains more than one element. Let  $Q = \Gamma_{i \in I} Q_i$ . Then  $Q$  satisfies (IP) if and only if all  $Q_i$  satisfy (IP) and for each couple  $(j, k)$  of elements of  $I$  such that  $k$  covers  $j$  it is true that if  $Q_k$  is not directed and  $Q_j$  is non-trivially ordered, then the quasigroup  $Q_j$  is a dense antilattice.*

PROOF: Let  $Q$  satisfy (IP). Let  $j, k$  be such elements of  $I$ , that  $k$  covers  $j$  and let  $Q_k$  be not directed,  $Q_j$  be non-trivially ordered. By 4.1 there exist  $e_k, t_k, r_k \in Q_k$  such that  $U(e_k, t_k) = \emptyset$  and  $L(e_k, r_k) = \emptyset$ . Take any element  $e \in Q$  with  $e(k) = e_k$ . To prove that  $Q_j$  is dense it is sufficient to verify that for each  $g_j \in Q_j$ ,  $e(j) < g_j$ , there exists  $h_j \in Q_j$  such that  $e(j) < h_j < g_j$ . Define elements  $a, u, v \in Q$  by  $a(j) = e(j)$ ,  $u(j) = v(j) = g_j$ ,  $a(k) = t_k$ ,  $u(k) = e_k$ ,  $v(k) = r_k$  and  $a(l) = u(l) = v(l) = e(l)$  for all  $l \in I - \{j, k\}$ . We have  $e < u, v$ ,  $a < u, v$ ,  $a \parallel e$ ,  $u \parallel v$ . By assumption there exists  $p \in Q$  such that  $e, a < p < u, v$ . Evidently  $p(i) = e(i)$  for all  $i < j$ ,  $e(j) \leq p(j) \leq g_j$ . If  $p(j) = e(j)$ , then  $p(k) \geq e_k, t_k$ , a contradiction. On the other hand, if  $p(j) = g_j$ , then  $p(k) \leq e_k, r_k$ , which is again a contradiction. So we have  $e(j) < p(j) < g_j$  which proves the density of  $Q_j$ .

The rest of the proof can be performed by using the same methods as in [6]. □

**4.7 Corollary** (cf. [6]). *Let  $I$  be a well-ordered set with the least element  $i_0$  and let  $\{Q_i : i \in I\}$  be a family of partially ordered quasigroups such that each  $Q_i$  contains more than one element. Then  $Q = \Gamma_{i \in I} Q_i$  is a Riesz quasigroup if and only if the following conditions are satisfied:*

- (i)  $Q_{i_0}$  is a directed quasigroup;
- (ii) all  $Q_i$ 's satisfy (IP);
- (iii) if  $j, k \in I$ ,  $k$  covers  $j$ ,  $Q_k$  is not directed,  $Q_j$  is non-trivially ordered, then  $Q_j$  is a dense antilattice.

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