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A note on finitely generated ideal-simple commutative semirings

Vítězslav Kala, Tomáš Kepka

Abstract. Many infinite finitely generated ideal-simple commutative semirings are additively idempotent. It is not clear whether this is true in general. However, to solve the problem, one can restrict oneself only to parasemifields.

Keywords: semiring, ideal, simple

Classification: 16Y60

It is known that every finitely generated commutative ring is a Hilbert ring. Using this (and some other classical results) one easily shows that a (commutative) field is finite provided that it is finitely generated as a ring. Now, a ring is finitely generated if and only if it is finitely generated as a semiring; a ring is ideal-simple if and only if it is congruence-simple. Of course, simple commutative rings are just fields and zero-multiplication rings of finite prime order. Consequently, every finitely generated simple commutative ring is finite. On the other hand, setting $a \oplus b = \min(a, b)$ and $a \odot b = a + b$ for all $a, b \in \mathbb{Z}$, we get an infinite commutative semiring that is both ideal- and congruence-simple and that is finitely generated. This semiring is additively idempotent and it is known that every infinite finitely generated congruence-simple commutative semiring is additively idempotent. On the other hand, it seems to be an open problem whether this remains true in the ideal-simple case. The aim of this short note is to reduce the question to a special case of semirings — those whose multiplicative semigroups are groups (such semirings are called parasemifields in the present note). We are going to show that the following two statements are equivalent.

(a) *Every infinite finitely generated ideal-simple commutative semiring is additively idempotent.*
(b) *Every (commutative) parasemifield that is finitely generated as a semiring is additively idempotent.*

(Notice that (a) implies (b) trivially.)

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1. Introduction

A *semiring* is a non-empty set supplied with two associative operations (addition and multiplication) where the addition is commutative and the multiplication distributes over the addition from both sides. A semiring is a *ring* if the addition defines an abelian group.

Let S be a semiring. A non-empty subset I of S is an *ideal* if $(I+I)\cup SI\cup IS\subseteq I$. The semiring is called *ideal-simple* if S is non-trivial and $I=S$ whenever I is an ideal containing at least two elements. The semiring S is called *congruence-simple* if there are just two congruences on S.

The following lemma is obvious.

1.1 Lemma. *The following conditions are equivalent for a ring R.*

(i) *R is ideal-simple as a ring.*
(ii) *R is ideal-simple as a semiring.*
(iii) *R is congruence-simple as a ring.*
(iv) *R is congruence-simple as a semiring.*

(And then R is called simple.)

Every two element semiring is both ideal- and congruence-simple and it is easy to see there are exactly ten two element semirings (up to isomorphism). The following eight of them are commutative:

$\mathbb{S}_1$

+	0	1
0	0	0
1	0	0

·	0	1
0	0	0
1	0	0

$\mathbb{S}_2$

+	0	1
0	0	0
1	0	0

·	0	1
0	0	0
1	0	1

$\mathbb{S}_3$

+	0	1
0	0	0
1	0	1

·	0	1
0	0	0
1	0	0

$\mathbb{S}_4$

+	0	1
0	0	0
1	0	1

·	0	1
0	1	1
1	1	1

$\mathbb{S}_5$

+	0	1
0	0	0
1	0	1

·	0	1
0	0	1
1	1	1

$\mathbb{S}_6$

+	0	1
0	0	0
1	0	1

·	0	1
0	0	0
1	0	1

$\mathbb{S}_7$

+	0	1
0	0	1
1	1	0

·	0	1
0	0	0
1	0	0

$\mathbb{S}_8$

+	0	1
0	0	1
1	1	0

·	0	1
0	0	0
1	0	1

Notice that $\mathbb{S}_1$ and $\mathbb{S}_2$ are additively constant, $\mathbb{S}_3,\mathbb{S}_4,\mathbb{S}_5$ and $\mathbb{S}_6$ are additively idempotent and $\mathbb{S}_7$ and $\mathbb{S}_8$ are rings. Moreover, $\mathbb{S}_1,\mathbb{S}_3,\mathbb{S}_4$ and $\mathbb{S}_7$ are multiplicatively constant and $\mathbb{S}_2,\mathbb{S}_5,\mathbb{S}_6$ and $\mathbb{S}_8$ are multiplicatively idempotent.

The following lemma is easy to prove.

1.2 Lemma. *Let S be a non-trivial semiring containing an element w such that $T = S\backslash\{w\}$ is a subgroup of the multiplicative semigroup of S.*

(i) *If w is multiplicatively neutral (i.e., $w = 1_S$), then T is a subsemiring of S.*

(ii) *If w is multiplicatively absorbing but not additively absorbing, then w is additively neutral (i.e., $w = 0_S$) and either S is a division ring or T is a subsemiring of S.*

(iii) *If $|S| \geq 3$ and w is neither multiplicatively neutral nor multiplicatively absorbing then there exists $v \in T$ such that $wx = vx$ and $xw = xv$ for every $x \in S$.*

2. Introduction continued

Only commutative semirings will be dealt with in the rest of the paper, and hence the word 'semiring' will always mean a commutative semiring.

In this note, a semiring S will be called a *parasemifield* if the multiplicative semigroup of S is a non-trivial group. Clearly, each parasemifield is ideal-simple (in fact, ideal-free).

A non-trivial semiring S will be called a *semifield* if there exists an element $w \in S$ such that w is multiplicatively absorbing (then w is determined uniquely) and the set $S\backslash\{w\}$ is a subgroup of the multiplicative semigroup of S. Clearly, every semifield is ideal-simple.

We have the following basic classification of ideal-simple semirings (see e.g. [1, 11.2]):

2.1 Theorem. *A semiring S is ideal-simple if and only if it is of at least (and then just) one of the following five types:*

(1) *$S \simeq \mathbb{S}_1, \mathbb{S}_3, \mathbb{S}_4$;*
(2) *S is a zero-multiplication ring of finite prime order;*
(3) *S is a field;*
(4) *S is a proper semifield;*
(5) *S is a parasemifield.*

2.2 Proposition ([1, 14.3])**.** *Every infinite finitely generated congruence-simple semiring is additively idempotent.*

2.3 Proposition ([1, 14.5])**.** *No infinite finitely generated ideal-simple semiring is additively cancellative.*

2.4 Example. (i) The parasemifield $\mathbb{Q}^+ \times \mathbb{Q}^+$ (where $\mathbb{Q}$ denotes the field of rational numbers) is ideal-simple but not congruence-simple.

(ii) Denote by W the set of real numbers of the form $m - n\sqrt{2}$, where m, n are non-negative integers and $m + n \geq 1$. Put $a \oplus b = \min(a, b)$ and $a \odot b = a + b$ for all $a, b \in W$. Then $W(\oplus, \odot)$ is an infinite finitely generated congruence-simple semiring that is not ideal-simple. This semiring is additively idempotent and multiplicatively cancellative.

3. Semifields

In the following three lemmas, let S be a non-trivial semiring and let $w \in S$ be such that $T = S\backslash\{w\}$ is a subgroup of the multiplicative semigroup $S(\cdot)$.

3.1 Lemma. *If $1_T w = w$ then $Sw = w$ (i.e., w is multiplicatively absorbing) and S is a semifield.*

PROOF: If $aw = v \neq w$ for some $a \in T$, then $w = 1_T w = a^{-1}aw = a^{-1}v \in T$, a contradiction. Consequently, $Tw = w$ and it remains to show that $ww = w$.

Assume that $ww = u \in T$. Then $1_T = u^{-1}u = u^{-1}ww = ww = u$ according to the preceding part of the proof, and therefore $ww = 1_T$ and $a = a1_T = aww = ww = 1_T$ for every $a \in T$. Thus we have shown that $S = \{w, 1_T\}$ and that S has the following multiplication table:

	w	1_T
w	1_T	w
1_T	w	1_T

Therefore $w(w + 1_T) = ww + w1_T = 1_T + w$, a contradiction since $wz \neq z$ for every $z \in S$. □

3.2 Lemma. *Assume that $1_T w = z \in T$ and $ww \in T$. Then*

(i) *T is a subsemiring of S;*
(ii) *if $|T| = 1$ then $S \simeq \mathbb{S}_1, \mathbb{S}_3, \mathbb{S}_4, \mathbb{S}_7$;.*
(iii) *if $|T| \geq 2$ then T is a parasemifield (and so T is infinite);*
(iv) *$aw = az$ for every $a \in T$;*
(v) *$ww = zz$;*
(vi) *$Sw \subseteq T$ and T is an ideal of S;*
(vii) *if $a \in T$ then either $w + a = z + a \in T$ or $w + a = w$ and $z + a = z$;*
(viii) *if $w + w \in T$ then $w + w = z + z$;*
(ix) *if $w + w = w$ then S is additively idempotent.*

PROOF: If $a, b \in T$ are such that $a + b = w$, then $w = a + b = a1_T + b1_T = (a + b)1_T = w1_T = z$, a contradiction. Thus $T + T \subseteq T$ and T is a subsemiring of S. Further, $aw = a1_T w = az, a \in T$, and $ww = ww1_T = wz = zz$. The rest is easy. □

3.3 Lemma. *Assume that $1_T w = z \in T$ and $ww = w$. Then*

(i) *T is a subsemiring of S;*
(ii) *if $|T| = 1$ then $S \simeq \mathbb{S}_2, \mathbb{S}_5, \mathbb{S}_6, \mathbb{S}_8$;*

(iii) *if* $|T| \geq 2$ *then* T *is a parasemifield* (*and so* T *is infinite*)*;*
(iv) $z = 1_T$*;*
(v) $wv = v$ *for every* $v \in S$ (*i.e.,* $w = 1_S$)*;*
(vi) T *is an ideal of* S*;*
(vii) *if* $a \in T$ *then either* $w + a = 1_T + a \in T$ *or* $w + a = w$ *and* $1_T + a = 1_T$*;*
(viii) *if* $w + w \in T$ *then* $w + w = 1_T + 1_T$*;*
(ix) *if* $w + w = w$ *then* S *is additively idempotent.*

Proof: Similar to that of 3.2. □

3.4 Lemma. *Let* S *be a non-trivial semiring and let* $w_1, w_2 \in S$ *be such that both* $T_1 = S\backslash\{w_1\}$ *and* $T_2 = S\backslash\{w_2\}$ *are subgroups of the multiplicative semigroup* $S(\cdot)$*. Then either* $w_1 = w_2$ *or* $|S| = 2$ *and* $S \simeq \mathbb{S}_2, \mathbb{S}_5, \mathbb{S}_6, \mathbb{S}_8$*.*

Proof: Assume that $w_1 \neq w_2$. If $|S| = 2$ then $S = \{1_{T_1}, 1_{T_2}\}$, and hence S is multiplicatively idempotent. If $|S| \geq 3$ then $T_1 \cap T_2 \neq \emptyset$. Now, $w_1 \in T_2$ and there is $a \in T_2$ such that $w_1 a \in T_1 \cap T_2$. Moreover, $w_1 ab = 1_{T_1}$ for some $b \in T_1$ and $cw_1 = 1_{T_2}$ for some $c \in T_2$. Then $c1_{T_1} = cw_1 ab = 1_{T_2} ab = ab$ and $1_{T_2} 1_{T_1} = w_1 c 1_{T_1} = w_1 ab = 1_{T_1}$. Similarly we get $1_{T_2} 1_{T_1} = 1_{T_2}$, and therefore $1_{T_1} = 1_{T_2} = 1_T$ is a multiplicatively neutral element of S. Then every element from S has an inverse, and so S is a group, a contradiction (see 3.1 and 3.2). □

3.5 Proposition. *Let* S *be a non-trivial semiring and let* $w \in S$ *be such that the set* $S\backslash\{w\}$ *is a subgroup of* $S(\cdot)$*. Then* S *is a semifield* (*i.e.,* $Sw = w$) *in each of the following cases:*

(1) $1_T w = w$*;*
(2) $ww = w$ *and* $1_T w \neq 1_T$*;*
(3) $S \not\simeq \mathbb{S}_1, \mathbb{S}_7$*,* S *is not additively idempotent and* $\mathbb{Q}^+$ *is not isomorphic to a subsemiring of* S*;*
(4) S *is finite,* $S \not\simeq \mathbb{S}_1, \mathbb{S}_7$ *and* S *is not additively idempotent.*

Proof: Combine 3.1, 3.2 and 3.3. □

4. Semifields continued

4.1. Let T be a parasemifield. Then $0 \notin T$; let $S = T \cup \{0\}, x + 0 = x = 0 + x$ and $x0 = 0 = 0x$ for every $x \in S$. In this way we get a semifield (containing T as a semiring), which will be denoted $\mathbb{X}(T)$ in the sequel.

4.1.1 Lemma. (i) $\mathbb{X}(T)$ *is additively idempotent* (*resp. additively cancellative*) *if and only if* T *is such.*

(ii) *A subset* M *of* $\mathbb{X}(T)$ *generates* $\mathbb{X}(T)$ *as a semiring if and only if* $0 \in M$ *and* $M \cap T$ *generates* T *as a semiring* (*then* $|M| \geq 2$)*.*
(iii) $\mathbb{X}(T)$ *is a finitely generated semiring if and only if* T *is such.*
(iv) $\mathbb{X}(T)$ *is not a one-generated semiring; it is a two-generated semiring if and only if* T *is a one-generated semiring.*

Proof: Easy to see. □

4.2. Let $A(\cdot)$ be a non-trivial abelian group, $o \notin A$, $S = A \cup \{o\}$, $x + o = o = o + x$, $x \in S$; $a + a = a$ and $a + b = o$, $a, b \in A$, $a \neq b$. Moreover, $xo = o = ox$, $x \in S$. In this way we get an additively idempotent semifield which will be denoted as $\mathbb{V}(A(\cdot))$.

4.2.1 Lemma. (i) *A subset M of $\mathbb{V}(A(\cdot))$ generates $\mathbb{V}(A(\cdot))$ as a semiring if and only if $M \cap A$ generates $A(\cdot)$ as a semigroup.*

(ii) *$\mathbb{V}(A(\cdot))$ is a finitely generated semiring if and only if $A(\cdot)$ is a finitely generated group.*

(iii) *$\mathbb{V}(A(\cdot))$ is a one-generated semiring if and only if $A(\cdot)$ is a one-generated semigroup. This is equivalent to the fact that $A(\cdot)$ is a finite cyclic group.*

(iv) *$\mathbb{V}(A(\cdot))$ is generated by a two-element set containing the unit element if and only if $A(\cdot)$ is a finite cyclic group (see (iii)).*

Proof: Easy to see. □

4.3. Let T be a parasemifield, $o \notin T$, $S = T \cup \{o\}, x + o = o + x = xo = ox = o$ for every $x \in S$. In this way we get a semifield which will be denoted as $\mathbb{U}(T)$.

4.3.1 Lemma. (i) *$\mathbb{U}(T)$ is additively idempotent if and only if T is such.*

(ii) *A subset M of $\mathbb{U}(T)$ generates $\mathbb{U}(T)$ as a semiring if and only if $o \in M$ and $M \cap T$ generates T as a semiring (then $|M| \geq 2$).*

(iii) *$\mathbb{U}(T)$ is a finitely generated semiring if and only if T is such.*

(iv) *$\mathbb{U}(T)$ is not a one-generated semiring; it is a two-generated semiring if and only if T is a one-generated semiring.*

Proof: Easy to see. □

4.4. Let T be a parasemifield and let the multiplicative group $T(\cdot)$ be a proper subgroup of an abelian group $A(\cdot)$, $o \notin A$. Put $S = A \cup \{o\}$ and define

a) $x + o = o = o + x$, $x \in S$;
b) $a + b = o$, $a, b \in A$, $a^{-1}b \notin T$;
c) $c + d = (1_T + c^{-1}d)c (= (1_T + d^{-1}c)d)$, $c, d \in A$, $c^{-1}d \in T$.

Moreover, put $xo = o = ox, x \in S$. In this way we get a semifield which will be denoted as $\mathbb{W}(T, A(\cdot))$.

4.4.1 Lemma. (i) *T is a subsemiring of $\mathbb{W}(T, A(\cdot))$.*

(ii) *$\mathbb{W}(T, A(\cdot))$ is additively idempotent if and only if T is such.*

(iii) *A subset M of $\mathbb{W}(T, A(\cdot))$ generates it as a semiring if and only if $M \setminus \{o\}$ generates S.*

Proof: Easy to see. □

4.4.2 Lemma. *If the semiring $\mathbb{W}(T, A(\cdot))$ is generated by $a_1, \dots, a_m \in A$, $m \geq 1$, then the factorgroup $A(\cdot)/T(\cdot)$ is generated by the cosets $a_1T, \dots, a_mT$ as a semigroup.*

Proof: Let $a \in A$. Then $a = b_1 + \dots + b_n$, $n \geq 1$, $b_j = a_1^{k_{1,j}} \cdots a_m^{k_{m,j}}$, $k_{i,j} \geq 0$. If $b_{j_1}^{-1} b_{j_2} \notin T$ for some $1 \leq j_1 < j_2 \leq n$, then $b_{j_1} + b_{j_2} = o$ and so $a = o$, a contradiction. Thus $b_{j_1}^{-1} b_{j_2} \in T$, and so $b_j = c_j b_1$, $c_j \in T$. Then $a = cb_1$, $c = c_1 + \dots + c_n$ and $aT = b_1T$. The rest is clear. □

4.4.3 Lemma. *Let $a_1, \dots, a_m \in A$, $m \geq 1$, be such that the factorgroup $A(\cdot)/T(\cdot)$ is generated by the cosets $a_1T, \dots, a_mT$ as a semigroup. Denote by B the subsemigroup of $A(\cdot)$ generated by the elements $a_1, \dots, a_m$. Then for every $a \in A$ there are $b \in B$ and $c \in T$ such that $a = bc$.*

Proof: Obvious. □

4.4.4 Lemma. *If $\mathbb{W}(T, A(\cdot))$ is a finitely generated semiring then T is also.*

Proof: Let the semiring be generated by $a_1, \dots, a_m \in A$, $m \geq 1$. Denote by B the subsemigroup of $A(\cdot)$ generated by these elements. Then $C = BB^{-1} \cap T$ is a finitely generated subgroup of $T(\cdot)$, and hence the subsemiring T_1 of T generated by C is a finitely generated semiring. It remains to show that $T = T_1$.

Let $a \in T$. Then $a = b_1 + \dots + b_n$, $n \geq 1$, $b_j \in B$, $b_j = c_j b_1$, $c_j = b_j b_1^{-1} \in C$ (see the proof of 4.4.2), and therefore $a = cb_1$, $c = c_1 + \dots + c_n \in T_1$. Of course, $b_1 = c^{-1}a \in B \cap T \subseteq C \subseteq T_1$ and so $a, b_1, \dots, b_n \in T_1$. □

4.4.5 Lemma. *$\mathbb{W}(T, A(\cdot))$ is a finitely generated semiring if and only if T is a finitely generated semiring and $A(\cdot)/T(\cdot)$ is a finitely generated group.*

Proof: Combine 4.4.2, 4.4.3 and 4.4.4. □

4.4.6 Remark. Assume that $\mathbb{W}(T, A(\cdot))$ is generated by a single element s as a semiring, denote $1_{\mathbb{W}} = 1_{\mathbb{W}(T,A(\cdot))}$. We have $s \in A$; $B = \{s, s^2, s^3, \dots\}$ is the subsemigroup of $A(\cdot)$ generated by s and $BB^{-1} = \{\dots, s^{-3}, s^{-2}, s^{-1}, 1_{\mathbb{W}}, s, s^2, s^3, \dots\}$ is the subgroup generated by s. Notice that $s \neq 1_{\mathbb{W}}$.

(i) For every $a \in A$ there are $m \geq 1$ and $1 \leq k_1 \leq \dots \leq k_m$ such that $a = s^{k_1} + s^{k_2} + \dots + s^{k_m} = s^{k_1} b$, $b = 1_{\mathbb{W}} + s^{k_2 - k_1} + \dots + s^{k_m - k_1}$. Since $a \neq o$, we have $s^{k_2 - k_1}, \dots, s^{k_m - k_1} \in T$ and so $b \in T$. Moreover, if $a \in T$ then $s^{k_1} = ab^{-1} \in T$ and consequently $s^{k_1}, s^{k_2}, \dots, s^{k_m} \in T$.

(ii) It follows from (i) that $D = B \cap T \neq \emptyset$ and so D is a subsemigroup and $C = DD^{-1}$ a subgroup of $T(\cdot)$. Consequently, there is $n \geq 0$ such that $C = \{\dots, s^{-3n}, s^{-2n}, s^{-n}, 1_{\mathbb{W}}, s^n, s^{2n}, s^{3n}, \dots\}$.

(iii) Denote by T_1 the subsemiring of T generated by s^{-n} and s^n. It follows from (i) and (ii) that $T_1 = T$. Consequently, $n \geq 1$ and T is a two-generated semiring.

(iv) The factorgroup $A(\cdot)/T(\cdot)$ is generated by the coset sT as a semigroup. Thus $A(\cdot)/T(\cdot)$ is a finite cyclic group.

(v) Proceeding similarly as above, one can show that (iii) and (iv) remain true if $\mathbb{W}(T, A(\cdot))$ is generated by $1_{\mathbb{W}}$ and s as a semiring.

4.5 Theorem. *Let S be a semifield and let $w \in S$ be such that w is multiplicatively absorbing and $T = S \setminus \{w\}$ is a subgroup of $S(\cdot)$. Then just one of the following eight cases takes place:*

(1) $S \simeq \mathbb{S}_2$ *(and w is bi-absorbing);*
(2) $S \simeq \mathbb{S}_5$ *(and w is additively neutral);*
(3) $S \simeq \mathbb{S}_6$ *(and w is bi-absorbing);*
(4) *T is a subparasemifield of S and $S \simeq \mathbb{X}(T)$ (and w is additively neutral);*
(5) *$|S| \geq 3$ and $S \simeq \mathbb{V}(T(\cdot))$ (and w is bi-absorbing and S is additively idempotent);*
(6) *T is a subparasemifield of S and $S \simeq \mathbb{U}(T)$ (and w is bi-absorbing);*
(7) *$T_1 = \{a \in T | a + 1_T \neq w\}$ is a subparasemifield of S, $T_1 \neq T$, and $S \simeq \mathbb{W}(T_1, T(\cdot))$ (and w is bi-absorbing);*
(8) *S is a field.*

Proof: Easy (use 3.1, 3.2 and 3.3). □

5. Summary

5.1 Summary. Combining 2.1, 4.5, 4.1.1 (i), (iii), 4.2, 4.3.1 (i), (iii), 4.4.1(ii) and 4.4.4, we conclude that the following two assertions are equivalent.

(a) *Every infinite finitely generated ideal-simple semiring is additively idempotent.*
(b) *Every parasemifield that is finitely generated as a semiring is additively idempotent.*

5.2 Remark. Let F be a field. If F is a finitely generated ring then F is finite. If F is finite then the multiplicative group $F \setminus \{0\}$ is cyclic, and hence F is generated by one element as a semiring.

5.3 Remark. Let S be a one-generated ideal-simple semiring. Combining 2.1, 4.5, 4.1.1(iv), 4.2.1(iii), 4.3.1(iv), 4.4.6 and 5.2, we get that one of the following cases takes place:

(1) $S \simeq \mathbb{S}_1, \mathbb{S}_3, \mathbb{S}_4$;
(2) S is a zero multiplication ring of finite prime order;
(3) S is a finite field;
(4) $S \simeq \mathbb{V}(A(\cdot))$, where $A(\cdot)$ is a non-trivial finite cyclic group;
(5) $S \simeq \mathbb{W}(T, A(\cdot))$, where T is a two-generated parasemifield and $A(\cdot)/T(\cdot)$ is a (non-trivial) finite cyclic group;
(6) S is a parasemifield.

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