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SP-scattered spaces; a new generalization of scattered spaces

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Abstract. The set of isolated points (resp. P-points) of a Tychonoff space X is denoted by $\mathrm{Is}(X)$ (resp. $P(X)$). Recall that X is said to be *scattered* if $\mathrm{Is}(A) \neq \emptyset$ whenever $\emptyset \neq A \subset X$. If instead we require only that $P(A)$ has nonempty interior whenever $\emptyset \neq A \subset X$, we say that X is *SP-scattered*. Many theorems about scattered spaces hold or have analogs for *SP-scattered* spaces. For example, the union of a locally finite collection of SP-scattered spaces is SP-scattered. Some known theorems about Lindelöf or paracompact scattered spaces hold also in case the spaces are SP-scattered. If X is a Lindelöf or a paracompact SP-scattered space, then so is its P-coreflection. Some results are given on when the product of two Lindelöf or paracompact spaces is Lindelöf or paracompact when at least one of the factors is SP-scattered. We relate our results to some on RG-spaces and z-dimension.

Keywords: scattered spaces, SP-scattered spaces, CB-index, sp-index, P-points, P-spaces, strong P-points, RG-spaces, z-dimension, locally finite, Lindelöf spaces, paracompact spaces, P-coreflection, G_δ-topology, product spaces

Classification: 54G10, 54G12

1. Introduction

A space X is said to be *scattered* if each of its nonempty subpaces has an isolated point. To determine if a space X is scattered, one begins by deleting its set $\mathrm{Is}(X)$ of isolated points, followed by deleting $\mathrm{Is}(X \setminus \mathrm{Is}(X))$ from $X \setminus \mathrm{Is}(X)$ and continuing the process possibly transfinitely until a stage is reached when there are no isolated points left to delete. As is well-known, X is scattered if and only if one reaches the empty set in this way. There is a very large mathematical literature on such spaces; see for example those listed in our references. (Scattered spaces are sometimes called *dispersed*.) The reason that these spaces appear in so many places is that this concept arises naturally in many parts of general topology and functional analysis, as does the process of creating them. There are also many generalizations and we create another one by removing P-points instead of isolated points. (Recall that if the set of open neighborhoods of a point p of X is closed under countable intersection, then p is called a *P-point*. If each

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point of X is a P-point, then X is called a *P-space*.) We denote the set of P-points of X by $P(X)$.

Trying naively to duplicate the procedure for creating scattered spaces by deleting P-points instead of isolated points is not productive because while all subsets of $\mathrm{Is}(X)$ are open, this is not the case for $P(X)$. Instead, we delete interiors of sets of P-points. That is we delete $\operatorname{int} P(X)$ from X, then $\operatorname{int}_{X \setminus \operatorname{int} P(X)} P(X \setminus \operatorname{int} P(X))$ from $X \setminus \operatorname{int} P(X)$ and continue (transfinitely when necessary) as before.

If this iteration eventually produces the empty set, we say that X is *SP-scattered*. Clearly every scattered space is SP-scattered. The converse fails as is witnessed by any (nonempty) P-space without isolated points.

The theme of this paper is that the behavior of SP-scattered spaces generalizes that of scattered spaces. In particular, it is shown that "scattered" can be replaced by "SP-scattered" in the hypotheses of several previously known theorems concerning Lindelöf and paracompact scattered spaces.

We summarize the contents of subsequent sections. In Section 2 we formally define SP-scattered spaces and study their internal structure. We show that the union of a locally finite family of SP-scattered subspaces of a space is SP-scattered. We also show that the property of being SP-scattered is preserved by perfect continuous (but not closed continuous) surjections.

In Section 3 we study Lindelöf SP-scattered spaces. Our principal result is that the P-coreflection of such a space is Lindelöf. We derive several consequences of this.

In Section 4 we prove analogously that the P-coreflection of an SP-scattered paracompact space must be paracompact. These two results generalize theorems, due to Levy and Rice [LR81], which state that P-coreflections of scattered Lindelöf (resp. paracompact) spaces are Lindelöf (resp. paracompact).

In Section 5 we apply our results to the class of RG-spaces introduced in [HRW02], and show that an SP-scattered Lindelöf space with finite sp-index (defined in 2.3 below) must be an RG-space. We also discuss the relationship between our new concepts and z-dimension in the sense of Martinez and Zenk (an analogue of Krull dimension in [MZ05]).

The paper concludes with a final section devoted to questions and some answers.

All topological spaces that appear are assumed to be Tychonoff unless the contrary is stated explicitly. A familiarity with results and terminology in the Gillman-Jerison text and the one by Engelking is assumed as well as standard results about scattered spaces. Some of these results will be repeated for the sake of completeness. See, in particular [E89], [GJ76], [LR81], and [S59].

2. The structure and images of SP-scattered spaces

2.1 Definitions. A point p in a topological space X is called a *strong P-point* if it has a neighborhood consisting of P-points. The set of all strong P-points of X is denoted by $\mathrm{SP}(X)$.

Observe that $\mathrm{SP}(X) = \mathrm{int}_X P(X)$.

We leave the verification of the following as exercises.

2.2 Proposition. (a) $\mathrm{SP}(X)$ *is an open subset of* X *containing the set* $\mathrm{Is}(X)$ *of isolated points of* X.

(b) *If* $p \in \mathrm{SP}(X)$ *has a compact neighborhood or if* $\{p\}$ *is a* G_δ, *then* $p \in \mathrm{Is}(X)$. *Hence if* X *has countable pseudocharacter* (*for instance, if* X *is countable, first countable, or perfectly normal*), *or if* X *is locally compact, then* $\mathrm{SP}(X) = \mathrm{Is}(X)$.

(c) $\mathrm{SP}(X) = X$ *if and only if* X *is a* P*-space.*

(d) $\mathrm{SP}(X) = \emptyset$ *if and only if* X *has a dense set of non* P*-points.*

2.3 Definitions. Let $S_0(X) = X$, $S_1(X) = X \setminus \mathrm{SP}(X)$, and let $S_{\alpha+1}(X) = S_1(S_\alpha(X))$ for any ordinal $\alpha \geq 1$. If λ is a limit ordinal λ, let $S_\lambda(X) = \bigcap_{\alpha<\lambda} S_\alpha(X)$.

Observe that $S_\alpha(X)$ is a closed subspace of X for each α.

If there is an ordinal δ such that $S_\delta(X) = \emptyset$, then X is said to be an *SP-scattered space* and the least ordinal δ for which $S_\delta(X) = \emptyset$ is called the *SP-index* of X and is denoted by $\mathrm{sp}(X)$.

The proofs of the following assertions are left as exercises.

2.4 Proposition. (a) $\mathrm{sp}(X) = 1$ *iff* X *is a* P*-space.*

(b) *If* X *has countable pseudocharacter or is locally compact, then* $\mathrm{sp}(X)$ *is the Cantor-Bendixson index* $\mathrm{CB}(X)$. (*See* [LR81] *or* [S59] *for the definition of* $\mathrm{CB}(X)$.)

(c) *If* X *is a* P*-space with no isolated points, then* $\mathrm{sp}(X) = 1$, *but* X *is not scattered.*

2.5 Proposition. *If* E *is a subspace of* X *and* $\alpha \geq 1$ *is an ordinal, then:*

(a) $E \cap P(X) \subset P(E)$, *and equality holds if* E *is open or dense in* X;

(b) $E \cap \mathrm{SP}(X) \subset \mathrm{SP}(E)$;

(b)' *if* E *is open in* X, *then* $E \cap \mathrm{SP}(X) = \mathrm{SP}(E)$;

(c) $S_1(E) \subset E \cap S_1(X)$;

(c)' *if* E *is open in* X, *then* $E \cap S_1(X) = S_1(E)$;

(d) $S_\alpha(E) \subset E \cap S_\alpha(X)$;

(d)' *if* E *is open in* X, *then* $E \cap S_\alpha(X) = S_\alpha(E)$.

Proof: (a) Verifying the inclusion is an exercise.

Suppose conversely that E is open, $x \in P(E)$, and G is a G_δ-set of X that contains x. Then $E \cap G$ is a G_δ-set of E that contains x. So by assumption and the fact that E is open, $x \in \mathrm{int}_E(E \cap G) = \mathrm{int}_X(E \cap G) \subset \mathrm{int}_X G$. So, $x \in E \cap P(X)$.

Suppose finally that E is dense in X, $p \in P(E)$, and G is a G_δ-set of X containing p. Then $G = \bigcap\{U(n) : n \in \omega\}$, where each $U(n)$ is open in X. We may assume that $\{U(n) : n \in \omega\}$ is a decreasing sequence. By using regularity and a straightforward induction, we can find a decreasing sequence $\{V(n) : n \in \omega\}$ of X-open sets such that $p \in \bigcap\{V(n) : n \in \omega\}$ and $\mathrm{cl}_X V(n+1) \subset V(n) \subset U(n)$ for each n.

Since $p \in P(E)$, there is an X-open set H such that $p \in E \cap H \subset \bigcap\{E \cap V(n) : n \in \omega\}$. It will be shown finally that $H \subset G$, from which it follows that p is a P-point of X. For otherwise, there is a $k \in \omega$ such that $H \setminus U(k) \neq \emptyset$, and so $H \setminus \mathrm{cl}_X[V(k+1)]$ is a nonempty X-open set. Because E is dense in X, this open set meets E, contrary to the definition of H.

(b) Each x in $E \cap \mathrm{SP}(X)$ has an X-open neighborhood W consisting of P-points of X, so $W \cap E$ is a neighborhood of x in E consisting of P-points of E.

(b)' Since E is open, $\mathrm{SP}(E) = \mathrm{int}_E P(E) = \mathrm{int}_X P(E) = \mathrm{int}_X(E \cap P(X)) = \mathrm{int}_X E \cap \mathrm{int}_X P(X) = E \cap \mathrm{SP}(X)$.

(c) By (b), $S_1(E) = E \setminus \mathrm{SP}(E) \subset E \setminus [E \cap \mathrm{SP}(X)] = E \cap (X \setminus \mathrm{SP}(X)) = E \cap S_1(X)$.

(c)' By (b)', the inclusion above is an equality.

(d) will be shown by transfinite induction.

By (c), this holds if $\alpha = 1$. Assume that it holds for $\alpha = \gamma$. Then we may apply (c) using $E \cap S_\gamma(X)$ in place of X and $S_\gamma(E)$ in place of E to obtain:

(i) $S_1(S_\gamma(E)) \subset S_\gamma(E) \cap S_1(E \cap S_\gamma(X))$.

Next we apply (c) again replacing X by $S_\gamma(X)$ and E by $E \cap S_\gamma(X)$ to obtain:

(ii) $S_1(E \cap S_\gamma(X)) \subset [E \cap S_\gamma(X)] \cap S_1(S_\gamma(X)) = E \cap S_\gamma(X) \cap S_{\gamma+1}(X) = E \cap S_{\gamma+1}(X)$.

Combining (i) and (ii) yields:

$$\begin{aligned} S_{\gamma+1}(E) = S_1(S_\gamma(E)) &\subset S_\gamma(E) \cap S_1(E \cap S_\gamma(X)) \\ &\subset S_\gamma(E) \cap E \cap S_{\gamma+1}(X) = E \cap S_{\gamma+1}(X). \end{aligned}$$

So (d) holds for $\alpha = \gamma + 1$. It remains to show that it holds for a limit ordinal λ if it holds for all ordinals $\alpha < \lambda$. By the preceding we have

$$S_\lambda(E) = \bigcap_{\alpha<\lambda} S_\alpha(E) \subset \bigcap_{\alpha<\lambda} [E \cap S_\alpha(X)] = E \cap \bigcap_{\alpha<\lambda} S_\alpha(E) = E \cap S_\lambda(X)$$

which completes the proof of (d).

(d)' This proof is identical to that of (d) except that inclusions are replaced (using (c)') by equalities. $\square$

2.6 Remark. If $X = N \cup \{\infty\}$ is the one-point compactification of a countably infinite discrete space N and $E = \{\infty\}$, then $P(X) = N$, while $P(E) = E$. So the equation $E \cap P(X) = P(E)$ in Proposition 2.5(a) need not hold if E fails to be open in X.

An immediate consequence of (d) is:

2.7 Theorem. *Each subspace of an SP-scattered space is SP-scattered.*

Next a characterization of SP-scattered spaces is given.

2.8 Theorem. *If X is any space, then the following are equivalent:*

(a) *X is SP-scattered;*
(b) *if $A \subset X$ is nonempty, then $\operatorname{int}_A P(A) \neq \emptyset$.*

Proof: Assume that (b) fails and A is a nonempty subset of X for which $\operatorname{int}_A P(A) = \emptyset$. Then $S_1(A) = A$, and more generally $S_\delta(A) = A$ for each ordinal δ. Thus A is not SP-scattered, so by Theorem 2.7, (a) fails.

Assume next that (b) holds. Since $S_{\gamma+1}(X) = S_\gamma(X) \setminus \operatorname{int}_{S_\gamma(X)} P(S_\gamma(X))$, it follows from (b) that the nonempty members of $\{S_\gamma(X) : \gamma \text{ an ordinal}\}$ form a strictly decreasing sequence of subspaces of X. Therefore, if $\gamma(0)$ is the least ordinal greater than $|X|$, it follows that $S_{\gamma(0)} = \emptyset$ and (a) holds. □

2.9 Corollary. *If X is SP-scattered, then $\mathrm{SP}(A)$ is dense in A for each subspace A of X.*

Proof: Suppose to the contrary that $A \setminus \operatorname{cl}_A(\operatorname{int}_A P(A)) = E$ is nonempty for some subspace A of X. By 2.5(a), $P(E) = E \cap P(A)$ since E is open in A. Consequently $\operatorname{int}_E P(E) = \emptyset$, so E is not SP-scattered. By Theorem 2.7, it follows that X is not SP-scattered. □

Next we record a series of results that culminate in showing that the union of a locally finite collection of SP-scattered subspaces is SP-scattered.

2.10 Lemma. *Suppose S_1 and S_2 are subspaces of a space X such that*

(i) *$S_1 \cup S_2 = X$;*
(ii) *$\mathrm{SP}(S_i)$ is dense in S_i for $i = 1, 2$;*
(iii) *$P(X)$ has empty interior.*

Then each S_i is dense in X.

Proof: If $S_1 = X$, then since the S_1-interior of $P(S_1)$ is nonempty and coincides with its X-interior, (iii) is violated. Hence $S_1 \neq X$. Similarly $S_2 \neq X$.

Suppose that S_1 is not dense in X. There is an open subset W of X such that $\operatorname{int}_{S_2} P(S_2) = W \cap S_2$. By (i), the nonempty open set $X \setminus \operatorname{cl}_X S_1$ is contained in S_2. So by (ii) it follows that

$$(X \setminus \operatorname{cl}_X S_1) \cap (S_2 \cap W) \neq \emptyset.$$

Since $X \setminus \mathrm{cl}_X S_1 \subset S_2$ and $S_2 \cap W$ is a P-space, it follows that

$$(X \setminus \mathrm{cl}_X S_1) \cap W = (X \setminus \mathrm{cl}_X S_1) \cap (S_2 \cap W)$$

is a nonempty open P-space contained in X contrary to (iii). Hence S_1 is dense in X, and a similar argument shows that S_2 is dense in X. $\square$

2.11 Lemma. *If S_1 and S_2 are subspaces of a space X such that* (i) *and* (ii) *of Lemma* 2.10 *hold, then $P(X)$ has a nonempty interior.*

PROOF: If instead $\mathrm{int}_X P(X) = \emptyset$, then by Corollary 2.9, each of the S_i is dense in X. Choose U_i open in X such that $U_i \cap S_i = \mathrm{int}_{S_i} P(S_i)$ for $i = 1, 2$. Then by (ii), each $U_i \cap S_i$ is a P-space that is dense in S_i and hence in X. Thus U_1 and U_2 are dense open subsets of X. It follows that $U_1 \cap U_2$ is a dense open subset of X. Since the intersection of an open dense subspace and a dense subspace is dense, it follows from 5(a) that for $i = 1, 2$, $U_1 \cap U_2 \cap S_i$ is a P-space that is dense in X.

Hence by 2.5(a), for $i = 1, 2$,

$$P(X) \cap (U_1 \cap U_2 \cap S_i) = P(U_1 \cap U_2 \cap S_i) = U_1 \cap U_2 \cap S_i.$$

Therefore, $U_1 \cap U_2$ is a dense open subset of X contained in $P(X)$, contrary to the assumption that $\mathrm{int}_X P(X) = \emptyset$. $\square$

2.12 Proposition. *The union of finitely many SP-scattered subspaces of a space X is SP-scattered.*

PROOF: It suffices to show this for the union of two such spaces. Suppose $X = S_1 \cup S_2$ where S_1 and S_2 are SP-scattered. If A is a nonempty subspace of X, then by Theorem 2.7, each $S_i \cap A$ is SP-scattered. So by Corollary 2.9, $\mathrm{SP}(A \cap S_i)$ is dense in $A \cap S_i$ for $i = 1, 2$. By Lemma 2.11, $\mathrm{SP}(A)$ is nonempty. Hence by Theorem 2.8, X is SP-scattered. $\square$

2.13 Theorem. *If X is a union of open SP-scattered subspaces, then it is SP-scattered.*

PROOF: Suppose $X = \bigcup\{T_\alpha : \alpha < \lambda\}$, where each T_α is an open SP-scattered subspace. By Theorem 2.8, for each ordinal $\alpha < \lambda$, there is an ordinal $\sigma(\alpha)$ such that $S_{\sigma(\alpha)}(T_\alpha) = \emptyset$. Let $\gamma = \sup\{\sigma(\alpha) : \alpha < \lambda\} + 1$. Clearly $S_\gamma(T_\alpha) = \emptyset$ for all $\alpha < \lambda$.

By Proposition 2.5(d)', for every $\alpha < \lambda$, we have $T_\alpha \cap S_\gamma(X) = S_\gamma(T_\alpha) = \emptyset$. Therefore:

$$\begin{aligned} S_\gamma(X) = X \cap S_\gamma(X) &= (\bigcup\{T_\alpha : \alpha < \lambda\}) \cap S_\gamma(X) \\ &= \bigcup\{T_\alpha \cap S_\gamma(X) : \alpha < \lambda\} = \emptyset. \end{aligned}$$

So X is SP-scattered. $\square$

2.14 Theorem. *If X is the union of a locally finite family $\mathcal{F}$ of SP-scattered subspaces, then X is SP-scattered.*

Proof: If $x \in X$, then there is an open neighborhood $V(x)$ of x such that $\mathcal{F}(x) = \{F \in \mathcal{F} : V(x) \cap F \neq \emptyset\}$ is a finite set. Then $V(x) = \bigcup\{V(x) \cap F : F \in \mathcal{F}(x)\}$. By Theorem 2.12, each $V(x) \cap F$ is SP-scattered, so by Theorem 2.13 above, each $V(x)$ is SP-scattered. But $X = \bigcup\{V(x) : x \in X\}$, so X is SP-scattered by Theorem 2.13. □

2.15 Remark. "Locally finite" cannot be replaced by "point finite" in the hypothesis of Theorem 2.14. To see this, let $\mathbb{Q}$ denote the space of rational numbers. If $q \in [0,1) \cap \mathbb{Q}$, let $F(q) = \{q + n : n \in \mathbb{Z}\}$ and note that each $F(q)$ is a discrete subspace of $\mathbb{Q}$. The family $\mathcal{F} = \{F(q) : q \in [0,1) \cap \mathbb{Q}\}$ partitions $\mathbb{Q}$ into countably many SP-scattered subspaces, but $\mathbb{Q} = \bigcup \mathcal{F}$ is not SP-scattered. Note that this collection is point finite since it is a partition, but not locally finite.

2.16 Definitions. Suppose $f : X \to Y$ is a closed continuous surjection.

(a) If $f^{\leftarrow}(y)$ is compact whenever $y \in Y$, then f is called a *perfect map*.
(b) If $f[A] \neq Y$ whenever A is a proper closed subset of X, then f is called an *irreducible map*.

2.17 Theorem. *A perfect image of an SP-scattered space is SP-scattered.*

Proof: Suppose $f : X \to Y$ is a perfect map; and X is SP-scattered. By 6.5(c) of [PW88], there is a closed subspace B of X such that $f|B : B \to Y$ is a perfect irreducible surjection. Moreover, by Theorem 2.7, B is SP-scattered. So we may assume that $f : X \to Y$ is a perfect irreducible surjection.

By Theorem 2.8(b), $\mathrm{int}_X\, P(X)$ is dense in X. So by 6.5(d)(2) of [PW88] its nowhere dense closed complement $X \setminus \mathrm{int}_X\, P(X)$ is mapped by f onto a nowhere dense closed subset of Y. Thus $S = Y \setminus f[X \setminus \mathrm{int}_X\, P(X)]$ is a dense open subset of Y. Observe next that $f^{\leftarrow}[S] \subset \mathrm{int}_X\, P(X)$ and hence is a P-space. Moreover, by 1.8(b)(2) of [PW88], the restriction of f to $f^{\leftarrow}[S]$ is a perfect surjection. Because closed continuous maps are quotient maps and a quotient image of a P-space is a P-space, (see 4K(5) of [GJ76]) it follows that S is a P-space and a dense open subset of Y. Hence $\mathrm{int}_Y\, P(Y)$ is dense in Y.

If T is a nonempty subset of Y, then using 1.8f(2) of [PW88], the restriction $f|f^{-1}[T]$ is a perfect map from $f^{-1}[T]$ onto T, so by 6.5(c) of [PW88], there is an $E \subset X$ such that $f|E \to T$ is a perfect irreducible surjection. By Theorem 2.7, E is SP-scattered. So by the conclusion of the previous paragraph, $\mathrm{int}_T\, P(T)$ is dense in T, and we may conclude from Theorem 2.8 that Y is SP-scattered. □

2.18 Remarks on closed continuous images of SP-scattered spaces

In [KV77], the authors construct, using CH, a locally compact, countably compact, first countable, separable scattered Hausdorff (and hence Tychonoff) space which is totally disconnected but not strongly zero-dimensional that admits a closed continuous map onto [0,1]. Thus, if CH holds, the property of being an SP-scattered space need not be preserved by closed continuous maps.

Without using CH, an example is given in [KV77] of a zero-dimensional scattered Hausdorff (and hence Tychonoff) space whose image under a closed continuous map is not scattered, and a simpler one is given in [T77]. We do not know if there is such an example in ZFC whose closed continuous image is not SP-scattered.

In [T68], it is shown that every paracompact scattered space is strongly zero-dimensional. We do not know if this conclusion holds for paracompact SP-scattered spaces.

3. Lindelöf SP-scattered spaces

3.1 Definitions. (a) For any space X, the topology obtained by letting every G_δ subset of X be open is called the *G_δ-topology* and the space so obtained is denoted by X_δ.

(b) A space X such that $|f(X)|$ is countable for every $f \in C(X)$ is said to be *functionally countable*.

Note that X_δ is always a P-space and that if X_δ is Lindelöf, then X is functionally countable. That the converse of this latter assertion is false is shown in Example 6 of [LR81].

The proof of the next result is modelled after that of 5.2 in [LR81], where our X_δ is denoted by bX.

3.2 Theorem. *If X is an SP-scattered Lindelöf space, then X_δ is a Lindelöf space.*

PROOF: Suppose not and let X denote a counterexample for which $\mathrm{sp}(X)$ is minimal. Thus we are assuming that:

(i) X is an SP-scattered Lindelöf space such that X_δ is not a Lindelöf space, and

(ii) if Y is any SP-scattered Lindelöf space and $\tau = \mathrm{sp}(Y) < \mathrm{sp}(X)$, then Y_δ is a Lindelöf space.

We will show first that τ is not a limit ordinal.

For otherwise, $\emptyset = S_\tau(X) = \bigcap_{\alpha<\tau} S_\alpha(X)$ is the intersection of a decreasing collection of nonempty closed subsets of the Lindelöf space X. So there is a countable subset $\{\alpha_i : i < \omega\}$ of $[0,\tau)$ such that $\bigcap_{i<\omega} S_{\alpha_i}(X) = \emptyset$, and $\tau = \sup\{\alpha_i : i < \omega\}$. Now $X = \bigcup_{i<\omega}(X \setminus S_{\alpha_i}(X))$. If $\mathcal{U}$ is a cover by cozerosets of X that refines $\{X \setminus S_{\alpha_i}(X) : i < \omega\}$, then since X is a Lindelöf space, there is a countable subcover $\{U_s : s < \omega\}$ of $\mathcal{U}$ whose union is X such that for all $s < \omega$, there is an $i(s) < \omega$ such that $U_{i(s)}$ is contained in $X \setminus S_{\alpha_{i(s)}}(X)$.

If $s < \omega$, then by Lemma 2.5(d)', $S_{\alpha_{i(s)}}(U_s) = U_s \cap S_{\alpha_{i(s)}}(X) = \emptyset$. Thus $\mathrm{sp}(U_s) \leq \alpha_{i(s)} < \tau$. So by the minimality of τ and the fact that U_s is a cozeroset of a Lindelöf space, it follows that $(U_s)_\delta$ is a Lindelöf space; consequently so is $\bigcup_{s<\omega}(U_s)_\delta$. This contradicts the choice of X, so τ is not a limit ordinal as claimed. Hence there exists an ordinal γ such that $\tau = \gamma + 1$, in which case $\emptyset = S_{\gamma+1} = S_1(S_\gamma(X))$. Therefore $S_\gamma(X)$ is a nonempty P-space and is a Lindelöf space since it is closed in X.

Suppose $\mathcal{C}$ is a cover of X by zerosets of X. To show that X_δ is Lindelöf, we must show that $\mathcal{C}$ has a countable subcover. Note first that the Lindelöf space $S_\gamma(X)$ is z-embedded in X. (See [B76].) Since $S_\gamma(X)$ is a Lindelöf P-space, there is a countable subfamily $\{Z_i : i < \omega\}$ of $\mathcal{C}$ that covers $S_\gamma(X)$.

We disjointify this family as follows. Let

$$T_1 = Z_1,\ T_2 = Z_2 \setminus T_1,\ T_3 = Z_3 \setminus (T_1 \cup T_2), \ldots, T_n = Z_n \setminus (\bigcup_{i=1}^{n-1} T_i), \ldots$$

Then $\{T_i \cap S_\gamma(X) : i < \omega\}$ is a countable partition of $S_\gamma(X)$ into clopen sets, and hence for each $i < \omega$, there is a $V_i \in \mathrm{coz}(X)$ such that $T_i \cap S_\gamma(X) = V_i \cap S_\gamma(X)$. If $B = X \setminus \bigcup_{i<\omega} V_i$, then B is Lindelöf. By Lemma 2.5(d), $S_\gamma(B) \subset B \cap S_\gamma(X) = \emptyset$, so $\mathrm{sp}(B) \leq \gamma < \tau$. Thus by the minimality of τ, B_δ is Lindelöf, and so there is a countable subfamily $\mathcal{E}$ of $\mathcal{C}$ that covers B_δ.

Now $V_i \setminus Z_i \subset X \setminus S_\gamma(X)$ since $V_i \cap S_\gamma(X) = T_i \cap S_\gamma(X) \subset Z_i$ and V_i is Lindelöf. Arguing as was done with B, we obtain $\mathrm{sp}(V_i \setminus Z_i) \leq \gamma < \tau$. Hence each $(V_i \setminus Z_i)_\delta$ is Lindelöf, so $T = \bigcup_{i<\omega}(V_i \setminus Z_i)_\delta$ is a Lindelöf space. Thus, there is a countable subfamily $\mathcal{A}$ of $\mathcal{C}$ that covers T. Hence $\mathcal{A} \cup \mathcal{E} \cup \{Z_i : i < \omega\}$ is the required countable subfamily of $\mathcal{C}$. □

3.3 Corollary. *Suppose X is SP-scattered. Then:*

(a) *if X is Lindelöf, then it is functionally countable;*
(b) *if X is locally Lindelöf, then it is locally functionally countable.*

Our next result generalizes Theorem 2.7 of [G84].

3.4 Theorem. (a) *A product $\prod_{n<\omega} X_n$ of countably many SP-scattered Lindelöf spaces is Lindelöf.*
(b) *The product of an SP-scattered Lindelöf space X and a Lindelöf space Y is Lindelöf.*

Proof: (a) By Theorem 3.2, each $(X_n)_\delta$ is a Lindelöf P-space. As noted in 2.6 of [G84], N. Noble showed in [N71] that a countable product of Lindelöf P-spaces is Lindelöf. Consequently its continuous image $\prod_{n<\omega} X_n$ is also Lindelöf.

(b) By Theorem 3.2, X_δ is a Lindelöf P-space. It is well-known that the product of a Lindelöf P-space and a Lindelöf space is Lindelöf. So $X_\delta \times Y$ is Lindelöf as is its continuous image $X \times Y$. □

3.5 Remark. The result of (b) is stated in [A87] and the reader is referred to [A88] for a proof. However, it is easier to prove (b) directly with an argument that apes the proof that the product of a compact space and a countably compact space is countably compact.

Corollary 3.3(a) cannot be generalized by replacing "Lindelöf" by "realcompact and weakly Lindelöf". This possibility is closed off by an example due to Alan Dow used in [HRW02] for other reasons.

3.6 Example. A realcompact weakly Lindelöf SP-scattered space that is not Lindelöf.

Strengthen the usual topology on $[0,1]$ by first making each rational point isolated, and for each irrational x, finding a sequence $s(x)$ of distinct rationals that converges to x. Define a new neighborhood base at x consisting of sets of the form $\{x\} \cup (s(x) \setminus F)$, where F is a finite subset of $s(x)$. X is locally compact, realcompact (by 8.17 of [GJ76]) and is scattered with $\mathrm{CB}(X) = \mathrm{sp}(X) = 2$. Because the set $\mathbb{Q}$ of rationals is dense in X, it is separable and hence weakly Lindelöf. But $X \setminus \mathbb{Q}$ is uncountable, closed, and discrete, so X is not Lindelöf. Observe also that X is locally Lindelöf. □

Recall that if a space Y is homeomorphic to a dense subspace of a space X, then X is called an *extension of* Y.

3.7 Theorem. (a) *Every locally Lindelöf but not Lindelöf space Y has a one-point Lindelöf extension $X = Y \cup \{q\}$ (where $q \notin Y$).*

(b) *If, in addition, Y is SP-scattered, then so is X, and hence X is functionally countable.*

Proof: (a) follows from results in [MRW72] by letting the property P_J in that paper stand for "Lindelöf space" and using their Theorems 3.1 and 4.1.

(b) follows by applying (a), Theorem 2.12, Theorem 3.2, and Corollary 3.3(a). □

3.8 Remark. It seems natural to ask whether every SP-scattered locally Lindelöf space has a one-point Lindelöf extension in which the added point is a P-point. The answer is "no" as is shown by the (Mrowka-) Isbell space Y in 5I of [GJ76]. It is scattered and locally compact, but if $Y \cup \{p\}$ were any one-point extension of Y, then p cannot be a P-point of this space. For if it were, $Y \cup \{p\} \setminus N$ would a $Y \cup \{p\}$-neighborhood of p, which it cannot be since N is dense in $Y \cup \{p\}$.

4. Paracompact SP-scattered spaces

Three well-known facts that will be used below are recalled in the next lemma.

4.1 Lemma. (a) *Every paracompact space has a locally finite refinement consisting of cozerosets.*

(b) *The union of a locally finite family of cozerosets is a cozeroset.*
(c) *An F_σ-subspace of a paracompact space is paracompact.*

Proof: (a) See Lemma 4.3 of [CN75].

(b) This is a special case Theorem 1.5 of [Na70]. For convenience we include a simple direct proof. Suppose $\{\operatorname{coz} f_i : i \in I\}$ is a locally finite family of cozerosets of functions $f_i \in C(X)$. We may assume $\mathbf{0} \le f_i \le \mathbf{1}$ for each $i \in I$. For each $x \in X$, let $f(x) = \sum_{i \in I} f_i(x)$. It follows easily from the local finiteness of this family that $f \in C(X)$ and $\operatorname{coz} f = \bigcup\{\operatorname{coz} f_i : i \in I\}$.

(c) See 5.12.8 of [E89]. □

The result that follows generalizes 5.1 of [LR81]. Its proof is patterned to some extent on the proof of this latter result.

4.2 Theorem. *If X is an SP-scattered paracompact space, then X_δ is a paracompact space.*

This will be shown by contradiction with the aid of a number of lemmas. That is, we will assume the theorem is false and show that it follows that there must be an example showing it to be false that is minimal in a sense we will describe next. Then we will show that this "minimal" counter example is paracompact; thereby proving the theorem. Suppose the contrary and let $\tau = \min\{\alpha :$ there is a paracompact SP-scattered space S such that S_δ is not paracompact and $\operatorname{sp}(S) = \alpha\}$. There is a space X such that:

(i) X is paracompact and SP-scattered, $\operatorname{sp}(X) = \tau$, and X_δ is not paracompact, and
(ii) if Y is paracompact and SP-scattered with $\operatorname{sp}(Y) < \operatorname{sp}(X)$, then Y_δ is paracompact.

4.3 Lemma. *τ is not a limit ordinal, so there is an ordinal $\gamma \ge 1$ such that $\tau = \gamma + 1$.*

Proof: For, otherwise, $\emptyset = \bigcap_{\alpha<\tau} S_\alpha(X)$. Since each $S_\alpha(X)$ is closed in X, $\bigcup_{\alpha<\tau}(X \setminus S_\alpha(X)) = X$. Because X is paracompact, the open cover $\mathcal{C} = \{X \setminus S_\alpha(X) : \alpha < \tau\}$ has a σ-discrete cozero refinement $\mathcal{U} = \bigcup_{i<\omega} \mathcal{U}_i$, where each $\mathcal{U}_i$ is a collection of cozerosets of X with the property that each $x \in X$ has a neighborhood that meets at most one member of $\mathcal{U}_i$. Each member of $\mathcal{U}$ is paracompact since it is an F_σ subset of a paracompact space. Because $\mathcal{U}$ is a refinement of $\mathcal{C}$, for each $U \in \mathcal{U}$, there is an $\alpha(U) < \tau$ such that $U \subset X \setminus S_{\alpha(U)}(X)$, so $S_{\alpha(U)}(U) = U \cap S_{\alpha(U)}(X) = \emptyset$. Therefore $\operatorname{sp}(U) \le \alpha(U) < \tau$, while U is SP-scattered and paracompact. So, by the minimality of τ, the space U_δ is paracompact. Moreover, because U is a cozeroset, U_δ is clopen in X_δ.

For each i, $\{U_\delta : U_i \in \mathcal{U}_i\}$ is a discrete (locally finite) family of clopen subsets of X_δ, so if $S_i = \bigcup \mathcal{U}_i$, then $(S_i)_\delta = \bigcup\{U_\delta : U \in \mathcal{U}\}$ is a clopen subset of X_δ that is a free union since the sets involved are pairwise disjoint. A free union of paracompact spaces is paracompact, so each $(S_i)_\delta$ is paracompact and clopen in X_δ.

Now let $\mathcal{K}$ denote an open cover of X_δ. For each $i < \omega$, $\{K \cap (S_i)_\delta : K \in \mathcal{K}\}$ is an open cover of $(S_i)_\delta$. So, by the paracompactness of $(S_i)_\delta$, this open cover has a locally finite open refinement $\mathcal{F}_i$. Since $(S_i)_\delta$ is open in X_δ, the members of $\mathcal{F}_i$ are also open in X_δ, and it is clear that $\mathcal{U} = \bigcup\{\mathcal{U}_i : i < \omega\}$ covers X. It follows that $\mathcal{F} = \{F : F \in \mathcal{F}_i$ for some $i < \omega\}$ is a σ-locally finite open refinement of $\mathcal{K}$. Therefore, each open cover of X_δ contains a σ-locally open refinement, so X_δ is paracompact. This contradiction establishes that τ is not a limit ordinal. To see that $\gamma \geq 1$, note that if $\tau = \mathrm{sp}(X) = 1$, then by 2.4(a), X is a P-space. So $X = X_\delta$ is paracompact, contrary to (i) above. Thus $\tau > 1$ and $\gamma \geq 1$. □

Henceforth we will denote $S_\gamma(X)$ by S.

4.4 Lemma. *If $T \subset X$ is paracompact and disjoint from S, then T_δ is paracompact.*

Proof: T is SP-scattered by Theorem 2.7 since it is a subspace of the SP-scattered space X. By Proposition 2.5(a), $S_\gamma(T) \subset T \cap S_\gamma(X) = \emptyset$, so $\mathrm{sp}(T) \leq \gamma < \gamma + 1 = \tau$. Thus T_δ is paracompact by the minimality of τ. □

The balance of the proof of 4.2 is modeled after the argument in lines 2–13 of page 233 of [LR81]. It will be complete when we show that the assumptions made above imply X_δ is paracompact.

Since $\mathcal{Z}(X)$ is an open base for X_δ, to show that X is paracompact it suffices to show that if $\mathcal{U}$ is an open cover of X by zerosets, then it has an open locally finite refinement in the topology of X_δ.

Each $x \in S$ is a member of some $Z_x \in \mathcal{U}$. Because S is closed in the (normal) paracompact space X, it is z-embedded. So because $Z_x \cap S$ is a zeroset of the P-space S, it is clopen in S, and hence its complement in S is in $\mathcal{Z}(S)$. By the z-embedding cited above, there is a cozeroset V_x of X such that $V_x \cap S = Z_x \cap S$.

Now $\{V_x : x \in S\} \cup \{X \setminus S\}$ is an open cover of the paracompact space X, so by Lemma 4.1(a), it has a locally finite refinement $\mathcal{C}$ consisting of cozerosets of X. If $x \in S$, there are $C_x \in \mathcal{C}$ and $a(x) \in S$ such that $x \in C_x \subset V_{a(x)}$. Thus

$$(\#) \qquad C_x \cap S \subset V_{a(x)} \cap S = Z_{a(x)} \cap S.$$

Since $\{C_x : x \in S\}$ is a locally finite collection of cozerosets of X, its union is a cozeroset of X by Lemma 4.1(b).

If $H = X \setminus \bigcup\{C_x : x \in S\}$, then $H \in \mathcal{Z}(X)$. So by our previous assumptions, H is paracompact, H_δ is clopen in X_δ, and $H \cap S = \emptyset$. So H_δ is paracompact by 4.4. Hence the open cover $\{U \cap H_\delta : U \in \mathcal{U}\}$ has a locally finite open refinement $\mathcal{W}$ in the topology of H_δ, each member of which is open in X_δ. It follows from the above that each member of $\mathcal{W}$ is a subset of some member of $\mathcal{U}$.

Next consider the collection $\mathcal{J} = \{\mathcal{C}_x \cap Z_{a(x)} : x \in S\}$. Each of its members is clopen in X_δ, and since $\{C_x : x \in S\}$ is X-locally finite, $\mathcal{J}$ is locally finite in X_δ. Finally, each member of $\mathcal{J}$ is a subset of $Z_{a(x)}$ which belongs to $\mathcal{U}$.

Now for each $x \in S$, the set $C_x \setminus Z_{a(x)}$ is a cozeroset (and hence an F_σ-set) of the paracompact space X and hence by Lemma 4.1(c) is paracompact as well as clopen in X_δ. By (#) above, $(C_x \setminus Z_{a(x)}) \cap S = \emptyset$, so $(C_x \setminus Z_{a(x)})_\delta$ is paracompact. Hence its open cover $\{(C_x \setminus Z_{a(x)}) \cap U : U \in \mathcal{U}\}$ has a locally finite open refinement in the topology of $(C_x \setminus Z_{a(x)})_\delta$, which will be denoted by $\mathcal{G}(x)$. It follows that $\mathcal{G}(x)$ is a locally finite collection of open sets of X_δ each of whose members is a subset of some member of $\mathcal{U}$.

Finally, let

$$\mathcal{B} = W \cup \mathcal{J} \cup [\bigcup\{\mathcal{G}(x) : x \in S\}]$$

and observe that it is a union of collections of sets. It has been shown above that each of the sets in the members of $\mathcal{B}$ is open in X_δ, and that each of them is a subset of some member of $\mathcal{U}$. To contradict the assumptions made at the beginning of the proof of Theorem 4.2, we need to show that

(1) $\mathcal{B}$ covers X_δ, and
(2) $\mathcal{B}$ is a locally finite collection of sets of X_δ.

To prove (1), we consider three cases.

(i) If $x \in H$, it follows from the definition of $\mathcal{W}$ that x is in some member of $\mathcal{W}$.
(ii) If $x \in C_x \cap Z_{a(x)}$ for some $x \in S$, then x is in some member of $\mathcal{J}$.
(iii) If $x \in C_x \setminus Z_{a(x)}$ for some $x \in S$, then x is in some member of $\mathcal{G}(x)$.

So (1) holds.

To prove (2), we suppose $b \in X$ and consider two cases.

(i) If $b \in H$, then since $\mathcal{W}$ is locally finite in X_δ, there is an X_δ-neighborhood G of b meeting only finitely many members of $\mathcal{W}$. Because H_δ is disjoint from each member of $\mathcal{J}$ and each member of each $\mathcal{G}(y)$ for each $y \in S$, it follows that $G \cap H_\delta$ is an X_δ-neighborhood G of b meeting only finitely many members of $\mathcal{B}$.

(ii) If $b \notin H$, then there is an $x(b) \in S$ such that $b \in C_{x(b)}$. Since $\{C_x : x \in S\}$ is X-locally finite (and hence point finite), there is a finite subset $F(b)$ of S such that $b \in C_x$ if and only if $x \in F(b)$. If $L = \bigcap\{C_x : x \in F(b)\}$, then L_δ is a neighborhood of b that meets no members of $\mathcal{W}$ and only finitely many members of $\mathcal{J}$.

For each $x \in F(b)$, there is a neighborhood $K(x)$ of b in X_δ that meets only finitely many members of $\mathcal{G}(x)$. Thus $\bigcap\{K(x) : x \in F(b)\}$ is an X_δ-neighborhood of b that meets only finitely many members of $\bigcup\{\mathcal{G}(x) : x \in F(b)\}$, finitely many members of $\mathcal{J}$, and no members of $\mathcal{W}$ — and thus of $\mathcal{B}$.

Hence our "minimal" counterexample is paracompact, so Theorem 4.2 holds.

□

Next, we turn to the question of when the product of an SP-scattered paracompact space and a paracompact space is paracompact. Our first partial answer follows.

4.5 Theorem. *If X is a paracompact P-space and Y is a Lindelöf space, then $X \times Y$ is paracompact.*

Informal conversation with K. Alster revealed that he was aware of 4.5, although it seems not to appear in the literature.

To prove this, we begin with a lemma that is well-known.

4.6 Lemma. *Every open cover $\mathcal{C}$ of a Lindelöf space Y has a countable locally finite open refinement.*

Proof: Since Lindelöf spaces are paracompact, $\mathcal{C}$ has an open locally finite refinement, which must be countable since Y is Lindelöf by 5.1.24 of [E89]. □

Some notational conventions follow. The projection map of the product $X \times Y$ onto X (resp. Y) is denoted by p_X (resp. p_Y). It is well-known that these mappings are continuous open surjections. The product $U \times V$ of an open subset U of X and an open subset V of Y is called a *rectangular open subset of* $X \times Y$. It is well-known and easily seen that $X \times Y$ is paracompact if and only of each open cover of $X \times Y$ by rectangular open sets has a locally finite open refinement.

4.7 Lemma. *If $\mathcal{S}$ is a cover of $X \times Y$ by rectangular open sets and $x \in X$, then there is an open neighborhood $W(x)$ of x and a countable locally finite open cover $\{D(x,j) : j \in \omega\}$ of Y such that for each $j \in \omega$, there is a $T(x,j) \in \mathcal{S}$ that satisfies $W(x) \times D(x,j) \subset T(x,j)$.*

Proof: Since $\mathcal{S}$ covers $X \times Y$, for each $y \in Y$, there is an $S(x,y) \in \mathcal{S}$ containing (x,y). It follows that $\{p_Y[S(x,y)] : y \in Y\}$ is an open cover $\mathcal{C}(x)$ of Y. By Lemma 4.6, since Y is a Lindelöf space, $\mathcal{C}(x)$ has a countable locally finite open refinement $\{D(x,j) : j \in \omega\}$. Thus $\bigcup_{j\in\omega} D(x,j) = Y$.

For each $j \in \omega$ there is a $y_j \in Y$ such that $S(x,y_j) \in \mathcal{S}$ and

$$D(x,j) \subset p_Y[S(x,y_j)]. \tag{1}$$

Now $(x,y_j) \in S(x,y_j)$ for each $j \in \omega$, and so x is in the open set $p_X[S(x,y_j)]$. Since X is a P-space, $W(x) = \bigcap_{j\in\omega}\{p_X[S(x,y_j)]\}$ is an open neighborhood of x and

$$W(x) \subset p_X[S(x,y_j)] \quad \text{for each} \quad j \in \omega. \tag{2}$$

Combining (1) and (2), we see that for each $j \in \omega$,

$$W(x) \times D(x,j) \subset p_X[S(x,y_j)] \times p_Y[S(x,y_j)] = S(x,y_j).$$

Hence $S(x,y_j)$ is the element $T(x,j)$ of $\mathcal{S}$ that is required. □

The notation in Lemma 4.7 will be used in the next two lemmas and the proof of Theorem 4.5.

4.8 Lemma. *There is a locally finite open cover $\mathcal{H} = \{H(x) : x \in X\}$ for which $x \in H(x) \subset W(x)$ for all $x \in X$.*

Proof: Since X is paracompact, the open cover $\{W(x) : x \in X\}$ has a locally finite open refinement $\mathcal{B}$. For each $x \in X$, choose $B_x \in \mathcal{B}$ such that $x \in B_x$. Let $H(x) = B_x \cap W(x)$. Then $x \in H(x) \subset W(x)$. If $\mathcal{H}$ =$\{H(x) : x \in X\}$, then $\mathcal{H}$ is an open cover of X. To see that $\mathcal{H}$ is locally finite, note first that any $a \in X$ has an X-neighborhood M that meets only finitely many members of $\mathcal{B}$ and hence only finitely many members of $\mathcal{H}$ since $H(x) \subset B_x$ for each $x \in X$. □

4.9 Lemma. *If $\mathcal{G} = \{H(x) \times D(x, j) : x \in X$ and $j \in \omega\}$, then $\mathcal{G}$ is a locally finite open refinement of $\mathcal{S}$.*

Proof: Clearly each member of $\mathcal{G}$ is open in $X \times Y$. If $(x, y) \in X \times Y$, then since $\{D(x, j) : j \in \omega\}$ covers Y, there is an $j_y \in \omega$ such that $y \in D(x, j_y)$. Clearly $(x, y) \in H(x) \times D(x, j_y) \in \mathcal{G}$, so $\mathcal{G}$ covers $X \times Y$.

It will be shown next that $\mathcal{G}$ is locally finite. Suppose that $(a, b) \in X \times Y$. Because $\mathcal{H}$ is locally finite, a has an X-neighborhood L such that $\{x \in X : L \cap H(x) \neq \emptyset\}$ is a finite subset F of X.

If $x \in F$, then b has a Y-neighborhood $J(x)$ such that $\{j \in \omega : D(x, j) \cap J(x) \neq \emptyset\}$ is a finite subset $A(x)$ of ω because each $\{D(x, j) : j \in \omega\}$ is locally finite. If $J = \bigcap\{J(x) : x \in F\}$, then J is a Y-neighborhood of b, so $L \times J$ is an $X \times Y$-neighborhood of (a, b).

Suppose next that $L \times J$ meets $H(z) \times D(z, j)$ for some $z \in X$ and $j \in \omega$. Then $L \cap H(z) \neq \emptyset$, so $z \in F$. Also $J \cap D(z, j) \neq \emptyset$, so since $z \in F$, it follows that $J(z) \cap D(z, j) \neq \emptyset$. Thus $j \in A(z)$. We may conclude that $H(z) \times D(z, j)$ is a member of the finite subfamily $\{H(x) \times D(x, j) : x \in F$ and $j \in \bigcup[A(x) : x \in F]\}$ of $\mathcal{G}$. Thus $L \times J$ is a neighborhood of (a, b) that meets only finitely many members of $\mathcal{G}$.

We will show finally that $\mathcal{G}$ refines $\mathcal{S}$. For any $x \in X$ and $j \in \omega$, the members of $\mathcal{G}$ have the form $H(x) \times D(x, j)$. By Lemma 4.8, $H(x) \subset W(x)$, and by Lemma 4.7, there is a $T(x, j)$ such that

$$W(x) \times D(x, j) \subset T(x, j).$$

Thus $H(x) \times D(x, j) \subset T(x, j) \in \mathcal{S}$, so $\mathcal{G}$ refines $\mathcal{S}$. □

Proof of Theorem 4.5: The lemmas above show that every cover of $X \times Y$ by rectangular open sets has a locally finite open refinement, so we may conclude that $X \times Y$ is paracompact. □

5. RG-spaces and spaces with finite z-dimension

In this section, the relationship between SP-scattered spaces and two other classes of spaces is studied.

5.1 Definitions. If $f \in C(X)$, let $f^*(x) = \frac{1}{f(x)}$ if $x \in \mathrm{coz}(f)$, and let $f^*(x) = 0$ if $x \in Z(f)$, and let $G(X)$ denote the set of finite sums of functions fg^* from X into R. It is not difficult to see that $G(X)$ is a subalgebra of $C(X_\delta)$. If $G(X) = C(X_\delta)$, then X is said to be an *RG-space*.

The class of RG-spaces includes all Lindelöf scattered spaces of finite Cantor-Bendixon index. See 2.12 of [HRW02].

If $f \in G(X)$, then we let $\mathrm{rg}(f) = \min\{n < \omega :$ there are f_i, g_i in $C(X)$ for $1 \leq i \leq n$ such that $f = \sum_{i=1}^n f_i, g_i^*\}$, and we let $\mathrm{rg}(X) = \sup\{\mathrm{rg}(f) : f \in C(X)\}$ if such a finite ordinal exists, or ∞ otherwise.

Next we generalize a portion of 2.11 and 2.12 of [HRW02]. Note that in that paper, the authors use D_α in place of our S_α.

5.2 Theorem. *If X is a Lindelöf SP-scattered space and $S_1(X) = X \setminus \mathrm{int}_X\, P(X)$ is an RG-space, then X is an RG-space.*

Proof: Suppose $s \in C(X_\delta)$. The second and third paragraphs of the proof of 2.11(a) in [HRW02] establish that $S_1(X) \subset Z(r)$ for some $r \in C(X_\delta)$. Because X is a Lindelöf SP-scattered space, X_δ is a Lindelöf P-space by Theorem 3.2. So, for all $x \in \mathrm{coz}(r)$, there is an $E(x) \in \mathcal{Z}(X)$ contained in $\mathrm{coz}(r)$. Using the fact that $\mathrm{coz}(r)$ is Lindelöf, it follows that there is a sequence $\{x_i\}_{i<\omega}$ of elements of $\mathrm{coz}(r)$ such that $\mathrm{coz}(r) = \bigcup_{i<\omega} E(x_i)$. Because each $E(x_i)$ is open in the P-space $\mathrm{int}_X\, P(X)$, it follows that each $E(x_i)$ is clopen in X.

Let $A_1 = E(x_1)$, and if $n > 1$, let $E(x_n) = E(x_n) \setminus \bigcup_{i=1}^{n-1} A_i$. Then $\{A_i\}_{i<\omega}$ partitions $\mathrm{coz}(r)$ into countably many clopen nonempty sets. Because $Z(r)$ is clopen in X, there is an $m \in C(X)$ whose cozero set is $\mathrm{coz}(r)$. Armed with this information, we may apply the argument beginning on line 5 of page 9 in [HRW02] to conclude that $s \in G(X)$ and hence that X is an RG-space. □

5.3 Theorem. *If X is a Lindelöf SP-scattered space with finite sp-index, then X is an RG-space and $\mathrm{rg}(X)$ is finite.*

Proof: Apply Theorem 5.2 and the proof of 2.11(b) in [HRW02]. □

In [MZ05], a space X is said to have *finite z-dimension* if there is a positive integer k such that no chain of prime z-ideals in the ring $C(X)$ has length exceeding k. (A precise definition of z-dimension is in this paper.) In Sections 4 and 5 these authors show that a compact space has finite z-dimension if and only if it is scattered and has finite CB-index. In [HRW03], it is shown that if X is compact or metrizable, then X is an RG-space with $\mathrm{rg}(X)$ finite if and only if X is scattered with finite CB-index. Hence we have:

5.4 Theorem. *A compact space is an RG-space if and only it has finite z-dimension.*

Martinez and Zenk note that no space containing a copy of the space of countable ordinals or of $\beta\omega$ has finite z-dimension. They note also that the finiteness requirement in Theorem 5.4 may not be omitted.

5.5 Theorem. *Suppose X is a locally compact metrizable SP-scattered space with finite CB-index, then X is a free union of locally compact metrizable spaces with finite z-dimension.*

Proof: Because X is locally compact and paracompact space, it is a free union of locally compact σ-compact (and hence Lindelöf) RG-spaces $\{X_i\}_{i\in I}$, and there is a positive integer k such that $\mathrm{rg}(X_i) \leq k$ for all $i \in I$. The conclusion follows from Section 3 of [HRW03], 4.7 of [MZ05], and Theorem 5.4 above. □

6. Questions and some answers

6.1 Question. *Must the product of a paracompact SP-scattered space and a paracompact space be paracompact?*

Answer

A negative answer to this question appears in Example 1 of [A06] where K. Alster shows that there is a Lindelöf (and hence paracompact) P-space X of weight ω_1 and a paracompact space Y such that $X \times Y$ is not normal.

6.2 Questions. (a) *Must the product of a paracompact scattered space and a paracompact space be paracompact?*

(b) *What can be said about countable products of scattered paracompact spaces?*

Answer

(a) A positive answer to this question may be deduced from results in [T71] as follows. Near the top of p. 60, the class of spaces whose product with every paracompact space is paracompact is denoted by Π. Near the bottom of this page, the class of C-scattered spaces is defined and it is noted that scattered spaces are C-scattered. On p. 68, the class of Π^* is introduced and it is observed that Π^* is contained in Π, and in Theorem 23, it is shown that every C-scattered paracompact space is in Π^*. Combining these results yields a positive answer to (a). The author shows also that no P-space without isolated points can be C-scattered.

(b) Also, it is shown in [RW83] that a countable product of scattered paracompact spaces is paracompact.

6.3 Question. *Must a product of paracompact P-spaces be paracompact?*

Answer

A negative answer to this question follows immediately from an example of a paracompact P-space X such that $X \times X$ is not normal whose existence follows from Theorem 6 in [AE72]. See also [A06].

6.4 Question. *Must the product of an SP-scattered paracompact space and a Lindelöf space be paracompact?*

Answer

Unknown except that by Theorem 4.5, the answer is yes if the first factor is a paracompact P-space.

6.5 Conjecture. *If X is a Lindelöf space, then the following are equivalent:*

(a) *X is an RG-space;*
(b) *X is SP-scattered and has finite SP-index;*
(c) *X is an RG-space and* $\mathrm{rg}(X)$ *is finite.*

6.6 Question. *If X is metacompact and scattered (or SP-scattered), must X_δ be scattered (or SP-scattered)?*

6.7 Question. *If X is a metacompact P-space and Y is Lindelöf, must $X \times Y$ be metacompact?*

Added in proof. The absolute $E(X)$ (see [PW88, Chapter 6]) of a P-space X without isolated points and of non-measurable cardinality witnesses the fact that the perfect irreducible continuous preimage of an SP-scattered space X need not be SP-scattered. This contrasts with Theorem 2.17.

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