Markku Niemenmaa
On the structure of finite loop capable Abelian groups

Commentationes Mathematicae Universitatis Carolinae, Vol. 48 (2007), No. 2, 217--224

On the structure of finite loop capable Abelian groups

Markku Niemenmaa

Abstract. Loop capable groups are groups which are isomorphic to inner mapping groups of loops. In this paper we show that abelian groups $C_p^k \times C_p \times C_p$, where $k \geq 2$ and p is an odd prime, are not loop capable groups. We also discuss generalizations of this result.

Keywords: loop, group, connected transversals

Classification: 20D10, 20N05

1. Introduction

If Q is a loop (a quasigroup with a neutral element e), then we can define two permutations L_a and R_a on Q by $L_a(x) = ax$ and $R_a(x) = xa$ for each $a \in Q$. We write $M(Q) = \langle L_a, R_a : a \in Q \rangle$ and say that $M(Q)$ is the multiplication group of Q. The stabilizer of the neutral element e is denoted by $I(Q)$ and we say that $I(Q)$ is the inner mapping group of Q. These two notions which link loop theory to group theory were introduced by Bruck [1] in 1946 and he was the first to investigate the structure of loops by using group theory.

If Q is a group, then $I(Q)$ is the group of all inner automorphisms of Q. Groups, which are isomorphic to inner automorphism groups of groups, are called capable groups. In this paper groups, which are isomorphic to inner mapping groups of finite loops, are called loop capable groups and we investigate the following problem: *Which finite abelian groups are loop capable groups*? Previous investigations [6] have shown that $I(Q)$ is cyclic if and only if Q is an abelian group. In [4], [7] it was shown that $I(Q)$ cannot be isomorphic to $C_n \times D$, where C_n is a cyclic group of order n and D is a finite abelian group such that $\gcd(n, |D|) = 1$. The result from [9, Corollary 4.1] says that the inner mapping group $I(Q)$ cannot be isomorphic to $C_{p^k} \times C_{p^l}$, where p is an odd prime number and $k > l \geq 0$. We now continue the tradition of proving results in the negative and the purpose of this paper is to show that the following results hold:

1) The direct product $C_{p^k} \times C_p \times C_p$, where p is an odd prime number and $k \geq 2$, is not a loop capable group.

2) Let p be an odd prime number and $k \geq 2$. Then the direct product $(C_{p^k} \times C_p) \times D$, where D is an abelian group whose order is not divisible by p, is not a loop capable group.

3) Let $p < q$ be two odd prime numbers and $k \geq 2$. Then the direct product $(C_{p^k} \times C_p \times C_p) \times D$, where $D > 1$ is an abelian q-group, is not a loop capable group.

When loops are studied by using their multiplication groups, one of the major tools is the notion of connected transversals (introduced by Kepka and Niemenmaa [5] in 1990). This paper also uses this notion and related results and therefore our Section 2 contains basic information about connected transversals and their role in this investigation. Sections 3 and 4 contain our main results in group theoretic terms and finally, in Section 5, we give the loop theoretical interpretation of our results.

2. From loops to groups and connected transversals

Let Q be a loop and consider the groups $M(Q)$ and $I(Q)$ and the left and right translations L_a and R_a defined in the introduction. If we write $A = \{L_a : a \in Q\}$ and $B = \{R_a : a \in Q\}$, then the commutator subgroup $[A, B] \leq I(Q)$ and A and B are left transversals to $I(Q)$ in $M(Q)$. If $1 < K \leq I(Q)$, then K is not a normal subgroup of $M(Q)$. Finally, $M(Q) = \langle A, B\rangle$.

We then consider the corresponding situation in groups in general. Let H be a subgroup of G and let A and B be two left transversals to H in G. We say that A and B are H-connected if the commutator subgroup $[A, B] \leq H$. By H_G we denote the core of H in G, i.e. the largest normal subgroup of G contained in H. If $H_G = 1$, we say that H is core-free in G. The relation between multiplication groups of loops and connected transversals is given by

Theorem 2.1. *A group G is isomorphic to the multiplication group of a loop if and only if there exist a subgroup H satisfying $H_G = 1$ and H-connected transversals A and B such that $G = \langle A, B\rangle$.*

For the proof, see [5, Theorem 4.1].

In the following lemmas we assume that A and B are H-connected transversals in G.

Lemma 2.1. *If $C \subseteq A \cup B$ and $K = \langle H, C\rangle$, then $C \subseteq K_G$.*

Lemma 2.2. *If $H_G = 1$, then $N_G(H) = H \times Z(G)$.*

Lemma 2.3. *If $H_G = 1$, then $Z(G) \subseteq A \cap B$.*

Lemma 2.4. *If H is a finite abelian group, then G is a solvable group.*

For the proofs, see [5, Lemma 2.5 and Proposition 2.7], [6, Lemma 1.4] and [10, Theorem 4.1].

In the following lemmas we assume in addition that $G = \langle A, B\rangle$. As usual, p denotes a prime number.

Lemma 2.5. *If H is a cyclic subgroup of G, then $G' \leq H$.*

Lemma 2.6. *If $H \cong C_p \times C_p$, then $G' \leq N_G(H)$.*

Lemma 2.7. *If G is a finite group and $H \cong C_p \times C_p \times C_p$, then $G' \leq N_G(H)$.*

Lemma 2.8. *If G is a finite group and $H \cong C_n \times D$, where $n > 1$, D is abelian and $\gcd(n, |D|) = 1$, then $H_G > 1$.*

Lemma 2.9. *If G is a finite group and H is abelian, then H is subnormal in G.*

For the proofs, see [6, Theorem 2.2], [10, Lemma 4.2], [2, Theorem 3.7], [7, Theorem 2.3] and [10, Proposition 6.3].

The following two results deal with the core of H in G.

Lemma 2.10. *Let $H \cong C_{p^k} \times C_p$, where p is an odd prime number and $k \geq 2$. Then H_G is not trivial.*

Lemma 2.11. *Let H be abelian and $H_G = 1$. Then the core of $HZ(G)$ in G contains $Z(G)$ as a proper subgroup.*

For the proofs, see [8, Theorem 3.1] and [9, Lemma 2.7].

The following well known result on commutator calculus is needed later.

Lemma 2.12. *If $[a, b]$ commutes with a and b, then $(ab)^n = a^n b^n [b, a]^{\binom{n}{2}}$.*

For the proof, see [3, pp. 253–254].

3. Main theorems

Throughout this section we assume that G is a finite group, H is an abelian subgroup of G with a special structure and there exist H-connected transversals A and B such that $G = \langle A, B \rangle$.

Theorem 3.1. *Let $H \cong C_{p^k} \times C_p \times C_p$ where p is an odd prime number and $k \geq 2$. Then H_G is not trivial.*

Proof: Let G be a minimal counterexample. Thus $H_G = 1$ and we conclude from Lemmas 2.2 and 2.9 that $Z(G) > 1$. Let $z \in Z(G)$ be an element of prime order q and consider the factor groups $G/\langle z \rangle$ and $H\langle z \rangle/\langle z \rangle$. As G is a minimal counterexample, it follows that there exists a normal subgroup K of G such that $\langle z \rangle < K \leq H\langle z \rangle$. Furthermore, K is the largest normal subgroup of G contained in $H\langle z \rangle$. Clearly, $K \leq H\langle z \rangle$ is an abelian group. If $q \neq p$, then K has a normal Sylow p-subgroup $P \leq H$ and as P is normal in G, it follows that $H_G > 1$. Thus we may assume that $q = p$ and it is also clear that $Z(G)$ is a p-group. The Frattini subgroup of K is normal in G, hence K has to be an elementary abelian p-group. Thus $K = H_1 \times \langle z \rangle$, where $H_1 = H \cap K$.

Then assume that $k \geq 3$ and consider the factor groups G/K and HK/K. By Lemmas 2.5 and 2.10 or by the fact that G is a minimal counterexample, we may conclude that there exists a normal subgroup N of G such that $K < N \leq HK =$

$H\langle z\rangle$, a contradiction. Thus we may assume that $k=2$. It is also clear that the order of H_1 is p or p^2 and HK/K is elementary abelian of order p^2 or p^3.

From Lemmas 2.6 and 2.7 it follows that $(G/K)' \le N_{G/K}(HK/K)$ and thus $G' \le N_G(HK)$, which means that $N_G(HK)$ is normal in G. By Lemma 2.2 we may conclude that $N_{G/K}(HK/K) = HK/K \times Z(G/K)$.

We write $M/K = Z(G/K)$ and then $N_G(HK) = HM$, where $H \cap M = H_1$ and M is a normal subgroup of G.

Now $C_G(H_1) = C_G(K)$ is normal in $N_G(K) = G$. In what follows we consider the subgroup $T = N_G(HK) \cap C_G(H_1) = HM \cap C_G(H_1) = H(M \cap C_G(H_1))$. Naturally, T is a normal subgroup of G and $K = H_1\langle z\rangle \le Z(T)$. We also see that $T' \le N_G(HK)' \le K$, which means that $T' \le Z(T)$.

Now denote $D = \langle g \in T : g^p = 1\rangle$. Clearly, D is a characteristic subgroup of T, hence D is normal in G. If t and d are elements of order p from T, then by Lemma 2.12 $(td)^p = t^p d^p [d,t]^{\binom{p}{2}} = 1$, since p is odd. Thus $D = \{g \in T : g^p = 1\}$. We then consider the factor groups G/D and HD/D. As HD/D is cyclic, it follows by Lemma 2.5 that HD is normal in G. If $h \in H$ and $d \in D$, then $(hd)^p = h^p d^p [d,h]^{\binom{p}{2}} = h^p$, hence $(HD)^p$ is a nontrivial subgroup of H. As $(HD)^p$ is characteristic in HD, we conclude that $(HD)^p$ is normal in G and thus $H_G > 1$, a contradiction. □

4. Generalized results

We again assume that G is a finite group, $H \le G$ is abelian and there exist H-connected transversals A and B such that $G = \langle A, B\rangle$.

The result of Lemma 2.10 says that $C_{p^k} \times C_p$ (p odd, $k \ge 2$) is not a loop capable group. The following theorem helps us to prove a more general result.

Theorem 4.1. *Let p be an odd prime number. If $H = C \times D$, where $C \cong C_{p^k} \times C_p$ ($k \ge 2$) and D is a finite abelian group, whose order is not divisible by p, then H_G is not trivial.*

Proof: Our proof is by induction on $|G|$. Thus we assume that G satisfies the conditions of the theorem but $H_G = 1$. As before, it follows that $N_G(H) = H \times Z(G)$ and $Z(G) > 1$. Let $z \in Z(G)$ be an element of prime order. Now the core of $H\langle z\rangle/\langle z\rangle$ in $G/\langle z\rangle$ is not trivial, hence the core of $H\langle z\rangle$ in G is $H_1\langle z\rangle$, where $1 < H_1 \le H$. By using Lemmas 2.5 and 2.10, induction and the fact that $H_G = 1$ we conclude that $|H_1| = p$ and it is also clear that $k = 2$ and $|z| = p$. Now we write $K = H_1\langle z\rangle$. Thus $HK/K \cong (C_p \times C_p) \times D$. From Lemma 2.11 it follows that the core of $HK/K \times Z(G/K)$ in G/K is $LK/K \times Z(G/K)$, where $H_1 < L \le H$. If p divides $|LK/K|$, then we use Lemma 2.8 and conclude that $L \cong (C_{p^2} \times C_p) \times E$ where $1 \le E \le D$ (thus $L = C \times E$). If p does not divide $|LK/K|$, then $L = EH_1$, where $1 < E \le D$. We now divide the proof into two parts according to the two different choices for L.

1) Let $L \cong (C_{p^2} \times C_p) \times E$ $(1 \leq E \leq D)$. We write $Z(G/K) = M/K$ and then $W = LM = CEM$ is normal in G. As K is normal in G, we conclude that $C_G(H_1)$ is normal in G. Then $F = W \cap C_G(H_1) = CE(M \cap C_G(H_1))$ is normal in G. Let $P \geq CK$ be a Sylow p-subgroup of F. As $F' \leq K$, it follows that P is normal in F, hence P is normal in G. Clearly, $Z(P) \geq K \geq F' \geq P'$ and we can proceed as in the proof of Theorem 3.1. Thus $T = \langle g \in P \mid g^p = 1 \rangle = \{g \in P \mid g^p = 1\}$ and naturally T is normal in G. Now $HT/T \cong C_p \times D$ and from Lemma 2.8 we conclude that $U = CD_1T$(where $D_1 \leq D$) is normal in G. Since CT is the unique Sylow p-subgroup of U, we may conclude that CT is normal in G. Now we again proceed as in the proof of Theorem 3.1 and we see that $C \geq (CT)^p > 1$, hence $H_G > 1$.

2) Assume that $L = EH_1$, where $1 < E \leq D$. Thus $EK/K \times Z(G/K)$ is normal in G. By denoting $Z(G/K) = M/K$, we see that $W = EM$ is normal in G. As $C_G(H_1)$ is normal in G, it follows that $F = W \cap C_G(H_1) = E(M \cap C_G(H_1))$ is normal in G. We denote $M \cap C_G(H_1)$ by R (then $F = ER$ and R is normal in G). From Lemma 2.3 it follows that $R = A_1H_1 = B_1H_1$, where $A_1 \subseteq A$ and $B_1 \subseteq B$. Since H_1 is normal in R, it follows that $R' \leq H_1$. As R is normal in G and $H_G = 1$, we conclude that R is an abelian group.

If Q is a Hall-subgroup (for those prime numbers that divide the order of D) of R then Q is normal in G. Consider the group G/Q and the subgroup HQ/Q. Clearly, the core of HQ/Q in G/Q is not trivial and thus the core of HQ in G is H_2E_1Q where H_2 is a nontrivial p-group and $E_1 \leq D$. If $H_1 \leq H_2$, then $H_2E_1Q \cap K = H_1$ is normal in G and $H_G > 1$, a contradiction. If H_1 is not a subgroup of H_2, then $S = H_2E_1QK$ is normal in G and p^2 divides $|S \cap C|$. By considering the group HS/S and by using Lemma 2.8, it follows that we have a normal subgroup of the form CE_2V (where $E_2 \leq D$ and $V \leq F$) in G. But then we can proceed as in the first part of the proof and conclude that $H_G > 1$, a contradiction.

It follows that $Q = 1$ and we may assume that E is a Hall-subgroup of ER (again for those primes that divide the order of D). Now EK is a normal subgroup of ER and as $K \leq Z(ER)$, it follows that E is a normal Hall-subgroup of ER. But then E is normal in G and again $H_G > 1$, a contradiction. □

We shall also generalize the result of Theorem 3.1 and for this purpose we need the following two lemmas.

Lemma 4.2. *Let $p \neq q$ be two prime numbers and let $H = P \times D$, where $P \cong C_p \times C_p$, $D > 1$ is an abelian q-group and q does not divide $|Z(G)|$. Then $H_G > 1$.*

Proof: Assume $H_G = 1$. By Lemmas 2.2 and 2.9, $N_G(H) = H \times Z(G)$ and $Z(G) > 1$. By Lemma 2.11, the core of $HZ(G)$ in G contains $Z(G)$ as a proper subgroup. From our assumptions and from Lemma 2.8 it follows that the core of $HZ(G)$ in G is $PZ(G)$. Then let $Q \geq D$ be a Sylow q-subgroup of G and write $F = PZ(G)N_G(Q)$. If $M \geq F$ is a maximal subgroup of G, then $M_G > 1$ by Lemma 2.1. From Lemma 2.9 it follows that $HM_G/M_G = M/M_G$ is normal in G/M_G, hence M is normal in G. By Frattini-lemma, $G = MN_G(Q) = M$, a contradiction. Thus $G = F = PZ(G)N_G(Q)$. But then $D^G = \langle D^g \mid g \in G\rangle \leq Q$ and D^G and $PZ(G) \times D^G$ are normal subgroups of G. Since $H_G = 1$, we know that $D < D^G$ and as D^G is a q-group, we know that $W = N_{D^G}(D) > D$. But then $N_G(H) \geq PZ(G)W > HZ(G)$, a contradiction. We conclude that $H_G > 1$. □

Lemma 4.3. *Let $p < q$ be two prime numbers and let $H = P \times D$, where $P \cong C_p \times C_p \times C_p$, $D > 1$ is an abelian q-group and q does not divide $|Z(G)|$. Then $H_G > 1$.*

Proof: Assume $H_G = 1$. As before $N_G(H) = H \times Z(G)$ and $Z(G) > 1$. From Lemmas 2.8 and 2.11 it follows that the core of $HZ(G)$ in G is $P_1Z(G)$, where either $P_1 \cong C_p$ or $P_1 \cong C_p \times C_p \times C_p$ (and then $P_1 = P$).

First assume that $P_1 \cong C_p$. Now $HZ(G)/P_1Z(G) \cong (C_p \times C_p) \times D$. If $Z(G/P_1Z(G))$ has a Sylow q-subgroup $QP_1Z(G)/P_1Z(G)$, then $K = QP_1Z(G)$ is normal in G. Now $[K : N_K(Q)]$ divides p and as $q > p$, it follows that Q is normal in K, hence Q is normal in G. But then $[G, Q] \leq P_1Z(G) \cap Q = 1$, hence $Q \leq Z(G)$. It follows that $Q = 1$ and thus q does not divide $|Z(G/P_1Z(G))|$. By Lemma 4.2, the core of $HZ(G)$ properly contains $P_1Z(G)$.

Thus we may assume that $P_1 = P$. Now we can proceed as in the proof of Lemma 4.2 : $G = PZ(G)N_G(Q)$, where $Q \geq D$ is a Sylow q-subgroup of G, D^G is a subgroup of Q and as in the proof of Lemma 4.2 we get a contradiction. □

Now we are ready to generalize the result of Theorem 3.1.

Theorem 4.4. *Let $p < q$ be two odd prime numbers and let $H = C \times D$, where $C \cong C_{p^k} \times C_p \times C_p$ ($k \geq 2$) and $D > 1$ is an abelian q-group. Then $H_G > 1$.*

Proof: Assume again that G is a minimal counterexample and $H_G = 1$. As before, $Z(G) > 1$ and we can consider $z \in Z(G)$ such that z has prime order. It follows that the core of $H\langle z\rangle$ in G is $K = H_1\langle z\rangle$, where either $H_1 \cong C_p$ or $H_1 \cong C_p \times C_p$ or $H_1 = C$. It is also clear that $k = 2$, $|z| = p$ and $Z(G)$ is a p-group.

If $H_1 \cong C_p$, then $HK/K \cong (C_p \times C_p \times C_p) \times D$. If $Z(G/K)$ has a Sylow q-subgroup QK/K, then QK is normal in G. Since $[QK : N_{QK}(Q)]$ divides p and $q > p$, it follows that Q is normal in QK, hence Q is normal in G. But then $[Q, G] \leq K \cap Q = 1$ and $Q \leq Z(G)$. Thus $Q = 1$ and we conclude that $Z(G/K)$ is not divisible by q. By Lemma 4.3, the core of HK/K in G/K is not trivial.

Then assume that $H_1 \cong C_p \times C_p$ and $HK/K \cong (C_p \times C_p) \times D$. If $Z(G/K)$ has a Sylow q-subgroup QK/K, then QK is normal in G. Now $t = [QK : N_{QK}(Q)]$ divides p^2. Clearly, $t = p$ is not possible as $p < q$. If $t = p^2$, then q divides

$p^2 - 1$, hence $q = p + 1$, a contradiction. It follows that $t = 1$ and Q is normal in G. Continue as in the previous part of the proof and it follows that $Q \leq Z(G)$. Thus q does not divide $|Z(G/K)|$ and from Lemma 4.2 it follows that the core of HK/K in G/K is not trivial. Thus we may assume that $H_1 = C$. But then $K^p = C^p < H$ is a nontrivial subgroup of G and $H_G > 1$. $\square$

5. Loop theoretical results

In Theorems 3.1, 4.1 and 4.4 we have proved results which are of purely group theoretical nature: these results tell us how the structure of an abelian subgroup with connected transversals determines the subgroup to have a nontrivial core. When these results are combined with Theorem 2.1, we immediately have the following results in loop theory.

Corollary 5.1. *Let p be an odd prime number and $k \geq 2$. Then the group $C_{p^k} \times C_p \times C_p$ is not a loop capable group (i.e., the inner mapping group $I(Q)$ of a finite loop Q cannot be isomorphic to $C_{p^k} \times C_p \times C_p$).*

Corollary 5.2. *Let p be an odd prime number and $k \geq 2$. If $H = C \times D$, where $C \cong C_{p^k} \times C_p$ and D is a finite abelian group whose order is not divisible by p, then H is not a loop capable group.*

Corollary 5.3. *Let $p < q$ be two odd prime numbers and $k \geq 2$. If $H = C \times D$, where $C \cong C_{p^k} \times C_p \times C_p$ and D is a finite abelian q-group, then H is not a loop capable group.*

Final remarks. If we look at the assumptions in our theorems, it is quite natural to ask whether these results hold when $p = 2$. It is also an interesting open problem to see if Corollary 5.3 could be proved in the following more general form: Let p be a prime number and $k \geq 2$. If $H = C \times D$, where $C \cong C_{p^k} \times C_p \times C_p$ and D is a finite abelian group such that p does not divide $|D|$, then H is not a loop capable group.

References

[1] Bruck R.H., *Contributions to the theory of loops*, Trans. Amer. Math. Soc. **60** (1946), 245–354.

[2] Csörgö P., *On connected transversals to abelian subgroups and loop theoretical consequences*, Arch. Math. **86** (2006), 499–516.

[3] Huppert B., *Endliche Gruppen I*, Springer, Berlin-Heidelberg-New York, 1967.

[4] Kepka T., *On the abelian inner permutation groups of loops*, Comm. Algebra **26** (1998), 857–861.

[5] Kepka T., Niemenmaa M., *On multiplication groups of loops*, J. Algebra **135** (1990), 112–122.

[6] Kepka T., Niemenmaa M., *On loops with cyclic inner mapping groups*, Arch. Math. **60** (1993), 233–236.

[7] Niemenmaa M., *On the structure of the inner mapping groups of loops*, Comm. Algebra **24** (1996), 135–142.

[8] Niemenmaa M., *On finite loops whose inner mapping groups are abelian*, Bull. Austral. Math. Soc. **65** (2002), 477–484.
[9] Niemenmaa M., *On finite loops whose inner mapping groups are abelian II*, Bull. Austral. Math. Soc. **71** (2005), 487–492.
[10] Niemenmaa M., Kepka T., *On connected transversals to abelian subgroups*, Bull. Austral. Math. Soc. **49** (1994), 121–128.

Department of Mathematical Sciences, University of Oulu, PL 3000, 90014 Oulu, Finland

(*Received* July 26, 2006, *revised* January 12, 2007)