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Intersections of minimal prime ideals in the rings of continuous functions

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Abstract. A space X is called μ-compact by M. Mandelker if the intersection of all free maximal ideals of $C(X)$ coincides with the ring $C_K(X)$ of all functions in $C(X)$ with compact support. In this paper we introduce ϕ-compact and ϕ'-compact spaces and we show that a space is μ-compact if and only if it is both ϕ-compact and ϕ'-compact. We also establish that every space X admits a ϕ-compactification and a ϕ'-compactification. Examples and counterexamples are given.

Keywords: minimal prime ideal, P-space, F-space, μ-compact space, ϕ-compact space, ϕ'-compact space, round subset, almost round subset, nearly round subset

Classification: Primary 54C40; Secondary 46E25

1. Introduction

By a space we always mean a completely regular Hausdorff space. It is well-known that if X is realcompact, then the intersection of all free maximal ideals of $C(X)$ coincides with the ring $C_K(X)$ of all functions in $C(X)$ with compact support ([1, 8.19]). A space with the latter property is called μ-compact by M. Mandelker in 1971 ([5]). A subset A of βX is called round by M. Mandelker in 1969 if for any zero set Z of X, $\mathrm{cl}_{\beta X} Z$ is a neighbourhood of A whenever $\mathrm{cl}_{\beta X} Z \supseteq A$ ([4, 4]). In 1973, D.G. Johnson and M. Mandelker have shown that for any space X, there is a smallest μ-compact space μX lying between X and βX ([3, 4.1]). They have also proved that μX is the smallest subspace of βX containing X for which $\beta X - \mu X$ is round ([3, 4.3]). We define ϕ-compact spaces in terms of intersections of minimal prime ideals of $C(X)$. The class of all ϕ-compact spaces extends the class of all μ-compact spaces. We prove that for any space X, there is a smallest ϕ-compact space ϕX lying between X and βX. Mandelker's definition of round subsets of βX characterizes P-spaces. In fact, X is a P-space if and only if every subset of βX is round ([4, 5.6]). The question is what type of subsets of βX characterize F-spaces? We define almost round subsets of βX. It turns out that a space X is an F-space if and only if every subset of βX is almost round. We also establish that ϕX is the smallest subspace of βX containing X for which $\beta X - \phi X$ is almost round. Our motivation to define ϕ'-compact spaces is the theorem in which we show that a space is μ-compact if and only if it is both ϕ-compact and ϕ'-compact. We prove that for any space X, there is a smallest ϕ'-compact space $\phi' X$ lying between X and βX. We define nearly round subsets of βX and similar results as for round and almost round subsets are established. Finally we show that an F-space X is a P-space if and only if every subset of βX is nearly round.

2. Maximal, prime and minimal prime ideals

As usual, βX is the Stone-Čech compactification of X. There is a one-one correspondence between the points of βX and the maximal ideals of $C(X)$, described in the following theorem ([1, 7.3]).

Theorem 2.1 ([Gelfand-Kolmogoroff]). *The maximal ideals of $C(X)$ are given by $M^p = \{f \in C(X) : p \in \mathrm{cl}_{\beta X} Z(f)\}(p \in \beta X)$, here $Z(f) = \{x \in X : f(x) = 0\}$ is the zero-set of f.*

Also the set $O^p = \{f \in C(X) : \mathrm{cl}_{\beta X} Z(f)$ is a neighbourhood of $p\}$ is an ideal of $C(X)$, for each $p \in \beta X$.

An ideal I of $C(X)$ is called a z-ideal if $Z(f) = Z(g)$ and $f \in I$ implies $g \in I$.

It is clear that for each $p \in \beta X$, M^p and O^p are z-ideals of $C(X)$.

We now write down the following important theorem given in [1, 7.15].

Theorem 2.2. *Every prime ideal P of $C(X)$ contains O^p for a unique p and M^p is the unique maximal ideal that contains P.*

It is well-known that X is an F-space if and only if O^p is prime for each $p \in \beta X$ ([1, 14.25]), and X is a P-space if and only if $O^p = M^p$ for each $p \in \beta X$ ([1, 14.29]). Clearly every P-space is an F-space, the converse is not true. The space $\beta\mathbb{R}\backslash\mathbb{R}$ is a compact F-space ([1, 14.27]). It fails to be a P-space since every compact P-space is finite ([1, 4k, 2]).

Every z-ideal in $C(X)$ is an intersection of prime ideals ([1, 2.8]). Since O^p is a z-ideal we have the following theorem.

Theorem 2.3. *The ideal O^p is the intersection of all minimal prime ideals containing it.*

Let $\mathcal{P}_{\min}(X)$ denote the class of all minimal prime ideals of $C(X)$. We define the relation '$\sim$' on $\mathcal{P}_{\min}(X)$ by $P \sim Q$ if and only if P, Q are contained in a same maximal ideal. Obviously '$\sim$' is an equivalence relation on $\mathcal{P}_{\min}(X)$. All the minimal prime ideals of $C(X)$ contained in M^p (i.e. containing O^p) for some $p \in \beta X$ form an equivalence class which will be denoted by E_p. We state the following important characterization of minimal prime ideals of $C(X)$ which is an immediate consequence of [2, Lemma 1.1].

Theorem 2.4. *Let P be a prime ideal of $C(X)$. Then P is minimal if and only if for any $f \in P$, there exists $g \in C(X) - P$ such that $fg = 0$.*

Notations 2.5. Let $X \subseteq Y \subseteq \beta X$ and $p \in \beta X$. The ideal $\{f \in C(X) : \mathrm{cl}_{\beta X} Z(f)$ is a neighbourhood of $p\}$ of $C(X)$ will be denoted by O^p_X and the ideal $\{f \in C(Y) : \mathrm{cl}_{\beta X} Z(f)$ is a neighbourhood of $p\}$ of $C(Y)$ will be denoted by O^p_Y.

We note that every minimal prime ideal in $C(X)$ is a z-ideal ([1, 14.7]). Now we prove the following theorem.

Theorem 2.6. *Let $X \subseteq Y \subseteq \beta X$ and $p \in \beta X$. If P_Y is a minimal prime ideal of $C(Y)$ with $P_Y \supseteq O^p_Y$ and if $f \in P_Y$ then there exists a minimal prime ideal P_X of $C(X)$ with $P_X \supseteq O^p_X$ such that $f|_X \in P_X$. Also if P_X is a minimal prime ideal of $C(X)$ with $P_X \supseteq O^p_X$ and if $f \in P_X$ with $f^Y \in C(Y)$ then there exists a minimal prime ideal P_Y of $C(Y)$ with $P_Y \supseteq O^p_Y$ such that $f^Y \in P^Y$, here f^Y is the continuous extension of f over Y.*

Proof: Let $f \in P_Y$ where P_Y is a minimal prime ideal of $C(Y)$ with $P_Y \supseteq O^p_Y$. Then there exists $g \in C(Y)$ such that $fg = 0$ and $g \notin P_Y$ (Theorem 2.4). Clearly, $g \notin O^p_Y$. Let $g' = g|_X$. Then $Z(g') \subseteq Z(g)$ and hence $g' \notin O^p_X$. Let $f' = f|_X$. Clearly, $f'g' = 0$. Now $g' \notin O^p_X$ implies that there exists a minimal prime ideal P_X of $C(X)$ with $P_X \supseteq O^p_X$ such that $g' \notin P_X$. Thus $f' = f|_X \in P_X$.

Conversely let, $f \in P_X$ with $f^Y \in C(Y)$ where P_X is a minimal prime ideal of $C(X)$ such that $P_X \supseteq O^p_X$. Now there exists $g \in C(X)$ with $fg = 0$ such that $g \notin P_X$ (Theorem 2.4). Let $h = g \wedge 1$. Since $g \notin P_X$ and P_X is a z-ideal, $h \notin P_X$. Clearly $fh = 0$. Let h^Y be the continuous extension of h over Y. Then, $f^Y h^Y = 0$. We claim that there exists a minimal prime ideal P_Y of $C(Y)$ with $P_Y \supseteq O^p_Y$ such that $h^Y \notin P_Y$. If not, then $h^Y \in O^p_Y$ and so there is a neighbourhood V of p in βX $(= \beta Y)$ such that $Z(h^Y) \supseteq V \cap Y$ ([1, 7.12(a)]). Thus, $Z(h) = X \cap Z(h^Y) \supseteq V \cap Y \cap X = V \cap X$ and so, $h \in O^p_X$ ([1, 7.12(a)]). Hence $g \in O^p_X$ since O^p_X is a z-ideal. This shows that $g \in P_X$, a contradiction. So, $h^Y \notin P_Y$ for some minimal prime ideal P_Y of $C(Y)$ with $P_Y \supseteq O^p_Y$ and thus $f^Y \in P_Y$. □

3. ϕ-compact spaces and almost round subsets

Recall the equivalence relation introduced in Section 2. Let us now give the following definition.

Definition 3.1. Let $A \subseteq \beta X$. A family $\mathcal{F}$ of minimal prime ideals of $C(X)$ is said to be adequate for A if $\mathcal{F} \cap E_p \neq \phi$ $\forall p \in A$. A space X is defined to be ϕ-compact if $\bigcap \mathcal{F} \subseteq C_K(X)$ for every family $\mathcal{F}$ of minimal prime ideals of $C(X)$, adequate for $\beta X - X$.

Examples 3.2. (a) Every F-space is ϕ-compact. In fact, if X is an F-space then $E_p = \{O^p\}$ $\forall p \in \beta X$. So if $\mathcal{F}$ is a family of minimal prime ideals of $C(X)$, adequate for $\beta X - X$ then $O^p \in \mathcal{F}\ \forall p \in \beta X - X$. Clearly, $\bigcap \mathcal{F} \subseteq \bigcap_{p\in\beta X-X} O^p = C_K(X)$ and thus X is ϕ-compact.

(b) Every μ-compact space is ϕ-compact (hence every realcompact space is ϕ-compact). In fact, if $\mathcal{F}$ is any family of minimal prime ideals of $C(X)$, adequate for $\beta X - X$ then $\bigcap \mathcal{F} \subseteq \bigcap_{p\in\beta X-X} M^p$. Now if X is μ-compact then $\bigcap_{p\in\beta X-X} M^p = C_K(X)$ and thus $\bigcap \mathcal{F} \subseteq C_K(X)$. So X becomes ϕ-compact.

(c) The Tychonoff plank T is not ϕ-compact. We know that there is only one free maximal ideal, say M^t in $C(T)$. Also O^t is not prime ([1, 8J, 6]). Thus if P is any minimal prime ideal of $C(T)$ with $P \subseteq M^t$ then $O^t \subsetneqq P$ and hence T cannot be ϕ-compact.

Our next theorem shows that every space X admits a ϕ-compactification.

Theorem 3.3. *For every space X, there is a smallest ϕ-compact space ϕX lying between X and βX. So X is ϕ-compact if and only if $X = \phi X$.*

Proof: Let Φ denote the set of all ϕ-compact spaces lying between X and βX. Clearly $\Phi \neq \emptyset$ since $\beta X \in \Phi$. Let $\phi X = \bigcap \Phi$. To complete the theorem we shall show that ϕX is ϕ-compact. Consider any family $\mathcal{F}$ of minimal prime ideals of $C(\phi X)$, adequate for $\beta(\phi X) - \phi X$ $(= \beta X - \phi X)$ and suppose $f \in \bigcap \mathcal{F}$. Let $Y \in \Phi$ and $p \in \beta X - Y$. Then $p \in \beta X - \phi X$. Since $\mathcal{F}$ is adequate for $\beta X - \phi X$, there is a minimal prime ideal $P_{\phi X}$ of $C(\phi X)$ in $\mathcal{F}$ with $P_{\phi X} \supseteq O^p_{\phi X}$. So $f \in P_{\phi X}$. Clearly $f \in C^*(\phi X)$ and let f^Y be the continuous extension of f over Y. By Theorem 2.6, there is a minimal prime ideal P_Y of $C(Y)$ with $P_Y \supseteq O^p_Y$ such that $f^Y \in P_Y$. Thus $\mathcal{F}' = \{P_Y : P_Y$ is a minimal prime ideal of $C(Y)$ with $f^Y \in P_Y\}$ is adequate for $\beta Y - Y$ and $f^Y \in \bigcap \mathcal{F}'$. Since Y is ϕ-compact, $f^Y \in C_K(Y)$. So, $\mathrm{cl}_Y(Y - Z(f^Y))$ is compact and hence so is $\bigcap_{Y\in\Phi} \mathrm{cl}_Y(Y - Z(f^Y))$. Clearly, $\mathrm{cl}_{\phi X}(\phi X - Z(f)) \subseteq \bigcap_{Y\in\Phi} \mathrm{cl}_Y(Y - Z(f^Y))$. Let $p \in \bigcap_{Y\in\Phi} \mathrm{cl}_Y(Y - Z(f^Y))$. Then $p \in Y\ \forall Y \in \Phi$ and so $p \in \phi X$. Take any neighbourhood U of p in ϕX. Then there is a neighbourhood V of p in Y (where $Y \in \Phi$) such that $V \cap \phi X = U$. Also, $V \cap (Y - Z(f^Y)) \neq \emptyset$. Thus, $V \cap (Y - Z(f^Y))$ is a non-void open set in Y. Since ϕX is dense in Y, $\phi X \cap V \cap (Y - Z(f^Y)) \neq \emptyset$ i.e. $U \cap (\phi X - Z(f)) \neq \emptyset$. So $p \in \mathrm{cl}_{\phi X}(\phi X - Z(f))$. Thus, $\mathrm{cl}_{\phi X}(\phi X - Z(f)) = \bigcap_{Y\in\Phi} \mathrm{cl}_Y(Y - Z(f^Y))$. Hence $f \in C_K(\phi X)$ and ϕX becomes ϕ-compact. □

We now define almost round subsets as follows.

Definition 3.4. A subset A of βX is said to be almost round if $\bigcap \mathcal{F} \subseteq \bigcap_{p\in A} O^p$ for every family $\mathcal{F}$ of minimal prime ideals of $C(X)$, adequate for A.

Obviously X is ϕ-compact if and only if $\beta X - X$ is almost round. We also note that the union of any collection of almost round subsets of βX is almost round.

We now prove the following two lemmas.

Lemma 3.5. *Let $X \subseteq Y \subseteq vX$. Then $f \in O^p_X$ if and only if $f^Y \in O^p_Y$ where f^Y is the continuous extension of f over Y.*

Proof: The lemma follows from the fact that $\mathrm{cl}_{\beta X} Z(f) = \mathrm{cl}_{\beta X} Z(f^Y)$. □

Lemma 3.6. *Let $X \subseteq Y \subseteq vX$. Then Y is ϕ-compact if and only if $\beta X - Y$ is almost round (with respect to X).*

Proof: Let Y be ϕ-compact and let $\mathcal{F}$ be a family of minimal prime ideals of $C(X)$, adequate for $\beta X - Y$. Suppose $f \in \bigcap \mathcal{F}$ and f^Y is the continuous extension of f over Y. If $p \in \beta X - Y$ then there is a minimal prime ideal $P_X \in \mathcal{F}$ with $P_X \supseteq O^p_X$, $\mathcal{F}$ being adequate for $\beta X - Y$. So by Theorem 2.6, there is a minimal prime ideal P_Y of $C(Y)$ with $P_Y \supseteq O^p_Y$ such that $f^Y \in P^Y$. Thus $\mathcal{F}' = \{P_Y : P_Y$ is a minimal prime ideal of $C(Y)$ with $f^Y \in P_Y\}$ is adequate for $\beta X - Y$ and $f^Y \in \bigcap \mathcal{F}'$. Since Y is ϕ-compact, $f^Y \in C_K(Y)$. Thus $f^Y \in O^p_Y$ $\forall p \in \beta X - Y$. So by Lemma 3.5, $f \in O^p_X$ $\forall p \in \beta X - Y$. Consequently, $\bigcap \mathcal{F} \subseteq \bigcap_{p \in \beta X - Y} O^p_X$ and so $\beta X - Y$ is almost round.

Conversely let $\beta X - Y$ be almost round. Suppose $\mathcal{F}'$ is any family of minimal prime ideals of $C(Y)$, adequate for $\beta Y - Y$ $(= \beta X - Y)$ and suppose $f \in \bigcap \mathcal{F}'$. Let $f_1 = f|_X$ and $p \in \beta X - Y$. Since $\mathcal{F}'$ is adequate for $\beta X - Y$, there is a minimal prime ideal $P_Y \in \mathcal{F}'$ such that $P_Y \supseteq O^p_Y$. Also $f \in P_Y$. By Theorem 2.6, there is a minimal prime ideal P_X of $C(X)$ with $P_X \supseteq O^p_X$ such that $f_1 \in P_X$. Thus $\mathcal{F} = \{P_X : P_X$ is a minimal prime ideal of $C(X)$ with $f_1 \in P_X\}$ becomes adequate for $\beta X - Y$ and $f_1 \in \bigcap \mathcal{F}$. Since $\beta X - Y$ is almost round, $f_1 \in O^p_X$ $\forall p \in \beta X - Y$ and so by Lemma 3.5, $f \in O^p_Y$ $\forall p \in \beta X - Y$. So $\bigcap \mathcal{F}' \subseteq \bigcap_{p \in \beta X - Y} O^p_Y = C_K(Y)$ and hence Y is ϕ-compact. □

Corollary 3.7. *For any space X, $\beta X - \phi X$ is almost round.*

We now use Lemma 3.6 to prove the following theorem.

Theorem 3.8. *For any space X, ϕX is the smallest subspace of βX containing X for which $\beta X - \phi X$ is almost round.*

Proof: Let $X \subseteq Y \subseteq \beta X$ such that $\beta X - Y$ is almost round. Then $(\beta X - \phi X) \cup (\beta X - Y) = \beta X - (\phi X \cap Y)$ is almost round. Clearly $X \subseteq \phi X \cap Y \subseteq vX$ and so Lemma 3.6 implies that $\phi X \cap Y$ is ϕ-compact. Since ϕX is the smallest ϕ-compact space between X and βX, $\phi X \subseteq \phi X \cap Y$. So $\phi X \subseteq Y$ and the theorem follows.
□

Almost round subsets characterize F-spaces in the following way.

Theorem 3.9. *X is an F-space if and only if every subset of βX is almost round.*

Proof: The necessity follows from the fact that for an F-space X, $E_p = \{O^p\}$ $\forall p \in \beta X$.

To prove the sufficiency let $p \in \beta X$. Since $\{p\}$ is almost round, $O^p = P$ for any minimal prime ideal P with $P \supseteq O^p$. Thus O^p is prime and so X is an F-space.

Let X be a ϕ-compact space. If $\tau : X \to Y$ is a homeomorphism then τ has an extension to a homeomorphism $\tau_1 : \beta X \to \beta Y$ such that $\tau|_{\beta X - X} : \beta X - X \to \beta Y - Y$ is also a homeomorphism. Also the map $\psi : C(Y) \to C(X)$ defined by $f \to f_\circ \tau$ is an isomorphism. If $\mathcal{F} = \{P_Y^\alpha : \alpha \in \wedge\}$ is a family of minimal prime ideals of $C(Y)$, adequate for $\beta Y - Y$ then clearly $\mathcal{F}_X = \{\psi(P_Y^\alpha) : \alpha \in \wedge\}$ becomes a family of minimal prime ideals of $C(X)$, adequate for $\beta X - X$. It is now easy to see that Y is ϕ-compact. Hence we have the following theorem. □

Theorem 3.10. *ϕ-compactness is a topological property.*

Example 3.11. Let $Y = \beta N - \{p\}$ where $p \in \beta N - N$. Then Y is an F-space and hence ϕ-compact. The lone free maximal ideal of $C(Y)$ is $M_Y^p = \{f \in C(Y) : p \in \mathrm{cl}_{\beta Y} Z(f)\}$. Clearly $p \in \mathrm{cl}_{\beta N}(Y - N)$. Define $f : N \to R$ by $f(n) = \frac{1}{n}$ and suppose $h = f^\beta|_Y$. Then $h \in C(Y)$ and $Z(h) = Y - N$. Thus $h \in M_Y^p$. Now $\mathrm{cl}_Y(Y - Z(h)) = \mathrm{cl}_Y N = Y$ which is not compact and so $h \notin C_K(Y)$. Hence Y is not μ-compact.

4. ϕ'-compact spaces and nearly round subsets

Recall the definition of a family $\mathcal{F}$ of minimal prime ideals of $C(X)$, adequate for $\beta X - X$ (Definition 3.1). Let us now give the following definition.

Definition 4.1. A space X is said to be ϕ'-compact if for any $f \in \bigcap_{p \in \beta X - X} M^p$, there is a family $\mathcal{F}$ of minimal prime ideals of $C(X)$, adequate for $\beta X - X$ such that $f \in \bigcap \mathcal{F}$.

Example 4.2. Every μ-compact space is ϕ'-compact (hence every realcompact space is ϕ'-compact). In fact, if X is μ-compact and if $f \in \bigcap_{p \in \beta X - X} M^p$ then $f \in C_K(X)$ and so f is in every free minimal prime ideal of $C(X)$. So if $\mathcal{F}$ is the collection of all free minimal prime ideals in $C(X)$ then $f \in \bigcap \mathcal{F}$. Clearly $\mathcal{F}$ is adequate for $\beta X - X$.

The following theorem relates μ-compact spaces, ϕ-compact spaces and ϕ'-compact spaces.

Theorem 4.3. *A space is μ-compact if and only if it is both ϕ-compact and ϕ'-compact.*

Proof: Necessity follows from 3.2(b) and 4.2.

For sufficiency we assume that X is both ϕ-compact and ϕ'-compact. Let $f \in \bigcap_{p \in \beta X - X} M^p$. Since X is ϕ'-compact, there is a family $\mathcal{F}$ of minimal prime ideals of $C(X)$, adequate for $\beta X - X$ such that $f \in \bigcap \mathcal{F}$. Now ϕ-compactness of X implies $\bigcap \mathcal{F} \subseteq C_K(X)$. Thus $f \in C_K(X)$ and so X is μ-compact. □

Example 4.4. Recall the space $Y = \beta N - \{p\}$ where $p \in \beta N - N$ given in 3.11. The space is ϕ-compact but not μ-compact. Hence the space is also not ϕ'-compact by the previous theorem.

Notations 4.5. Let $X \subseteq Y \subseteq \beta X$ and $p \in \beta X$. The maximal ideal $\{f \in C(X) : p \in \mathrm{cl}_{\beta X} Z(f)\}$ of $C(X)$ will be denoted by M^p_X and the maximal ideal $\{f \in C(Y) : p \in \mathrm{cl}_{\beta Y} Z(f)\}$ of $C(Y)$ will be denoted by M^p_Y.

In our next theorem we shall show that every space X admits a ϕ'-compactification.

Theorem 4.6. *For any space X, there is a smallest ϕ'-compact space $\phi' X$ lying between X and βX. Thus X is ϕ'-compact if and only if $X = \phi' X$.*

Proof: Let Φ' be the family of all ϕ'-compact spaces lying between X and βX. Then $\Phi' \neq \emptyset$ since $\beta X \in \Phi'$. Let $\phi' X = \bigcap \Phi'$. To prove the theorem we shall show that $\phi' X$ is ϕ'-compact. So let $f \in \bigcap_{p \in \beta X - \phi' X} M^p_{\phi' X}$ and let $p \in \beta X - \phi' X$. Then there is $Y \in \Phi'$ such that $p \in \beta X - Y$. Now $f \in C^*(\phi' X)$ and let f^Y be the continuous extension of f over Y. Let $q \in \beta X - Y$. Clearly $q \in \beta X - \phi' X$. So $f \in M^q_{\phi' X}$. Hence $q \in \mathrm{cl}_{\beta X} Z(f) \subseteq \mathrm{cl}_{\beta X} Z(f^Y)$. Thus $f^Y \in M^q_Y$. So $f^Y \in \bigcap_{q \in \beta X - Y} M^q_Y$. Since Y is ϕ'-compact and $p \in \beta X - Y$, there is a minimal prime ideal P_Y of $C(Y)$ with $P_Y \supseteq O^p_Y$ such that $f^Y \in P_Y$. So by Theorem 2.6, there is a minimal prime ideal $P_{\phi' X}$ of $C(\phi' X)$ with $P_{\phi' X} \supseteq O^p_{\phi' X}$ such that $f \in P_{\phi' X}$. So $\mathcal{F} = \{P_{\phi' X} : P_{\phi' X}$ is a minimal prime ideal of $C(\phi' X)$ with $f \in P_{\phi' X}\}$ is adequate for $\beta X - \phi' X$ and $f \in \bigcap \mathcal{F}$. Thus $\phi' X$ is ϕ'-compact. $\square$

We now define nearly round subsets as follows.

Definition 4.7. A subset A of βX is said to be nearly round if $f \in \bigcap_{p \in A} M^p$ implies $f \in \bigcap \mathcal{F}$ for some family $\mathcal{F}$ of minimal prime ideals of $C(X)$, adequate for A.

Obviously X is ϕ'-compact if and only if $\beta X - X$ is nearly round. We note that the union of any collection of nearly round subsets of βX is nearly round. We also note that a subset of βX is round if and only if it is both almost round and nearly round.

We now prove the following lemma.

Lemma 4.8. *Let $X \subseteq Y \subseteq vX$. Then Y is ϕ'-compact if and only if $\beta X - Y$ is nearly round (with respect to X).*

Proof: Let Y be ϕ'-compact and let $f \in \bigcap_{p \in \beta X - Y} M^p_X$. Let f^Y be the continuous extension of f over Y. Then $\mathrm{cl}_{\beta X} Z(f^Y) = \mathrm{cl}_{\beta X} Z(f)$ and thus $f^Y \in \bigcap_{p \in \beta X - Y} M^p_Y$. Suppose $p \in \beta X - Y$. Now ϕ'-compactness of Y implies that there is a minimal prime ideal P_Y of $C(Y)$ with $P_Y \supseteq O^p_Y$ such that

$f^Y \in P_Y$. So by Theorem 2.6, there is a minimal prime ideal P_X of $C(X)$ with $P_X \supseteq O_X^p$ such that $f \in P_X$. Thus $\mathcal{F} = \{P_X : P_X$ is a minimal prime ideal of $C(X)$ with $f \in P_X\}$ is adequate for $\beta X - Y$ and $f \in \bigcap \mathcal{F}$. Consequently $\beta X - Y$ is nearly round.

Conversely let $\beta X - Y$ be nearly round and let $f \in \bigcap_{p\in\beta X - Y} M_Y^p$. Let $f|_X = g$. Then $\mathrm{cl}_{\beta X} Z(f) = \mathrm{cl}_{\beta X} Z(g)$ and so $g \in \bigcap_{p\in\beta X-Y} M_X^p$. Let $q \in \beta X - Y$. Since $\beta X - Y$ is nearly round, there is a minimal prime ideal P_X of $C(X)$ with $P_X \supseteq O_X^q$ such that $g \in P_X$. Hence by Theorem 2.6, there is a minimal prime ideal P_Y of $C(Y)$ with $P_Y \supseteq O_Y^q$ such that $f \in P_Y$. Thus $\mathcal{F}' = \{P_Y : P_Y$ is a minimal prime ideal of $C(Y)$ with $f \in P_Y\}$ is adequate for $\beta X - Y$ and $f \in \bigcap \mathcal{F}'$. Thus Y is ϕ'-compact. □

Corollary 4.9. *For any space X, $\beta X - \phi' X$ is nearly round.*

We now use Lemma 4.8 to prove the following theorem.

Theorem 4.10. *For any space X, $\phi' X$ is the smallest subspace of βX containing X for which $\beta X - \phi' X$ is nearly round.*

PROOF: Let $X \subseteq Y \subseteq \beta X$ such that $\beta X - Y$ is nearly round. Then $(\beta X - \phi' X) \cup (\beta X - Y) = \beta X - (\phi' X \cap Y)$ is nearly round. Clearly $X \subseteq \phi' X \cap Y \subseteq vX$ and so by Lemma 4.8, $\phi' X \cap Y$ is ϕ'-compact. Since $\phi' X$ is the smallest ϕ'-compact space between X and βX, $\phi' X \subseteq \phi' X \cap Y$. So $\phi' X \subseteq Y$ and the proof is complete. □

The following theorem gives a necessary and sufficient condition for an F-space to be a P-space.

Theorem 4.11. *An F-space X is a P-space if and only if every subset of βX is nearly round.*

PROOF: Let X be a P-space and $A \subseteq \beta X$. Suppose $f \in \bigcap_{p\in A} M^p$. Then $f \in \bigcap_{p\in A} O^p$. Thus $\mathcal{F} = \{O^p : p \in A\}$ is a family of minimal prime ideals of $C(X)$, adequate for A with $f \in \bigcap \mathcal{F}$. So A is nearly round.

Conversely let X be an F-space and every subset of βX be nearly round. Let $p \in \beta X$ and suppose $f \in M^p$. Since $\{p\}$ is nearly round there is a minimal prime ideal P of $C(X)$ with $P \supseteq O^p$ such that $f \in P$. Also since X is an F-space, $P = O^p$ and thus $f \in O^p$. So $O^p = M^p$ and hence X is a P-space.

Let X be a ϕ'-compact space. If $\tau : X \to Y$ is a homeomorphism then τ has an extension to a homeomorphism $\tau_1 : \beta X \to \beta Y$ such that $\tau_1|_{\beta X - X} : \beta X - X \to \beta Y - Y$ is also a homeomorphism. Also the map $\psi : C(Y) \to C(X)$ defined by $f \to f_\circ \tau$ is an isomorphism. If f is in the intersection of all free maximal ideals of $C(Y)$ then $\psi(f)$ is in the intersection of all free maximal ideals of $C(X)$. Now ϕ'-compactness of X implies that there is a family $\mathcal{F}_X = \{P_X^\alpha : \alpha \in \wedge\}$ of minimal prime ideals of $C(X)$, adequate for $\beta X - X$ with $\psi(f) \in \bigcap \mathcal{F}_X$. Then $\mathcal{F}_Y = \{\psi^{\leftarrow}(P_X^\alpha) : \alpha \in \wedge\}$ becomes a family of minimal prime ideals of $C(Y)$

adequate for $\beta Y - Y$ and $f \in \bigcap \mathcal{F}_Y$. Thus Y is also ϕ'-compact. So we have the following theorem. □

Theorem 4.12. *ϕ'-compactness is a topological property.*

Notation 4.13. Let ω_1 denote the space of all countable ordinals. Let $T^* = (\omega_1 + 1) \times (\omega_0 + 1)$ and $T = T^* - \{(\omega_1, \omega_0)\}$ be the Tychonoff plank.

Let us denote for computational convenience, $(\alpha, \omega_1) \times \{n\}$ $((\alpha, \omega_1] \times \{n\})$ by $(\alpha, \{n\})$ $((\alpha, \{n\}]$, respectively), where $\alpha \lneqq \omega_1$ and $n \in (\omega_0 + 1)$.

Lemma 4.14. *For each $f \in M^t - O^t$, there exists $g \notin O^t$ such that $fg = 0$ where $t = \{(\omega_1, \omega_0)\}$.*

Proof: Since $f \in M^t$, i.e. $t \in \mathrm{cl}_{\beta T} Z(f)$, every neighbourhood of t must meet $Z(f)$. Also $f \notin O^t$ and so $\mathrm{cl}_{\beta T} Z(f)$ is not a neighbourhood of t. Now any neighbourhood of t is of the form $(\alpha, \omega_1] \times N'$, where $N' \subseteq \omega_0 + 1$, $\alpha \lneqq \omega_1$ and $(\omega_0 + 1) - N'$ is at most a finite set. Thus there exist infinite subsets N_1, N_2 of ω_0 with $N_1 \cup N_2 = \omega_0$ and $\alpha \lneqq \omega_1$, such that, for each $n \in N_1$, $f((\alpha, \{n\}]) = 0$ and for each $n \in N_2$, $f((\alpha, \{n\}]) \neq 0$. The choice of single α is possible here because of the non-cofinality character of any denumerable subset of ω_1. Also $f((\alpha, \{\omega_0\})) = 0$. Choose $g : T \to \mathbb{R}$ by defining $g((\alpha, \{n\}]) = \frac{1}{n}$, for each $n \in N_1$, $g((\alpha, \{n\}]) = 0$ for each $n \in N_2$ and assign 0 on rest of the region. Clearly, g is continuous in $[0, \alpha] \times (\omega_0 + 1)$. Choose $(\gamma, n) \in (\alpha, \{n\}]$, $n \in \omega_0$. Then $(\alpha, \{n\}]$ is an open neighbourhood of (γ, n) and $g((\alpha, \{n\}])$ is either $= 0$ or $\frac{1}{n}$. Thus f is continuous at (γ, n). If now $(\gamma, \omega_0) \in (\alpha, \{\omega_0\})$, then $g((\gamma, \omega_0)) = 0$. Choose any $\epsilon \gneqq 0$. Then there exists $n \in \omega_0$ such that $\frac{1}{n} \lneqq \epsilon$. Take $M = (\omega_0 + 1) - \{r \in \omega_0 : r \leqq n\}$. Then $(\alpha, \omega_1] \times M - \{t\}$ is an open neighbourhood of (γ, ω_0) and $g(((\alpha, \omega_1] \times M) - \{t\})$ is contained in $(-\epsilon, \epsilon)$. Hence g is continuous at (γ, ω_0). Thus g is continuous on T. Also since $T - Z(g)$ contains $(\alpha, \omega_1] \times N_1$, $g \notin O^t$. Clearly, $fg = 0$. □

Using the above lemma, we now show that the Tychonoff plank T is ϕ'-compact but not μ-compact.

Example 4.15. Since T is not ϕ-compact (Example 3.2(c)), it is neither μ-compact. We now show that T is ϕ'-compact. So let $f \in \bigcap_{p \in \beta T - T} M^p$ i.e. $f \in M^t$. We have to produce a family $\mathcal{F}$ of minimal prime ideals of $C(T)$, adequate for $\beta T - T = \{t\}$ such that $f \in \bigcap \mathcal{F}$. If $f \in O^t$, then it becomes obvious, if not then $fg = 0$ for some $g \notin O^t$ by Lemma 4.14. Since O^t is the intersection of all minimal prime ideals containing it, there is a minimal prime ideal, say P containing O^t such that $g \notin P$. So $f \in P$ since P is prime. Let $\mathcal{F} = \{P\}$. Clearly $\mathcal{F}$ is adequate for $\beta T - T$ and $f \in \bigcap \mathcal{F}$.

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