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Commentationes Mathematicae Universitatis Carolinae, Vol. 41 (2000), No. 3, 619--629

Persistent URL: http://dml.cz/dmlcz/119195

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Giorgio Nordo*, Boris A. Pasynkov†

Abstract. We prove that the maximal Hausdorff compactification χf of a T_2-compactifiable mapping f and the maximal Tychonoff compactification βf of a Tychonoff mapping f (see [P]) are perfect. This allows us to give a characterization of all perfect Hausdorff (respectively, all perfect Tychonoff) compactifications of a T_2-compactifiable (respectively, of a Tychonoff) mapping, which is a generalization of two results of Skljarenko [S] for the Hausdorff compactifications of Tychonoff spaces.

Keywords: Hausdorff (Tychonoff) mapping, compactification of a mapping, maximal Hausdorff (Tychonoff) compactification of a mapping, perfect compactification of a mapping

Classification: Primary 54C05, 54C10, 54C20, 54C25; Secondary 54D15, 54D30, 54D35

1. Introduction

In 1961, E.G. Skljarenko introduced the notion of the *perfect compactification* of a Tychonoff space. Given a Tychonoff space X, we say that a compactification γX of X is *perfect* if $cl_{\gamma X}(bd_X(U)) = bd_{\gamma X}(\langle U\rangle_{\gamma X})$ for every open set U of X, where $\langle U\rangle_{\gamma X}$ denotes the *maximal extension* of U relatively to γX, that is the maximal open set of γX whose trace on X is U.

In [S], Skljarenko, using proximal techniques, gave some characterizations of the perfect compactifications and he proved that γX is a perfect compactification of X if and only if the canonical map $\varphi_\gamma : \beta X \to \gamma X$ is monotone (i.e. every its fibre is connected) and so — in particular — that the Stone-Čech compactification βX is a perfect compactification of X.

Further results concerning this class of compactifications were given by Diamond in [D].

Recently, the first author [N] has generalized the notion of perfectness from a Hausdorff compactification of a Tychonoff space to a generic extension of an arbitrary space simplifying the treatment in a more general setting and obtaining several new characterizations.

Since it is clear now what is the compactification of a continuous mapping and since the notion of a topological space is the simplest case of the notion of a

* This research was supported by a grant from the C.N.R. (G.N.S.A.G.A.) and M.U.R.S.T. through "Gruppo Topologia e Geometria" (Italy).

† This paper was written while the second author was supported by a grant of the Mathematics Department of Messina University. He was also supported by a KCFE and RFFI.

continuous mapping (a space is its mapping to the one-point space), it is natural to extend to continuous mappings some results concerning compactifications of spaces.

The study of compactifications (= perfect extensions) of a continuous mapping was started in 1953 by Whyburn [W].

In [P], using techniques of partial topological products, Pasynkov described a general way to obtain all Tychonoff (i.e. completely regular, T_0-) compactifications of Tychonoff mappings between arbitrary spaces and he proved that the poset $TK(f)$ of all the Tychonoff compactifications of a Tychonoff mapping $f : X \to Y$ admits the maximal compactification $\beta f : \beta_f X \to Y$ which is the exact analogue of the Stone-Čech compactification of a Tychonoff space (since if $|Y| = 1$, X becomes a Tychonoff space and the domain $\beta_f X$ of βf coincides with βX).

The following similar result is obtained in [BN]:
If a continuous mapping $f : X \to Y$ is T_2-compactifiable (i.e. f has some Hausdorff compactification) then it has the maximal compactification $\chi f : \chi_f X \to Y$ in the poset $HK(f)$ of all Hausdorff compactifications of f.

Let us note in this connection that — unlike the corresponding case for spaces — there exist Hausdorff compact mappings which are not Tychonoff ([HI], [C]). Thus, it is necessary to consider the cases of Tychonoff and T_2-compactifiable mappings separately. It would be interesting
to find wide enough conditions when every Hausdorff compactification of a Tychonoff mapping is Tychonoff.

In this paper, we generalize to continuous mappings two extrinsic characterizations of perfect compactifications of spaces obtained by Skljarenko in [S].

We will prove that:

(1) the maximal Hausdorff (maximal Tychonoff) compactification χf (respectively βf) of a T_2-compactifiable (Tychonoff) mapping f is a perfect extension of f (Theorems 3.1, 3.9);
(2) a Hausdorff (Tychonoff) compactification bf of a T_2-compactifiable (Tychonoff) mapping f is a perfect extension of f if and only if the canonical morphism of χf (respectively βf) to bf is monotone (Theorems 3.6, 3.11).

2. Preliminaries

Throughout the paper, the word "space" will mean "topological space".

If X is a space, $\tau(X)$ will denote the set of all the open subsets of X while $\sigma(X)$ will denote the set of all the closed subsets of X.

As usual, for any pair of spaces X and Y, $C(X, Y)$ denotes the set of all continuous mappings from X to Y and $C^*(X)$ is the set of all continuous real bounded functions on X.

Undefined notions are used as in [E].

Definitions ([N], [S]). Let Y be an extension of a space X, $U \in \tau(X)$ and $x \in Y \backslash X$.

We say that the pair (x, U) is *perfect* if $x \in cl_Y(bd_X(U))$ provided $x \in bd_Y(\langle U\rangle_Y)$, where $\langle U\rangle_Y = \bigcup\{V \in \tau(Y) : V \cap X = U\}$ is the *maximal extension of U in Y*, i.e. the maximum open set of Y whose trace on X is U.

We say that Y is a *perfect extension of X relatively to x* if for every $W \in \tau(X)$ the pair (x, W) is perfect.

We say that Y is a *perfect extension of* X if it is a perfect extension of X relatively to every point of its remainder $Y \backslash X$.

Definition ([N], [S]). Let Y be an extension of X and $x \in Y \backslash X$. We say that $Y \backslash X$ *cuts X at x* if there exists some neighborhood O of x in Y and a pair U, V of disjoint open sets of X such that $O \cap X = U \cup V$ and $x \in cl_Y(U) \cap cl_Y(V)$.

The following characterization is given in [N].

Proposition 2.1. *Let Y be an extension of a space X and $x \in Y \backslash X$. Then Y is a perfect extension of X relatively to x if and only if $Y \backslash X$ does not cut X at x.*

Now, we define our framework.

For any fixed space Y, we consider the category $\mathbf{Top}_Y$, where

$$Ob(\mathbf{Top}_Y) = \{f \in C(X, Y) : \quad X \in Ob(\mathbf{Top})\}$$

is the class of the *objects* and, for every pair $f : X \to Y$, $g : Z \to Y$ of objects,

$$M(f, g) = \{\lambda \in C(X, Z) : \quad g \circ \lambda = f\}$$

is the class of the *morphisms* from f to g, whose generic representant is denoted for short by $\lambda : f \to g$.

A morphism $\lambda : f \to g$ from $f : X \to Y$ to $g : Z \to Y$ will be called *surjective* (resp. *dense*) if $\lambda(X) = Z$ (resp. if $\lambda(X)$ is dense in Z).

If $\lambda : f \to g$ is a surjective morphism, we say that g is the *image* of f (*by the morphism* λ) and we write $g = \lambda(f)$.

Moreover, we say that a morphism $\lambda : f \to g$ from $f : X \to Y$ to $g : Z \to Y$ is an *embedding* (resp. a *homeomorphism*) if the mapping $\lambda : X \to Z$ is an embedding.

A mapping $g : Z \to Y$ is called an *extension* of $f : X \to Y$ if some dense embedding $\lambda : f \to g$ is fixed (as usual X and f are identified with $\lambda(X)$ and $g_{|\lambda(X)}$ respectively).

A morphism $\lambda : g \to h$ between two extensions $g : Z \to Y$ and $h : W \to Y$ of a mapping $f : X \to Y$ will be called *canonical* if $\lambda_{|X} = id_X$.

Now, let us recall some other definitions.

Definitions. A mapping $f : X \to Y$ is said to be T_0 ([P]) if for any $x, x' \in X$ such that $x \neq x'$ and $f(x) = f(x')$ there exist either a neighborhood of x in X which does not contain x' or a neighborhood of x' in X not containing x.

A mapping $f : X \to Y$ is said to be *Hausdorff* (or T_2) [P] if for every $x, x' \in X$ such that $x \neq x'$ and $f(x) = f(x')$ there are disjoint neighborhoods of x and x' in X.

We shall say that $f : X \to Y$ is *compact* if it is perfect (i.e. closed and all its fibres are compact).

A mapping $f : X \to Y$ is said to be *completely regular* [P] if for every $F \in \sigma(X)$ and $x \in X \setminus F$ there exists a neighborhood O of $f(x)$ in Y and a continuous mapping $\varphi : f^{-1}(O) \to [0, 1]$ such that $\varphi(x) = 1$ and $\varphi(F \cap f^{-1}(O)) \subseteq \{0\}$.

A completely regular, T_0 mapping is called *Tychonoff* (or $T_{3\frac{1}{2}}$) [P].

The following lemma is evident.

Lemma 2.2. *Every morphism defined on a Hausdorff mapping is a Hausdorff mapping too.*

The next lemma from [P] will be useful in the following.

Lemma 2.3. *Let $f : X \to Y$ be a Hausdorff mapping, $y \in Y$ and let K_1, K_2 be two disjoint compact subsets of X such that $K_1 \cup K_2 \subseteq f^{-1}(\{y\})$. Then K_1 and K_2 have disjoint neighborhoods in X.*

Corollary 2.4. *If $f : X \to Y$ is a Hausdorff compact mapping, $y \in Y$ and K_1, K_2 are closed disjoint subsets of $f^{-1}(\{y\})$ then K_1 and K_2 have disjoint neighborhoods in X.*

Definition. A restriction $f_{|X'} : X' \to Y$ to $X' \subseteq X$ of a mapping $f : X \to Y$ is called a *closed submapping* of f if X' is a closed subset of X.

Obviously every closed submapping of a compact mapping is compact too.

Many well-known statements which hold in the category **Top** have their analogue (and hence a generalization) in $\mathbf{Top}_Y$. The following properties were given in [P].

Proposition 2.5. *Let λ and μ be morphisms from a mapping $f : X \to Y$ to a Hausdorff mapping $h : Z \to Y$ and D be a dense subset of X. Then, if $\lambda_{|D} = \mu_{|D}$, the morphisms λ and μ coincide.*

Proposition 2.6. *The composition of two compact Hausdorff mappings is compact Hausdorff.*

Proposition 2.7. *Every image $\lambda(k)$ of a compact mapping $k : X \to Y$ (under a morphism λ) is compact.*

Proposition 2.8. *Every compact submapping $h_{|X'} : X' \to Y$ of a Hausdorff mapping $h : X \to Y$ is a closed submapping of h.*

Proposition 2.9. *Every morphism $\lambda : k \to h$ from a compact mapping $k : X \to Y$ to a Hausdorff mapping $h : Z \to Y$ is a perfect mapping.*

Definition. We say that a mapping $c : X^c \to Y$ is a *compactification* of a mapping $f : X \to Y$ if it is a compact extension of f.

Definitions. Let $c : X^c \to Y$ and $d : X^d \to Y$ be compactifications of a mapping $f : X \to Y$. We say that:

- *c is projectively larger than d* (relatively to f) and we write that $c \geq_f d$ (or $c \geq d$, for short) if there exists some canonical morphism $\lambda : c \to d$;
- *c is equivalent to d* (relatively to f) and we write that $c \equiv_f d$ (shortly, $c \equiv d$) if there exists a canonical homeomorphism $\lambda : c \to d$.

In [BN], the following useful result is obtained:

Proposition 2.10. *Let $c : X^c \to Y$ and $d : X^d \to Y$ be Hausdorff compactifications of a mapping $f : X \to Y$. Then $c \equiv_f d$ if and only if $c \geq d$ and $d \geq c$.*

Definition. A Hausdorff mapping $f : X \to Y$ will be called *T_2-compactifiable* (or *Hausdorff compactifiable*) if it has some Hausdorff compactification.

All Hausdorff compactifications of any T_2-compactifiable mapping form a set up to their equivalence (see [BN]).

Definition. If $f : X \to Y$ is a T_2-compactifiable mapping, $HK(f)$ will denote the set of all Hausdorff compactifications of f (up to the equivalence $\equiv_f$).

So, by 2.10, it follows that $(HK(f), \geq)$ is a poset and, for any pair of Hausdorff compactifications $c, d \in HK(f)$, we can write $c = d$ instead of $c \equiv_f d$, that is, we do not distinguish between equivalent Hausdorff compactifications.

In [BN], the following is proved:

Theorem 2.11. *For any T_2-compactifiable mapping $f : X \to Y$, there is in the poset $(HK(f), \geq)$* a maximal Hausdorff compactification *$\chi f : \chi_f X \to Y$ of f.*

From 2.5 it follows — in particular — that for any Hausdorff compactification $bf : X^b \to Y$ of a T_2-compactifiable mapping $f : X \to Y$ there exists a unique canonical morphism $\lambda_b : \chi f \to bf$.

The following useful property can be found in [P].

Proposition 2.12. *Let $bf : X^b \to Y$ and $bg : Z^b \to Y$ be Hausdorff compactifications of $f : X \to Y$ and $g : Z \to Y$ respectively, $\lambda : f \to g$ be a perfect morphism and $\widetilde{\lambda} : bf \to bg$ be a morphism such that $\widetilde{\lambda}_{|X} = \lambda$. Then $\widetilde{\lambda}(X^b \backslash X) \subseteq Z^b \backslash Z$.*

In [P], Pasynkov proved that any Tychonoff mapping $f : X \to Y$ has a Tychonoff (and hence Hausdorff) compactification.

Definition. For any Tychonoff mapping $f : X \to Y$, we will denote by $TK(f)$ the set of all Tychonoff compactifications of f (up to the equivalence $\equiv_f$).

In [P], it is shown that, for any Tychonoff mapping $f : X \to Y$, there exists in $(TK(f), \geq)$ a *maximal Tychonoff compactification* $\beta f : \beta_f X \to Y$ of f.

Definition. For any mapping $g : T \to Y$ and any $U \in \tau(Y)$, let $C^*(U, g) = C^*(g^{-1}(U))$.

The following characterization of βf is given in [P].

Theorem 2.13. *For any Tychonoff compactification $bf : X^b \to Y$ of a Tychonoff mapping $f : X \to Y$, the following conditions are equivalent:*

(1) $bf = \beta f$;
(2) *for every $U \in \tau(Y)$ and $\varphi \in C^*(U, f)$, there exists a unique extension $\widetilde{\varphi} \in C^*(U, bf)$;*
(3) *for every compact Tychonoff mapping $k : Z \to Y$ and every morphism $\lambda : f \to k$ there exists a morphism $\widetilde{\lambda} : bf \to k$ which extends λ.*

Proposition 2.14. ([P]). *For any Tychonoff compactification $bf : X^b \to Y$ of a Tychonoff mapping $f : X \to Y$ there exists a unique (perfect) canonical morphism $\mu_b : \beta f \to bf$ and it results $\mu_b(\beta_f X \backslash X) = X^b \backslash X$.*

3. Perfectness of the maximal compactifications of a mapping

In [S], Skljarenko proved that a compactification γX of a Tychonoff space X is perfect if and only if the canonical map $\varphi_\gamma : \beta X \to \gamma X$ is monotone (that is, every its fibre is connected) and hence — in particular — that the Stone-Čech compactification βX of X is a perfect compactification of X.

In the following we will obtain similar (and more general) results for compactifications of a mapping.

Definition. Let $\widetilde{f} : \widetilde{X} \to Y$ be an extension of a mapping $f : X \to Y$. We say that $\widetilde{f}$ is a *perfect extension* of f if its domain $\widetilde{X}$ is a perfect extension of the space X.

Theorem 3.1. *The maximal Hausdorff compactification $\chi f : \chi_f X \to Y$ of a T_2-compactifiable mapping $f : X \to Y$ is a perfect extension of f.*

Proof: Suppose by contradiction that χf is not a perfect extension of f. By 2.1, there exists some $x \in \chi_f X \backslash X$ such that $\chi_f X \backslash X$ cuts X at x, i.e. there are a neighborhood U of x in $\chi_f X$ and a pair U_0, U_1 of disjoint open subsets of X such that $x \in cl_{\chi_f X}(U_0) \cap cl_{\chi_f X}(U_1)$ and $U \cap X = U_0 \cup U_1$. Note that $G = cl_U(U_0) \cap cl_U(U_1) \subseteq \chi_f X \setminus X$.

Let X' be the disjoint union of $\chi_f X \setminus U$ and $U_i' = cl_U(U_i)$ (for $i = 0, 1$). The copy of G lying in U_i' will be denoted by G_i and the copy of a point $t \in G$ lying in G_i will be denoted by t_i (for $i = 0, 1$). In particular, we have $x_i \in U_i'$ (for $i = 0, 1$). Set $\lambda(t) = t$ for $t \in X' \setminus (G_0 \cup G_1)$ and $\lambda(t_i) = t$ for $t_i \in G_i$ (for $i = 0, 1$). Hence, $\lambda(x_i) = x$ (for $i = 0, 1$) and $X \subseteq X'$, $\lambda|_X = id_X$.

Let θ consist of inverse images of all open sets of $\chi_f X$ by the mappings λ and $\lambda_i \equiv \lambda|_{U_i'}$ (for $i = 0, 1$). Evidently, θ is a topology on X', U_i' is open in X' (for $i = 0, 1$), λ is continuous and $\lambda : X' \setminus (G_0 \cup G_1) \to \chi_f X \setminus G$ is a homeomorphism.

In particular, $\lambda|_X$ is the identical homeomorphism of X. Since $\lambda^{-1}(\{t\})$ consists of two points for $t \in G$, all fibres of λ are compact.

Since $X' \setminus U_i'$ is closed in X', the corestriction of λ to this set is a homeomorphism and $\lambda(X' \setminus U_i') = (\chi_f X \setminus U) \cup cl_U(U_j)$ (where $j = 1$ when $i = 0$ and $j = 0$ when $i = 1$) is closed in $\chi_f X$ (for $i = 0, 1$), λ is closed and so perfect. Evidently, X is dense in X' and λ is Hausdorff.

Thus, $bf = \chi f \circ \lambda$ is a compact Hausdorff mapping (by 2.6) and $bf_{|X} = f$. So, bf is a Hausdorff compactification of f and λ is a canonical morphism from bf to χf, i.e. $bf \geq \chi f$.

Moreover, λ is not 1–1 because $x = \lambda(x_0) = \lambda(x_1)$. Thus $bf > \chi f$ which is a contradiction to the maximality of χf. □

To obtain an extrinsic characterization of the perfect Hausdorff compactification, we need two lemmas.

Lemma 3.2. *Let Y_1 and Y_2 be extensions of a space X, $x \in Y_1 \backslash X$ and $f : Y_2 \to Y_1$ a continuous mapping closed at x such that $f_{|X} = id_X$ and $f^{-1}(\{x\})$ is connected. Then, if Y_2 is a perfect extension of X relatively to any point of $F = f^{-1}(\{x\})$, Y_1 is a perfect extension of X relatively to x.*

Proof: First, we observe that $f^{-1}(\{x\}) \neq \emptyset$ as otherwise by the closedness of f at x, there exists some neighborhood N of x such that $f^{-1}(N) \subseteq \emptyset$.

Now, suppose — by contradiction — that Y_1 is not a perfect extension of X relatively to x. By 2.1, $Y_1 \backslash X$ cuts cut X at x, i.e. there exist a neighborhood O of x in Y_1 and disjoint open sets U, V of X such that $O \cap X = U \cup V$ and $x \in cl_{Y_1}(U) \cap cl_{Y_1}(V)$.

We claim that $F \cap cl_{Y_2}(U) \cap cl_{Y_2}(V) = \emptyset$. In fact, if there exists some $t \in F \cap cl_{Y_2}(U) \cap cl_{Y_2}(V)$, by continuity of f, $W = f^{-1}(O)$ is a neighborhood of t in Y_2 and, from $f_{|X} = id_X$ and $O \cap X = U \cup V$, it follows that $W \cap X = U \cup V$. But this means that $Y_2 \backslash X$ cuts X at $t \in F$ and by 2.1, Y_2 is not a perfect extension of X relatively to $t \in F$, which is a contradiction.

Moreover, $x \in O$ implies $F \subseteq W \subseteq cl_{Y_2}(W) = cl_{Y_2}(W \cap X) = cl_{Y_2}(U) \cup cl_{Y_2}(V)$. So, $(cl_{Y_2}(U) \cap F) \cup (cl_{Y_2}(V) \cap F) = F$ and, as F is connected, one of these two closed sets must be empty. Suppose that $cl_{Y_2}(U) \cap F = \emptyset$. Since $f : Y_2 \to Y_1$ is closed at x, there is some neighborhood N of x in Y_1 such that $f^{-1}(N) \subseteq Y_2 \backslash cl_{Y_2}(U)$. So, $cl_{Y_2}(U) \cap f^{-1}(N) = \emptyset$ and $U \cap N = U \cap X \cap N = U \cap f^{-1}(X \cap N) \subseteq cl_{Y_2}(U) \cap f^{-1}(N) = \emptyset$ imply $U \cap N = \emptyset$. This contradicts $x \in cl_{Y_1}(U)$.

Thus, it is proved that Y_1 is a perfect extension of X relatively to x. □

We recall that a mapping is called *monotone* if every its fibre is connected.

Corollary 3.3. *Let Y_1 and Y_2 be extensions of a space X and $f : Y_2 \to Y_1$ be a continuous, closed and monotone mapping such that $f_{|X} = id_X$. Then, if Y_2 is a perfect extension of X, Y_1 is a perfect extension of X too.*

Definition. Let S be a subspace of a space T. We say that S is *normally situated* (strongly normal in the terminology of [A]) in T if every pair of disjoint closed sets of S can be separated by a pair of disjoint open sets of T.

Remark. It follows from Corollary 2.4 that every fibre of a compact Hausdorff mapping is normally situated in its domain.

Lemma 3.4. *Let Y_1 and Y_2 be extensions of X, $x \in Y_1 \backslash X$ and $f : Y_2 \to Y_1$ be a continuous mapping closed at x, such that $F = f^{-1}(\{x\})$ is normally situated in Y_2 and $f_{|X} = id_X$. If Y_1 is a perfect extension of X relatively to x then F is connected.*

Proof: Suppose, by contradiction, that F is not connected, i.e. that there are disjoint non-empty closed sets C_1, C_2 of F such that $C_1 \cup C_2 = F$.

Since F is normally situated in Y_2, there are disjoint open sets U_1, U_2 of Y_2 such that $C_i \subseteq U_i$ (for $i = 1, 2$). So $F \subseteq U_1 \cup U_2$ and, by the closedness of f, there exists an open neighborhood O of x in Y_1 such that $f^{-1}(O) \subseteq U_1 \cup U_2$.

We may suppose that $f^{-1}(O) = U_1 \cup U_2$.

Since X is dense in Y_2, $V_i = U_i \cap X$ for $i = 1, 2$ are non-empty disjoint open sets of X and $O \cap X = f^{-1}(O) \cap X = V_1 \cup V_2$.

On the other hand, $x \in cl_{Y_1}(V_1) \cap cl_{Y_1}(V_2)$ because (for $i = 1, 2$) $U_i \subseteq cl_{Y_2}(U_i) = cl_{Y_2}(U_i \cap X) = cl_{Y_2}(V_i)$ and $x \in f(U_i) \subseteq f(cl_{Y_2}(V_i)) \subseteq cl_{Y_1}(f(V_i)) = cl_{Y_1}(V_i)$.

Thus $Y_1 \backslash X$ cuts X at x. This contradicts that Y_1 is a perfect extension of X relatively to x. Hence, F is connected. □

Corollary 3.5. *Let Y_1 and Y_2 be extensions of X and $f : Y_2 \to Y_1$ be a continuous closed mapping such that $f_{|X} = id_X$, $f^{-1}(X) = X$ and every its fibre is normally situated in Y_2. Then, if Y_1 is a perfect extension of X, the mapping f is monotone.*

Theorem 3.6. *Let $bf : X^b \to Y$ be a Hausdorff compactification of a mapping $f : X \to Y$ and let $\chi f : \chi_f X \to Y$ be the maximal Hausdorff compactification of f. Then bf is a perfect extension of f if and only if the canonical morphism $\lambda_b : \chi f \to bf$ is monotone.*

Proof: Suppose that bf is a perfect compactification of f, i.e. that X^b is a perfect extension of X. From 2.9, λ_b is perfect and, since χf is Hausdorff, by 2.2, λ_b is Hausdorff, too. Hence (see Remark before Lemma 3.4), every fibre of λ_b is normally situated in $\chi_f X$. By Corollary 3.5, λ_b is monotone.

Conversely, suppose that $\lambda_b : \chi_f X \to X^b$ is monotone. Since χf is a perfect extension of f, i.e. $\chi_f X$ if a perfect extension of X, 3.3 implies that X^b is a perfect extension of X. Hence bf is a perfect extension of f. □

If X is a Tychonoff space and $|Y| = 1$, every compactification γX of X corresponds to the (Tychonoff) compactifications $\gamma f : \gamma X \to Y$ of f, the domain

$\chi_f X$ of the maximal Hausdorff compactification of f coincides with the Stone-Čech compactification βX of X, the canonical morphism $\lambda : \chi f \to \gamma f$ becomes the usual canonical map $\varphi_\gamma : \beta X \to \gamma X$ and so the previous theorem gives as corollary the following proposition for spaces proved in [S].

Theorem 3.7. *A compactification γX of a Tychonoff space X is a perfect extension of X if and only if the canonical mapping $\varphi_\gamma : \beta X \to \gamma X$ is monotone.*

Remark. Let us observe that weaker versions of Theorems 3.1 and 3.6 were proved by Mazroa [M] by means of the notion of proximity for mappings (see [No]) only for the particular case of (Tychonoff) compactifications of a surjective (Tychonoff) mapping between T_3-spaces.

Theorem 3.8. *Let $f : X \to Y$ be a Tychonoff mapping, $\beta f : \beta_f X \to Y$ be its maximal Tychonoff compactification and $\chi f : \chi_f X \to Y$ be its maximal Hausdorff compactification. Then the canonical morphism $\lambda : \chi f \to \beta f$ is monotone.*

Proof: Since χf is compact and βf is Hausdorff, by 2.9, λ is perfect. From 2.12 it follows that $\lambda(\chi_f X\backslash X) \subseteq \beta_f X\backslash X$ and as λ is canonical, $\lambda^{-1}(X) = X$ and $\lambda(\chi_f X\backslash X) = \beta_f X\backslash X$.

Now, suppose — by contradiction — that $\lambda : \chi_f X \to \beta_f X$ is not monotone, i.e. that there is some $x \in \beta_f X\backslash X$ such that $\lambda^{-1}(\{x\})$ is not connected. So, there are non-empty disjoint closed sets B, C of $\lambda^{-1}(\{x\})$ such that $B \cup C = \lambda^{-1}(\{x\})$. Since $\lambda^{-1}(\{x\})$ is normally situated in $\chi_f X$ (see Remark before Lemma 3.4), there are disjoint open sets U, V of $\chi_f X$ such that $B \subseteq U$ and $C \subseteq V$. So, $U \cup V$ is an open neighborhood of $\lambda^{-1}(\{x\})$ and as $\lambda : \chi_f X \to \beta_f X$ is closed, there exists an open neighborhood W of x in $\beta_f X$ such that $\lambda^{-1}(W) \subseteq U \cup V$.

Since $\beta_f X\backslash W$ is a closed subset of $\beta_f X$ which does not contain the point x and $\beta f : \beta_f X \to Y$ is a Tychonoff mapping, there exist an open neighborhood H of $\beta f(x)$ in Y and a continuous mapping $\varphi : (\beta f)^{-1}(H) \to [0,1]$ such that $(\beta f)^{-1}(H) \cap (\beta_f X\backslash W) = (\beta f)^{-1}(H)\backslash W \subseteq \varphi^{-1}(\{0\})$ and $\varphi(x) = 1$.

Hence, $W_\beta = W \cap (\beta f)^{-1}(H)$ is an open neighborhood of x in $\beta_f X$ and $W_\chi = \lambda^{-1}(W_\beta)$ is an open set of $\chi_f X$. Obviously, $W_\beta \subseteq W$ and $W_\chi \subseteq U \cup V$.

Let us note that $W_\chi \cap X = \lambda^{-1}(W_\beta) \cap \lambda^{-1}(X) = \lambda^{-1}(W_\beta \cap X) = W_\beta \cap X$.

Now, $W_1 = U \cap W_\chi$ and $W_2 = V \cap W_\chi$ are non-empty disjoint open sets of $\chi_f X$ such that $W_\chi = W_1 \cup W_2$.

Let $O_i = W_i \cap X$ (for $i = 1, 2$). Since X is dense in $\chi_f X$, O_1 and O_2 are non-empty disjoint open sets of X such that $O_1 \cup O_2 = W_\chi \cap X = W_\beta \cap X$, $B \subseteq cl_{\chi_f X}(O_1)$ and $C \subseteq cl_{\chi_f X}(O_2)$.

Moreover, since $\beta f \circ \lambda = \chi f$ and $\chi f_{|X} = f$, we have $O_1 \cup O_2 = W_\chi \cap X = \lambda^{-1}(W_\beta) \cap X \subseteq \lambda^{-1}((\beta f)^{-1}(H)) \cap X = (\chi f)^{-1}(H) \cap X = f^{-1}(H)$.

Since both B and C are contained in the fibre $\lambda^{-1}(\{x\})$, we obtain $x \in \lambda(B) \cap \lambda(C) \subseteq \lambda(cl_{\chi_f X}(O_1)) \cap \lambda(cl_{\chi_f X}(O_2)) \subseteq cl_{\beta_f X}(\lambda(O_1)) \cap cl_{\beta_f X}(\lambda(O_2)) = cl_{\beta_f X}(O_1) \cap cl_{\beta_f X}(O_2)$.

There exists an open neighborhood O of x in $(\beta f)^{-1}(H)$ such that $\varphi(O) \subseteq]\frac{1}{2}, 1]$.

Define the mapping $\psi : f^{-1}(H) \to [-1, 1]$ by setting:

$$\psi(t) = \begin{cases} \varphi(t) & \text{if } t \in f^{-1}(H)\backslash O_2 \\ -\varphi(t) & \text{if } t \in cl_{f^{-1}(H)}(O_2). \end{cases}$$

It is continuous by the *Pasting Theorem* for closed sets because $cl_{f^{-1}(H)}(O_2) \cap (f^{-1}(H)\backslash O_2) = bd_{f^{-1}(H)}(O_2)$ and $O_1 \cap bd_{f^{-1}(H)}(O_2) = \emptyset$, $O_2 \cap bd_{f^{-1}(H)}(O_2) = \emptyset$ imply $(O_1 \cup O_2) \cap bd_{f^{-1}(H)}(O_2) = \emptyset$ and, hence,

$$\begin{aligned} bd_{f^{-1}(H)}(O_2) &\subseteq f^{-1}(H)\backslash(O_1 \cup O_2) \\ &= f^{-1}(H)\backslash(W_\beta \cap X) \\ &\subseteq (\beta f)^{-1}(H)\backslash W_\beta \\ &= (\beta f)^{-1}(H)\backslash W \\ &\subseteq \varphi^{-1}(\{0\}). \end{aligned}$$

Then, by 2.13, there is a continuous extension $\widetilde{\psi} : (\beta f)^{-1}(H) \to [-1, 1]$ of ψ to $(\beta f)^{-1}(H)$. Obviously, it results $\widetilde{\psi}(O_1 \cap O) \subseteq]\frac{1}{2}, 1]$ and $\widetilde{\psi}(O_2 \cap O) \subseteq [-1, -\frac{1}{2}[$.

On the other hand, since $x \in cl_{\beta_f X}(O_1) \cap cl_{\beta_f X}(O_2)$, $O_1 \cup O_2 \subseteq (\beta f)^{-1}(H)$ and $x \in (\beta f)^{-1}(H)$, it follows that $x \in cl_{(\beta f)^{-1}(H)}(O_1) \cap cl_{(\beta f)^{-1}(H)}(O_2)$ and as O is a neighborhood of x in $(\beta f)^{-1}(H)$, $x \in cl_{(\beta f)^{-1}(H)}(O_1 \cap O) \cap cl_{(\beta f)^{-1}(H)}(O_2 \cap O)$. So, by continuity of $\widetilde{\psi}$, we have $\widetilde{\psi}(x) \in \widetilde{\psi}(cl_{(\beta f)^{-1}(H)}(O_1 \cap O)) \cap \widetilde{\psi}(cl_{(\beta f)^{-1}(H)}(O_2 \cap O)) \subseteq cl_{[-1,1]}(\widetilde{\psi}(O_1 \cap O)) \cap cl_{[-1,1]}(\widetilde{\psi}(O_2 \cap O)) = \emptyset$.

A contradiction which proves that the canonical morphism $\lambda : \chi f \to \beta f$ is monotone. □

Theorems 3.6 and 3.8 allow us to obtain immediately the following:

Theorem 3.9. *The maximal Tychonoff compactification $\beta f : \beta_f X \to Y$ of a Tychonoff mapping $f : X \to Y$ is a perfect extension of f.*

Remark. If X is a Tychonoff space and $|Y| = 1$ then, for the maximal Tychonoff compactification $\beta f : \beta_f X \to Y$ and for the maximal Hausdorff compactification $\chi f : \chi_f X \to Y$, $\beta_f X$ and $\chi_f X$ coincide with the Stone-Čech compactification βX of X and so Theorems 3.1 and 3.9 give us as simple corollary the following proposition for spaces proved in [S].

Theorem 3.10. *The Stone-Čech compactification of a Tychonoff space X is a perfect extension of X.*

Theorems 3.6, 3.9 and Corollary 3.3 imply

Theorem 3.11. *A Tychonoff compactification bf of a Tychonoff mapping f is perfect if and only if the canonical morphism $\mu_b : \beta f \to bf$ is monotone.*

Acknowledgments. The authors would like to thank the referee for his very careful work.
The first author wishes to express his deep gratitude to aunt Nuccia and uncle Bruno for their affectionate support during the final preparation of this paper.

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(*Received* February 23, 1999, *revised* January 24, 2000)