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A-loops close to code loops are groups

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Abstract. Let Q be a diassociative A-loop which is centrally nilpotent of class 2 and which is not a group. Then the factor over the centre cannot be an elementary abelian 2-group.

Keywords: A-loop, central nilpotency, Osborn problem

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This modest note concerns diassociative A-loops that are centrally nilpotent of class 2. Its basic result can be expressed in the following way:

Proposition 1. *Suppose that Q is a diassociative A-loop with a central subloop N . If Q/N is a group of exponent 2, then Q is a group.*

The result is an offshoot of my interest in code loops and I hope that it will help to stimulate further research in the indicated direction.

It can be also understood in the context of the *Osborn problem*: Decide if every (finite) diassociative A-loop is a Moufang loop.

Osborn [6] solved this problem affirmatively for commutative, diassociative A-loops, and some progress in the general case has been recently reported by J.D. Phillips [7].

A subloop N of a loop Q is *central*, if all its elements associate and commute with all elements of Q . Suppose that $Q/N \simeq V$ and that N and V are fixed. The group N is abelian by definition, and we shall assume that V is an abelian group as well (though some of our initial observations can be easily generalized to the case of non-abelian groups). The loop Q is obviously isomorphic to one of the loops $Q(\vartheta)$, where $\vartheta : V \times V \rightarrow N$ is a mapping with $\vartheta(u, 0) = \vartheta(0, u) = 0$ for all $u \in V$, and where the binary operation of $Q(\vartheta)$ is defined by

$$(a, u) \cdot (b, v) = (a + b + \vartheta(u, v), u + v),$$

for all $a, b \in N$ and $u, v \in V$.

A loop Q is said to be a *code loop* ([5], [4], [1]), if $|N| = 2$ and V is an elementary abelian 2-group. Statements about code loops are often proved by computations in $Q(\vartheta)$, and this approach will be used also here. (It is clear that one could

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construct a proof which would not resort to ϑ -techniques. However, it is not clear if such proofs would be more transparent or more readable. Description of extensions by means of factor systems ϑ is for loops and quasigroups very natural and has been used many times, starting from [2].)

For a loop Q and $\alpha \in Q$ denote by L_α the left translation $\beta \mapsto \alpha\beta$, $\beta \in Q$, and by R_α the right translation $\beta \mapsto \beta\alpha$, $\beta \in Q$. Call Q a *left* (or *right*, or *middle*) *A-loop*, if $L_{\alpha\beta}^{-1}L_\alpha L_\beta$ (or $R_{\beta\alpha}^{-1}R_\alpha R_\beta$, or $R_\alpha^{-1}L_\alpha$) is an automorphism for all $\alpha, \beta \in Q$, respectively. A loop Q is an *A-loop* if it is simultaneously a left A-loop, a right A-loop and a middle A-loop. The primary source for A-loops is [3].

Fix now $\vartheta : V \times V \rightarrow N$, $\vartheta(u, 0) = \vartheta(0, u) = 0$ for all $u \in V$, and for all $u, v, w \in V$ put

$$C(u, v) = \vartheta(u, v) - \vartheta(v, u), \quad \text{and}$$

$$A(u, v, w) = -\vartheta(u, v) + \vartheta(u, v + w) - \vartheta(u + v, w) + \vartheta(v, w).$$

Suppose that $Q = Q(\vartheta)$, $\alpha = (a, u)$ and $\beta = (b, v)$. Then $L_\alpha L_\beta$ sends (c, x) to $(a + b + c + \vartheta(u, v + x) + \vartheta(v, x), u + v + x)$, and $L_{\alpha\beta}$ sends (c, x) to $(a + b + c + \vartheta(u, v) + \vartheta(u + v, x), u + v + x)$. Hence $L_\alpha L_\beta$ equals $L_{\alpha\beta}$ if and only if $A(u, v, x) = 0$ for all $x \in V$, and we see that $Q(\vartheta)$ is a group if and only if $A(u, v, w) = 0$ for all $u, v, w \in V$.

It is now also clear that $L_{\alpha\beta}^{-1}L_\alpha L_\beta(c, x)$ equals $(A(u, v, x) + c, x)$ for all $(c, x) \in Q$. Put $\varphi = L_{\alpha\beta}^{-1}L_\alpha L_\beta$ and consider $\gamma = (c, x) \in Q$ and $\delta = (d, y) \in Q$. Then $\varphi(\gamma)\varphi(\delta)$ equals $(A(u, v, x) + A(u, v, y) + c + d + \vartheta(x, y), x + y)$, while $\varphi(\gamma\delta)$ is equal to $(A(u, v, x + y) + c + d + \vartheta(x, y), x + y)$. Comparison of the both results yields claim (i) of the following lemma. Claims (ii) and (iii) can be obtained in a similar way.

Lemma 1. (i) $Q(\vartheta)$ is a left A-loop if $A(u, v, x + y) = A(u, v, x) + A(u, v, y)$ for all $x, y, u, v \in V$;

(ii) $Q(\vartheta)$ is a right A-loop if $A(x + y, v, u) = A(x, v, u) + A(y, v, u)$ for all $x, y, u, v \in V$;

(iii) $Q(\vartheta)$ is a middle A-loop if $C(u, x + y) = C(u, x) + C(u, y)$ for all $x, y, u \in V$.

The loop $Q(\vartheta)$ is a group, if $A(u, v, w) = 0$ for all $u, v, w \in V$. If $Q(\vartheta)$ is a left A-loop, then each pair $u, v \in V$ determines, by Lemma 1 (i), a group homomorphism $V \rightarrow N$. All such homomorphisms are trivial if and only if the loop $Q(\vartheta)$ is a group. Since finite groups of coprime order admit only trivial homomorphisms, Lemma 1 yields the following corollary:

Corollary 1. *Let Q be a finite centrally nilpotent loop of class 2 which is not a group, and let Z be its centre. If $|Z|$ and $|Q/Z|$ are coprime, then Q is neither a left A-loop, nor a right A-loop. Furthermore, if $|Z|$ and $|Q/Z|$ are coprime, then Q is a middle A-loop if and only if it is commutative.*

It will be useful to express the equation of Lemma 1 (i) just in terms of ϑ . We

obtain the equality:

$$\begin{aligned}
 (1) \quad & \vartheta(u, v) + \vartheta(u, v + x + y) - \vartheta(u + v, x + y) + \vartheta(v, x + y) \\
 & - \vartheta(u, v + x) + \vartheta(u + v, x) - \vartheta(v, x) \\
 & - \vartheta(u, v + y) + \vartheta(u + v, y) - \vartheta(v, y) = 0.
 \end{aligned}$$

Lemma 2. *Suppose that V is of exponent 2. Then $Q(\vartheta)$ is diassociative if and only if*

$$\begin{aligned}
 \vartheta(u, u + v) = \vartheta(u, u) - \vartheta(u, v), \quad \vartheta(u + v, u) = \vartheta(u, u) - \vartheta(v, u), \quad \text{and} \\
 2\vartheta(v, u) = 2\vartheta(u, v)
 \end{aligned}$$

are true for all $u, v \in U$.

PROOF: Consider $u, v \in V$. By our assumption, $2u = 2v = 0$, and hence the equations $A(u, u, v) = 0$ and $A(v, u, u) = 0$ are equivalent to equations $\vartheta(u, u) - \vartheta(u, u + v) - \vartheta(u, v) = 0$ and $-\vartheta(v, u) - \vartheta(v + u, u) + \vartheta(u, u) = 0$, respectively. If these equalities are true, then $A(u, v, u) = 0$ is equivalent to $2\vartheta(v, u) = 2\vartheta(u, v)$, as $A(u, v, u) = -\vartheta(u, v) + \vartheta(u, v + u) - \vartheta(u + v, u) + \vartheta(v, u) = (\vartheta(v, u) - \vartheta(u, v)) + (\vartheta(u, u) - \vartheta(u, v) - \vartheta(u, u) + \vartheta(v, u)) = 2\vartheta(v, u) - 2\vartheta(u, v)$.

It is now clear that the direct implication of the lemma holds. To prove the converse implication means to show that the ϑ -equalities imply $A(u, v, w) = 0$ whenever u, v and w belong to a subgroup of V with ≤ 4 elements. One clearly has $A(u, v, w) = 0$ if $0 \in \{u, v, w\}$. From the first part of the proof one gets the cases $u = v, v = w$ and $u = w$, and so the only remaining case is the case when $0, u, v$ and w are pairwise different. However, the assumed equalities yield $A(u, u + v, v) = -\vartheta(u, u + v) + \vartheta(u, u) - \vartheta(v, v) + \vartheta(u + v, v) = \vartheta(u, v) - \vartheta(u, v) = 0$. □

We are now ready to prove Proposition 1. Assume that V is of exponent 2 and that $Q(\vartheta)$ is a diassociative A-loop.

Consider (1) for the case $v + x + y = 0$, and note that the first row of (1) then yields $\vartheta(u, v) - \vartheta(u + v, v) + \vartheta(v, v)$, which is, by Lemma 2, equal to $2\vartheta(u, v) = 2\vartheta(u, x + y)$. Furthermore, $\vartheta(u, v + x) + \vartheta(u, v + y)$ equals $\vartheta(u, x) + \vartheta(u, y)$, and $\vartheta(v, x) + \vartheta(v, y) - \vartheta(u + v, x) - \vartheta(u + v, y)$ equals $\vartheta(x + y, x) + \vartheta(x + y, y) - \vartheta(x + y + u, x) - \vartheta(y + x + u, y) = \vartheta(x, x) - \vartheta(y, x) + \vartheta(y, y) - \vartheta(x, y) - \vartheta(x, x) + \vartheta(y + u, x) - \vartheta(y, y) + \vartheta(x + u, y) = -\vartheta(y, x) - \vartheta(x, y) + \vartheta(y + u, x) + \vartheta(x + u, y)$. Hence

$$\begin{aligned}
 (2) \quad & 2\vartheta(u, x + y) = \\
 & \vartheta(u, x) - \vartheta(y, x) + \vartheta(u, y) - \vartheta(x, y) + \vartheta(y + u, x) + \vartheta(x + u, y)
 \end{aligned}$$

holds for all $x, y, u \in V$.

By Lemma 1 (iii), $C(x, u + y)$ equals $C(x, u) + C(x, y)$, and hence $\vartheta(y + u, x) = \vartheta(x, u + y) + \vartheta(u, x) - \vartheta(x, u) + \vartheta(y, x) - \vartheta(x, y)$. Substitute now this expression of

$\vartheta(y+u, x)$ to (2). One gets $2\vartheta(u, x+y) = 2\vartheta(u, x) + \vartheta(u, y) - 2\vartheta(x, y) + \vartheta(x, u+y) + \vartheta(x+u, y) - \vartheta(x, u)$, and so we see that

$$(3) \quad 2(-\vartheta(u, x) + \vartheta(u, x+y) + \vartheta(x, y)) = \\ -\vartheta(x, u) + \vartheta(x, u+y) + \vartheta(x+u, y) + \vartheta(u, y)$$

holds for all $u, x, y \in V$.

By adding $-2\vartheta(x+u, y)$ to the both sides of (3), we obtain

$$2A(u, x, y) = A(x, u, y) \quad \text{for all } x, y, u \in V.$$

However, $2A(u, x, y) = A(u, x, y) + A(u, x, y) = A(u, x, y+y) = A(u, x, 0) = 0$ by Lemma 1(i), and so we see that $A(x, u, y)$ is always zero, and that is what was needed to prove Proposition 1.

Note that our proof did not use the fact that Q is a right A-loop. In our last statement we give an explanation.

Proposition 2. *Let Q be a loop that is centrally nilpotent of class 2. If Q is simultaneously a left A-loop and a middle A-loop, then it is also a right A-loop.*

PROOF: One can obviously assume that Q equals $Q(\vartheta)$. When the equality $A(x+y, v, u) - A(x, v, u) - A(y, v, u) = 0$ is expressed in terms of ϑ , one gets

$$(4) \quad -\vartheta(v, u) - \vartheta(x+y+v, u) + \vartheta(x+y, u+v) - \vartheta(x+y, v) \\ + \vartheta(x+v, u) - \vartheta(x, u+v) + \vartheta(x, v) \\ + \vartheta(y+v, u) - \vartheta(y, u+v) + \vartheta(y, v) = 0.$$

Summing up (1) and (4) one obtains

$$(5) \quad C(u, v) + C(u, v+x+y) - C(u+v, x+y) + C(v, x+y) \\ - C(u, v+x) + C(u+v, x) - C(v, x) \\ - C(u, v+y) + C(u+v, y) - C(v, y) = 0.$$

Since $C(x, u+v) = C(x, u) + C(x, v)$ and $C(u+v, x) = C(u, x) + C(v, x)$ are true, by Lemma 1(iii), for all $x, u, v \in V$, we see that (5) is satisfied. Since (1) holds by Lemma 1(i), the equality (4) must be also true (for all $x, y, u, v \in V$), and hence Q is a right A-loop, by Lemma 1(ii). \square

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