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Commentationes Mathematicae Universitatis Carolinae, Vol. 41 (2000), No. 1, 9--24

Persistent URL: <http://dml.cz/dmlcz/119137>

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Totality of product completions

JIŘÍ ADÁMEK, LURDES SOUSA, WALTER THOLEN

Abstract. Categories whose Yoneda embedding has a left adjoint are known as total categories and are characterized by a strong cocompleteness property. We introduce the notion of multitotal category \mathcal{A} by asking the Yoneda embedding $\mathcal{A} \rightarrow [\mathcal{A}^{op}, \mathit{Set}]$ to be right multiadjoint and prove that this property is equivalent to totality of the formal product completion $\Pi\mathcal{A}$ of \mathcal{A} . We also characterize multitotal categories with various types of generators; in particular, the existence of dense generators is inherited by the formal product completion iff measurable cardinals cannot be arbitrarily large.

Keywords: multitotal category, multisolid functor, formal product completion

Classification: 18A05, 18A22, 18A40

1. Introduction

The concept of totality, introduced by Street and Walters [15], is a strong property of categories (implying completeness and cocompleteness — and more, see [14], [11]) which, nevertheless, most “current” categories enjoy. Recall that a category \mathcal{A} is called *total* if its Yoneda embedding $Y_{\mathcal{A}} : \mathcal{A} \rightarrow [\mathcal{A}^{op}, \mathit{Set}]$ is right adjoint. Since solid functors (i.e., “good” faithful, right adjoint functors) $\mathcal{A} \rightarrow \mathcal{X}$ lift totality from \mathcal{X} to \mathcal{A} (see [17]), totality of Set and of its small-indexed powers is responsible for the totality of many important types of categories. For example, it allows us to conclude that all cocomplete, cowellpowered categories with a generator are total. These include locally presentable categories and monadic or topological categories over Set .

In the present paper we investigate the totality of the free product completion $\Pi\mathcal{A}$ of a category \mathcal{A} (dual to the free coproduct completion $\mathit{Fam}\mathcal{A}^{op}$). The motivation is to describe the appropriate strong property of categories which are not cocomplete, but only *multicocomplete*, i.e., every small diagram has a multicolimit (as introduced by Y. Diers); examples are the category of linearly ordered sets, the category of fields, the category of local rings, all locally multipresentable categories in the sense of Diers [8], etc. Usually, “multi-concepts” for \mathcal{A} are easily seen to be equivalent to the corresponding concepts for $\Pi\mathcal{A}$, e.g., a category \mathcal{A} is

* Supported by the Grant Agency of the Czech Republic, grant no. 201/99/0310.

† The hospitality of York University and financial support by NSERC and by Center of Mathematics of the University of Coimbra is gratefully acknowledged.

‡ The hospitality of Technische Universität of Braunschweig and partial financial assistance by NSERC is gratefully acknowledged.

multicocomplete iff $\Pi\mathcal{A}$ is cocomplete, and a functor $F : \mathcal{A} \rightarrow \mathcal{B}$ is *right multiadjoint* iff $\Pi F : \Pi\mathcal{A} \rightarrow \Pi\mathcal{B}$ is right adjoint, see [7]. We call a category *multitotal* if its Yoneda embedding $Y_{\mathcal{A}}$ is right multiadjoint; the question naturally arising is whether \mathcal{A} is multitotal iff $\Pi\mathcal{A}$ is total. The affirmative answer in Theorem 3.6 of this paper unexpectedly turns out to have a somewhat involved proof. This theorem enables us to establish quite easily analogues of the results of [17], [3] in the “multi” context. Namely, we are able to characterize multitotal categories with various types of generators; in particular the above mentioned examples of multicocomplete categories are all multitotal. In fact, any multicocomplete, cocomplete category with a generator is multitotal. While it is easy to see that the existence of a (strong or regular) generator in a category with a multi-initial object gives the same for its product completion, the corresponding property for dense generator turns out to be more involved. In fact, the question of whether a dense generator exists in $\Pi(\mathcal{A})$ whenever it exists in \mathcal{A} depends on the set-theoretical assumption (M) that measurable cardinals are not arbitrarily large. One direction follows from Isbell’s result in [10] that Set^{op} has a dense generator iff (M) holds; the converse direction is more difficult.

Analogously to the role of solid functors for totality, multisolid functors play an essential role in detecting multitotal categories. Multisolid functors $U : \mathcal{A} \rightarrow \mathcal{X}$ were already introduced (under a different name) and characterized in [18] by the property that the product-preserving extension $\Pi U : \Pi\mathcal{A} \rightarrow \Pi\mathcal{X}$ is solid. For \mathcal{X} multicocomplete and \mathcal{A} cocomplete, they are precisely the faithful right multiadjoint functors for which \mathcal{A} is multicocomplete, as shown more recently in [13].

2. Review of total categories and solid functors

2.1. A diagram $H : \mathcal{D} \rightarrow \mathcal{A}$ in a category \mathcal{A} is said to be *small-partitioned* [12] if for all $A \in \mathcal{A}$ the comma category $(A \downarrow H)$ has only a small set of connected components. Thus, every small diagram is small-partitioned. An example of a (generally large) small-partitioned diagram is the diagram of elements of any functor from \mathcal{A} to Set . Recall from [11] that the following conditions are equivalent for \mathcal{A} :

- (i) \mathcal{A} is total, i.e., the Yoneda embedding $Y_{\mathcal{A}} : \mathcal{A} \rightarrow [\mathcal{A}^{op}, Set]$ has a left adjoint;
- (ii) $\text{colim } H$ exists in \mathcal{A} whenever the diagram $H : \mathcal{D} \rightarrow \mathcal{A}$ is small-partitioned;
- (iii) $\text{colim } H$ exists in \mathcal{A} whenever, for all $A \in \mathcal{A}$, $\text{colim } \mathcal{A}(A, H-)$ exists in Set .

2.2. Total categories are trivially cocomplete (i.e., have colimits of all small diagrams), but they are also complete — indeed, they are “as complete as a category with small hom-sets can possibly be”. In fact, recall that a category \mathcal{A} is called *hypercomplete* [5] if every diagram $H : \mathcal{D} \rightarrow \mathcal{A}$ for which $\text{lim } \mathcal{A}(A, H-)$ exists in Set for all $A \in \mathcal{A}$, has a limit in \mathcal{A} . The following conditions are equivalent ([4]):

- (i) \mathcal{A} is hypercomplete;

- (ii) $\lim H$ exists in \mathcal{A} for every diagram $H : \mathcal{D} \rightarrow \mathcal{A}$ with the property that for all $A \in \mathcal{A}$, there is only a small set of cones $\Delta A \rightarrow H$ in \mathcal{A} .

Every total category is hypercomplete, see [4].

2.3. Since solid functors detect totality, we briefly recall this notion. For a functor $U : \mathcal{A} \rightarrow \mathcal{X}$, a U -sink σ with codomain $X \in \mathcal{X}$ is a (possibly large) family of \mathcal{A} -objects A_i and of \mathcal{X} -morphisms $x_i : UA_i \rightarrow X$ ($i \in I$). Let $(X \downarrow U)_\sigma$ be the full subcategory of $(X \downarrow U)$ of all objects $y : X \rightarrow UB$ such that for every $i \in I$ there exists a morphism $f_i : A_i \rightarrow B$ in \mathcal{A} with $Uf_i = y \cdot x_i$. The functor U is *solid* (formerly semi-topological [16]) if U is faithful and if for every U -sink σ the category $(X \downarrow U)_\sigma$ has an initial object. The following conditions are equivalent for every functor $U : \mathcal{A} \rightarrow \mathcal{X}$ (see [16], [4], [11]):

- (i) U is solid;
- (ii) U has a left adjoint, and there is a class \mathcal{E} of morphisms in \mathcal{A} containing all isomorphisms and being closed under composition with them, such that
 1. the counits of U belong to \mathcal{E} ;
 2. \mathcal{A} is \mathcal{E} -cocomplete, that is: a. the pushout of a morphism in \mathcal{E} along any morphism exists in \mathcal{A} and belongs to \mathcal{E} , and b. the cointersection of a (possibly large) family of morphisms in \mathcal{E} with common domain exists in \mathcal{A} and belongs to \mathcal{E} .

In particular, every faithful right adjoint functor $U : \mathcal{A} \rightarrow \mathcal{X}$ defined on a cocomplete and cowellpowered category \mathcal{A} is solid. The connection with totality is given by the following

Theorem. *Let $U : \mathcal{A} \rightarrow \mathcal{X}$ be a functor. Then:*

- (1) *If \mathcal{X} is total and U solid, then also \mathcal{A} is total (see [17]).*
- (2) *If \mathcal{A} is total and U faithful and right adjoint, then U is solid (see [3]).*

Recall that a *generator* in a category \mathcal{A} is a set \mathcal{G} of objects such that the canonical functor $\mathcal{A} \rightarrow \text{Set}^{\mathcal{G}}$, $A \mapsto (\mathcal{A}(G, A))_{G \in \mathcal{G}}$, is faithful. Since *Set* and all its small-indexed powers are total, the above theorem shows in particular:

Corollary ([4]). *Every cocomplete, cowellpowered category with a generator is total.*

2.4. Recall that the *free product completion* $\Pi\mathcal{A}$ of a category \mathcal{A} has objects $A = (A_i)_{i \in I}$ given by small-indexed families of \mathcal{A} -objects A_i , and a morphism $f : A \rightarrow B = (B_j)_{j \in J}$ in $\Pi\mathcal{A}$ is given by a function $\varphi : J \rightarrow I$ and a family $f_j : A_{\varphi(j)} \rightarrow B_j$ ($j \in J$) of \mathcal{A} -morphisms (with the obvious composition and identity maps), see [7]. Writing $SA = I$ and $Sf = \varphi$, one has a functor

$$S : (\Pi\mathcal{A})^{op} \longrightarrow \text{Set}.$$

Whenever necessary we write $S_{\mathcal{A}}$ instead of S for distinction. In the terminology of [6], $\Pi\mathcal{A} = (\text{Fam}(\mathcal{A}^{op}))^{op}$. We denote by

$$J_{\mathcal{A}} : \mathcal{A} \longrightarrow \Pi\mathcal{A}$$

the canonical embedding which identifies objects of \mathcal{A} with singleton families. Every functor $F : \mathcal{A} \rightarrow \mathcal{B}$ into a category with products has a product-preserving extension $\overline{F} : \Pi\mathcal{A} \rightarrow \mathcal{B}$ which is unique up to natural isomorphism; hence, $[\mathcal{A}, \mathcal{B}]$ is equivalent to the full subcategory of product-preserving functors in $[\Pi\mathcal{A}, \mathcal{B}]$.

2.5. A *multicolimit* of a diagram $H : \mathcal{D} \rightarrow \mathcal{A}$ in \mathcal{A} is given by a colimit of $J_{\mathcal{A}}H$ in $\Pi\mathcal{A}$; hence, it is a small-indexed family $(L_i)_{i \in I}$ of \mathcal{A} -objects together with cocones $\lambda_i : H \rightarrow \Delta L_i$, such that every cocone $H \rightarrow \Delta B$ in \mathcal{A} factors uniquely through λ_i , for a unique $i \in I$. Already Diers [7] proved that every small diagram in \mathcal{A} has a multicolimit if and only if $\Pi\mathcal{A}$ is cocomplete. As we shall deal with large diagrams, we need a precise analysis of this result. For $H : \mathcal{D} \rightarrow \Pi\mathcal{A}$ first assume that the limit $K = \lim SH^{op}$ exists in Set ; hence, every element $\alpha \in K$ is given by a compatible family of elements $\alpha_D \in SHD$, $D \in \mathcal{D}$. Every $\alpha \in K$ defines a diagram

$$H_{\alpha} : \mathcal{D} \rightarrow \mathcal{A} \quad \text{with} \quad H_{\alpha}D = (HD)_{\alpha_D}, \quad H_{\alpha}d = (Hd)_{\alpha_{D'}},$$

for all $d : D \rightarrow D'$ in \mathcal{D} . (Note that $H_{\alpha}d$ is well-defined since $(SHd)(\alpha_{D'}) = \alpha_D$.)

Lemma ([7]). *A diagram $H : \mathcal{D} \rightarrow \Pi\mathcal{A}$ has a colimit in $\Pi\mathcal{A}$ if SH^{op} has a limit K in Set and H_{α} has a multicolimit in \mathcal{A} , for every $\alpha \in K$.*

PROOF: For every $\alpha \in K$, a multicolimit of H_{α} is given by a small family of cocones $\lambda_{\alpha,i} : H_{\alpha} \rightarrow \Delta L_{\alpha,i}$, $i \in I_{\alpha}$. With $L = (L_{\alpha,i})_{\alpha \in K, i \in I_{\alpha}}$, this defines a cocone $\lambda : H \rightarrow \Delta L$ when we put

$$(\lambda_D)_{\alpha,i} = (\lambda_{\alpha,i})_D : (HD)_{\alpha_D} = H_{\alpha}D \rightarrow L_{\alpha,i}$$

for every $D \in \mathcal{D}$. In order to see that every cocone $\beta : H \rightarrow \Delta B$ factors uniquely through $\lambda_{\alpha,i}$, for a unique pair (α, i) , we may without loss of generality assume $B \in \mathcal{A}$. The naturality of the family β_D ($D \in \mathcal{D}$) defines a uniquely determined element $\alpha \in K$, and the morphisms $\beta_D : (HD)_{\alpha_D} \rightarrow B$ define in fact a cocone $\overline{\beta} : H_{\alpha} \rightarrow \Delta B$. Hence, there are uniquely defined $i \in I_{\alpha}$ and $f : L_{\alpha,i} \rightarrow B$ in \mathcal{A} with $\overline{\beta} = \Delta f \cdot \lambda_{\alpha,i}$, which gives the desired factorization $\beta = \Delta f \cdot \lambda$. \square

2.6. A functor $U : \mathcal{A} \rightarrow \mathcal{X}$ is *right multiadjoint* if its extension

$$\Pi U : \Pi\mathcal{A} \longrightarrow \Pi\mathcal{X}$$

with $\Pi U \cdot J_{\mathcal{A}} = J_{\mathcal{X}} \cdot U$ is right adjoint ([7]). Analogously one defines U to be *multisolid* (“strongly localizing semitopological” in [18]) if ΠU is solid; this means that U is faithful, and that for every U -sink σ as in 2.3 the category $(X \downarrow U)_{\sigma}$ has a multi-initial object (i.e., a multicolimit of the empty diagram).

The characterization 2.3 of solid functors should lead to a characterization of multisolid functors when we exploit it for ΠU in lieu of U . In fact, let us agree that a *counit* of a right multiadjoint functor $U : \mathcal{A} \rightarrow \mathcal{X}$ at $A \in \mathcal{A}$ is simply the counit

of ΠU at $A \in \Pi \mathcal{A}$. For a class \mathcal{E} of morphisms in \mathcal{A} we call \mathcal{A} *multi- \mathcal{E} -cocomplete* if the multipushout of a morphism in \mathcal{E} along any morphism exists in \mathcal{A} , with every component of it belonging to \mathcal{E} , and if the multicointersection of any family of morphisms in \mathcal{E} with common domain exists in \mathcal{A} , with every component of it belonging to \mathcal{E} . Hence, if \mathcal{A} is multi- \mathcal{E} -cocomplete, then $\Pi \mathcal{A}$ is \mathcal{E}^Π -cocomplete, with \mathcal{E}^Π those morphisms of $\Pi \mathcal{A}$ whose components lie in \mathcal{E} ; conversely if $\Pi \mathcal{A}$ is \mathcal{F} -cocomplete for a class of morphisms in $\Pi \mathcal{A}$, then \mathcal{A} is multi- \mathcal{F}^1 -cocomplete, with \mathcal{F}^1 the class of those morphisms of \mathcal{A} which appear as a component of some morphism in \mathcal{F} . These observations prove the following results which essentially appeared in [13]:

Proposition. *Equivalent are for a functor $U : \mathcal{A} \rightarrow \mathcal{X}$:*

- (i) $U : \mathcal{A} \rightarrow \mathcal{X}$ is *multisolid*;
- (ii) $U : \mathcal{A} \rightarrow \mathcal{X}$ is *right multiadjoint*, and there is a class \mathcal{E} of morphisms in \mathcal{A} containing all isomorphisms and being closed under composition with them, such that
 1. the counits of U lie in \mathcal{E} ,
 2. \mathcal{A} is multi- \mathcal{E} -cocomplete.

Corollary ([13]). *If \mathcal{X} is multicocomplete and \mathcal{A} cowellpowered, a faithful functor $U : \mathcal{A} \rightarrow \mathcal{X}$ is multisolid if and only if \mathcal{A} is multicocomplete.*

3. Multitotal categories

3.1 Definition. A category \mathcal{A} is called *multitotal* if the Yoneda embedding $Y_{\mathcal{A}} : \mathcal{A} \rightarrow [\mathcal{A}^{op}, Set]$ is right multiadjoint.

It is easy to prove a “multiversion” of the characterization 2.1 of total categories:

3.2 Proposition. *The following conditions are equivalent for a category \mathcal{A} :*

- (i) \mathcal{A} is *multitotal*;
- (ii) every *small-partitioned diagram* $H : \mathcal{D} \rightarrow \mathcal{A}$ has a *multicolimit* in \mathcal{A} ;
- (iii) every diagram $H : \mathcal{D} \rightarrow \mathcal{A}$ for which $\text{colim } \mathcal{A}(A, H-)$ exists in *Set* for all $A \in \mathcal{A}$, has a *multicolimit* in \mathcal{A} .

PROOF: (i) \Rightarrow (ii). A small-partitioned diagram $H : \mathcal{D} \rightarrow \mathcal{A}$ defines a functor $E : \mathcal{A}^{op} \rightarrow Set$ which assigns to an object A the set of connected components of $(A \downarrow H)$. A multicolimit $\lambda_i : H \rightarrow \Delta L_i$ ($i \in I$) is obtained from a $\Pi Y_{\mathcal{A}}$ -universal arrow $(\eta_i : E \rightarrow Y_{\mathcal{A}} L_i)_{i \in I}$ for $E \in [\mathcal{A}^{op}, Set]$: one just evaluates $(\eta_i)_{HD} : EHD \rightarrow \mathcal{A}(HD, L_i)$ at the component $(D, 1_{HD})$ in $(HD \downarrow H)$ to define $(\lambda_i)_D$, for every $i \in I$ and $D \in \mathcal{D}$.

(ii) \Rightarrow (i). For any $E \in [\mathcal{A}^{op}, Set]$ one considers a multicolimit $(\lambda_i)_{i \in I}$ of the small-partitioned forgetful functor $H : \text{el } E \rightarrow \mathcal{A}$, where $\text{el } E$ is the “element category” with objects (A, x) , $A \in \mathcal{A}$, $x \in EA$. Then the $\Pi Y_{\mathcal{A}}$ -universal arrow

$(\eta_i)_{i \in I}$ for E is obtained as $(\eta_i)_A(x) = (\lambda_i)_{(A,x)}$, for every $i \in I$, $A \in \mathcal{A}$ and $x \in EA$.

(ii) \Leftrightarrow (iii) follows from the fact that, given a diagram $H : \mathcal{D} \rightarrow \mathcal{A}$ and $A \in \mathcal{A}$, $\text{colim } \mathcal{A}(A, H-)$ exists in Set iff $(A \downarrow H)$ has just a set of connected components. \square

3.3 Corollary. *If $\Pi\mathcal{A}$ is total, then \mathcal{A} is multitotal.*

PROOF: We check condition (ii) of 3.2. Given a small-partitioned diagram $H : \mathcal{D} \rightarrow \mathcal{A}$, it is easy to see that then also $J_{\mathcal{A}}H : \mathcal{D} \rightarrow \Pi\mathcal{A}$ is small-partitioned, so that $\text{colim } J_{\mathcal{A}}H$ exists in $\Pi\mathcal{A}$, by hypothesis. But this is, by definition, a multicolimit of H in \mathcal{A} . \square

3.4. The question which remains is whether multitotality of \mathcal{A} is also a sufficient condition for $\Pi\mathcal{A}$ to be total. In other words: does right adjointness of $\Pi Y_{\mathcal{A}}$ imply right adjointness of $Y_{\Pi\mathcal{A}}$? For every category \mathcal{A} denote by $[(\Pi\mathcal{A})^{op}, \text{Set}]_{\Pi}$ the full subcategory of $[(\Pi\mathcal{A})^{op}, \text{Set}]$ of all coproduct-preserving functors; then one has a functor Σ which makes the upper triangle of the diagram (1) below commutative (up to natural isomorphism):

$$(1) \quad \begin{array}{ccc} & \Pi\mathcal{A} & \\ \Pi Y_{\mathcal{A}} \swarrow & & \searrow Y_{\Pi\mathcal{A}} \\ \Pi[\mathcal{A}]^{op}, \text{Set} & \xrightarrow{\Sigma} & [(\Pi\mathcal{A})^{op}, \text{Set}] \\ \uparrow \text{J} & & \uparrow \text{J} \\ [\mathcal{A}]^{op}, \text{Set} & \xrightarrow{\sim} & [(\Pi\mathcal{A})^{op}, \text{Set}]_{\Pi} \end{array}$$

Σ is the product-preserving extension of the functor that assigns to $E \in [\mathcal{A}^{op}, \text{Set}]$ the functor ΣE with $(\Sigma E)(C_k)_K = \prod_{k \in K} EC_k$; of course, $\Sigma E \in [(\Pi\mathcal{A})^{op}, \text{Set}]_{\Pi}$. It is clear that the restriction of Σ creates an equivalence of categories, as indicated in (1). Now it is a straightforward exercise to show that the existence of a left adjoint to $\Pi Y_{\mathcal{A}}$ gives a $(Y_{\Pi\mathcal{A}})$ -universal arrow for every $F \in [\mathcal{A}^{op}, \text{Set}]_{\Pi}$, hence a “partial left adjoint” to $Y_{\Pi\mathcal{A}}$. We now establish the existence of “a totally defined” left adjoint.

3.5 Example. The category ΠSet is total. To see this, we check the conditions of Corollary 2.3. First observe that ΠSet is certainly cocomplete (see 2.5). Now, a morphism $f : A \rightarrow B$ in ΠSet is an epimorphism if and only if $\varphi = Sf : SB \rightarrow SA$ is injective and every map $f_j : A_{\varphi(j)} \rightarrow B_j$ ($j \in J = SB$) is epic in Set (see 6.3 in [18]). This characterization shows that wellpoweredness and cowellpoweredness of Set give cowellpoweredness of ΠSet . Finally, ΠSet has a (single-object) generator, namely the triple $(1, \emptyset, \emptyset)$: the singleton set 1 and (one copy of) the empty set are needed to distinguish distinct morphisms $f, g : A \rightarrow B$ in ΠSet with $Sf = Sg$, and two copies of \emptyset are needed to distinguish f and g in case $Sf \neq Sg$.

3.6 Proposition. *The functor $S_{\mathcal{A}} : (\Pi\mathcal{A})^{op} \rightarrow \mathit{Set}$*

- *has a left adjoint $L_{\mathcal{A}}$ if \mathcal{A} has a terminal object;*
- *has a right adjoint $R_{\mathcal{A}}$ if \mathcal{A} has a multi-initial object.*

PROOF: For 1 terminal in \mathcal{A} and every set I , let $L_{\mathcal{A}}I = (1)_I$ be the constant I -indexed family with value 1. Then $\text{id}_I : I \rightarrow S_{\mathcal{A}}L_{\mathcal{A}}I$ serves as an $S_{\mathcal{A}}$ -universal arrow. For an initial object $O = (O_t)_{t \in T}$ in $\Pi\mathcal{A}$ and every set I , let $R_{\mathcal{A}}I = (O_i)_{(t,i) \in T \times I}$. Then the projection $S_{\mathcal{A}}R_{\mathcal{A}}I \rightarrow I$ is an $S_{\mathcal{A}}$ -couniversal arrow. \square

3.7 Theorem. *A category \mathcal{A} is multitotal if and only if $\Pi\mathcal{A}$ is total.*

PROOF: We need to prove necessity (see 3.3). To this end we assume \mathcal{A} to be multitotal and prove totality of $\Pi\mathcal{A}$ by checking condition 2.1(ii). Hence, let $H : \mathcal{D} \rightarrow \Pi\mathcal{A}$ be a small-partitioned diagram in $\Pi\mathcal{A}$. From 3.6 one has adjoint situations

$$L_{\mathit{Set}} \dashv S_{\mathit{Set}} \quad \text{and} \quad S_{\mathcal{A}} \dashv R_{\mathcal{A}},$$

and we can form

$$\overline{H} = L_{\mathit{Set}}^{op} S_{\mathcal{A}}^{op} H : \mathcal{D} \rightarrow \Pi\mathit{Set}.$$

By adjointness, for all $X \in \Pi\mathit{Set}$ there are canonical isomorphisms

$$(X \downarrow \overline{H}) \cong (S_{\mathit{Set}}X \downarrow S_{\mathcal{A}}^{op}H) \cong (R_{\mathcal{A}}S_{\mathit{Set}}X \downarrow H),$$

so that \overline{H} must be small-partitioned since H is. Consequently, $\text{colim} \overline{H}$ exists in $\Pi\mathit{Set}$, by 3.5. This is a limit of \overline{H}^{op} in $(\Pi\mathit{Set})^{op}$, which is preserved by the right adjoint functor S_{Set} . Since

$$S_{\mathit{Set}}\overline{H}^{op} = S_{\mathcal{A}}H^{op},$$

we see that a limit of $S_{\mathcal{A}}H^{op}$ exists in Set .

In order to see that $\text{colim} H$ exists in $\Pi\mathcal{A}$, according to Lemma 2.5 it is sufficient to show that for every $\alpha \in K = \lim S_{\mathcal{A}}H^{op}$ in Set , the diagram $H_{\alpha} : \mathcal{D} \rightarrow \mathcal{A}$ has a multicolimit in \mathcal{A} . In fact, since \mathcal{A} is multitotal, thanks to 3.2 it suffices to prove that H_{α} is small-partitioned. Hence, given $A \in \mathcal{A}$, we must show that $A \downarrow H_{\alpha}$ has only a small set of connected components.

Let $O = (O_t)_{t \in T}$ be initial in $\Pi\mathcal{A}$ and form $\overline{A} = A \times O$ in $\Pi\mathcal{A}$. Every object (D, f) , with $f : A \rightarrow H_{\alpha}D$, of $(A \downarrow H_{\alpha})$ defines an object (D, \overline{f}) of $(\overline{A} \downarrow H)$, as follows: for each $i \in S_{\mathcal{A}}HD - \{\alpha_D\}$, let $\overline{f}_i : O_t \rightarrow (HD)_i$ be the morphism determined by initiality of O , and for $i = \alpha_D$ let $\overline{f}_i = f$. Since H is small-partitioned, $(\overline{A} \downarrow H)$ has only a set of connected components. Hence, it now suffices to show that, for any pair of objects (D, f) , (D', f') in $(A \downarrow H_{\alpha})$, any zig-zag of $(\overline{A} \downarrow H)$ between (D, \overline{f}) , (D', \overline{f}') gives a zig-zag of $(A \downarrow H_{\alpha})$ between (D, f) , (D', f') .

Consider the first step of the zig-zag between \bar{f} , \bar{f}' , which is given by one of the following commutative triangles:

$$(2) \quad \begin{array}{ccc} & \bar{A} & \\ \bar{f} \swarrow & & \searrow g_1 \\ HD & \xrightarrow{Hd_1} & HD_1 \end{array} \qquad \begin{array}{ccc} & \bar{A} & \\ \bar{f} \swarrow & & \searrow g_1 \\ HD & \xleftarrow{Hd_1} & HD_1 \end{array}$$

(a) Case $g_1 = Hd_1 \cdot \bar{f}$. Since $(S_{\mathcal{A}}Hd_1)(\alpha_{D_1}) = \alpha_D$, we know that $S_{\mathcal{A}}g_1 = S_{\mathcal{A}}\bar{f} \cdot S_{\mathcal{A}}Hd_1$ takes α_{D_1} to the index of A . Hence, the morphism $(g_1)_{\alpha_{D_1}} : A \rightarrow (HD_1)_{\alpha_{D_1}} = H_{\alpha}D_1$ gives us the first step of the zig-zag between (D, f) , (D', f') .

(b) Case $\bar{f} = Hd_1 \cdot g_1$. Since $(S_{\mathcal{A}}Hd_1)(\alpha_D) = \alpha_{D_1}$, we know that, again, $S_{\mathcal{A}}g_1$ takes α_{D_1} to the index of A . Hence, also in this case $(g_1)_{\alpha_{D_1}}$ gives the first step of the zig-zag between (D, f) , (D', f') .

Inductively one finishes the proof of the last claim, so that the proof of the Theorem is complete. \square

3.8 Corollary ([18]). *Every multitotal category \mathcal{A} is connectively hypercomplete, that is, every connected diagram $H : \mathcal{D} \rightarrow \mathcal{A}$ for which $\lim \mathcal{A}(A, H-)$ exists in \mathbf{Set} for all $A \in \mathcal{A}$, has a limit in \mathcal{A} .*

PROOF: With 3.7 and 2.2, $\Pi\mathcal{A}$ is total and therefore hypercomplete. Consider a connected diagram $H : \mathcal{D} \rightarrow \mathcal{A}$ with the indicated property; equivalently, with the property that there is only a small set of cones $\Delta A \rightarrow H$ for every $A \in \mathcal{A}$. Since \mathcal{D} is connected, it is easy to see that there is only a small set of cones $\Delta A \rightarrow J_{\mathcal{A}}H$ for every $A \in \Pi\mathcal{A}$, so that $(L_i)_{i \in I} = \lim J_{\mathcal{A}}H$ exists in $\Pi\mathcal{A}$. Again, connectedness of \mathcal{D} determines a unique index i_0 such that $L_{i_0} = \lim H$. \square

Corollary. *Every multitotal category \mathcal{A} has equalizers, pullbacks and intersections of (arbitrarily large) families of monomorphisms.*

3.9. Theorem 3.7 makes it easy to establish the interrelationship between multitotal categories and multisolid functors:

Theorem. *Let $U : \mathcal{A} \rightarrow \mathcal{X}$ be a functor. Then:*

- (1) *if \mathcal{X} is multitotal and U multisolid, then also \mathcal{A} is multitotal;*
- (2) *if \mathcal{A} is multitotal and U faithful and right multiadjoint, then U is multisolid.*

PROOF: (1) The hypotheses together with Theorem 3.7 imply that $\Pi\mathcal{X}$ is total and $\Pi U : \Pi\mathcal{A} \rightarrow \Pi\mathcal{X}$ is solid, so that $\Pi\mathcal{A}$ is total, see Theorem 2.5. Hence, \mathcal{A} is multitotal.

(2) Using again Theorem 3.7 we see that $\Pi\mathcal{A}$ is total and ΠU is faithful and right adjoint, by hypothesis, so that ΠU is solid by Theorem 2.5. Hence, U is multisolid. \square

3.10. If \mathcal{A} has a generator \mathcal{G} , one may apply Proposition 2.6 and then Theorem 3.9(1) to the canonical functor $\mathcal{A} \rightarrow \text{Set}^{\mathcal{G}}$ to obtain:

Corollary. *Every cowellpowered category with a generator and multicolimits is multitotal.*

In the next section, we study multitotal categories with various types of generators.

4. Multitotal categories with generators

4.1. Recall that a generator \mathcal{G} of \mathcal{A} is *strong* if a morphism $f : A \rightarrow B$ in \mathcal{A} is an isomorphism whenever all maps $\mathcal{A}(G, f) : \mathcal{A}(G, A) \rightarrow \mathcal{A}(G, B)$ are bijective, $G \in \mathcal{G}$; in other words, if the canonical functor $\mathcal{A} \rightarrow \text{Set}^{\mathcal{G}}$ is conservative. In order to relate a (strong) generator of a category \mathcal{A} to the category $\Pi\mathcal{A}$, denote by O an initial object of $\Pi\mathcal{A}$ (i.e., a multi-initial object of \mathcal{A}), provided that it exists. For every $A \in \mathcal{A}$, we put $\overline{A} = A \times O$ (product in $\Pi\mathcal{A}$). For \mathcal{G} a set of objects in \mathcal{A} , let

$$\overline{\mathcal{G}} = \{\overline{G} \mid G \in \mathcal{G}\} \cup \{O \times O\};$$

and for \mathcal{H} a set of objects in $\Pi\mathcal{A}$, let \mathcal{H}^1 be the set of objects of \mathcal{A} which are components of objects in \mathcal{H} . Then we have the following:

Lemma. (1) *If \mathcal{G} is a (strong) generator of \mathcal{A} and if \mathcal{A} has a multi-initial object, then $\overline{\mathcal{G}}$ is a (strong) generator of $\Pi\mathcal{A}$.*

(2) *If \mathcal{H} is a (strong) generator of $\Pi\mathcal{A}$, then \mathcal{H}^1 is a (strong) generator of \mathcal{A} .*

PROOF: (1) If \mathcal{G} is a generator of \mathcal{A} and if $p, q : A \rightarrow B$ are distinct morphisms of $\Pi\mathcal{A}$, then there is $k \in SB$ such that either $Sp(k) \neq Sq(k)$, or $Sp(k) = l = Sq(k)$ and $p_k \neq q_k : A_l \rightarrow B_k$. In the former case, consider $h : O \times O \rightarrow A$ with $Sh(Sp(k)) \neq Sh(Sq(k))$, then $ph \neq qh$; in the latter case, choose $h_0 : G \rightarrow A_l$ with $p_k h_0 \neq q_k h_0$ and $G \in \mathcal{G}$, and let $h : \overline{G} \rightarrow A$ be a morphism with l -component h_0 , then $ph \neq qh$.

If \mathcal{G} is a strong generator of \mathcal{A} and if $f : A \rightarrow B$ is a morphism in $\Pi\mathcal{A}$ such that $\Pi\mathcal{A}(G, f)$ is bijective for all $G \in \overline{\mathcal{G}}$, then (a) Sf is bijective since, on the one hand, the bijectivity of $\Pi\mathcal{A}(O \times O, f)$ is clearly equivalent to the bijectivity of $\text{Set}^{op}(2, Sf)$, and, on the other hand, $\text{Set}^{op}(2, Sf)$ is surjective (injective) iff Sf is injective (surjective, respectively); and (b) each component f_k of f is an isomorphism since, thanks to the morphisms from $G \times O$, $G \in \mathcal{G}$, to B , $\mathcal{A}(G, f_k)$ is bijective. Thus f is an isomorphism, hence, $\overline{\mathcal{G}}$ is a strong generator.

(2) If \mathcal{H} is a generator of $\Pi\mathcal{A}$ and $p, q : A \rightarrow B$ are distinct morphisms of \mathcal{A} , choose $h : H \rightarrow A$ with $ph \neq qh$ in $\Pi\mathcal{A}$. The unique component $h_1 : H_1 \rightarrow A$ of h then fulfills $ph_1 \neq qh_1$ and $H_1 \in \mathcal{H}^1$. If \mathcal{H} is a strong generator of $\Pi\mathcal{A}$, it follows easily that also \mathcal{H}^1 is a strong generator of \mathcal{A} , from the observation that, given $H \in \mathcal{H}$ and a morphism $f : A \rightarrow B$ in \mathcal{A} , $\Pi\mathcal{A}(H, f)$ is essentially the map $\prod_{i \in SH} \mathcal{A}(H_i, f)$. \square

4.2. We use the notation of [GU] and denote by $\widetilde{\text{Set}}$ any small-indexed discrete power of Set .

Theorem. *The following conditions on a category \mathcal{A} are equivalent:*

- (i) \mathcal{A} is multitotal and has a generator,
- (ii) $\Pi\mathcal{A}$ is total and has a generator,
- (iii) $\Pi\mathcal{A}$ admits a solid functor into $\widetilde{\text{Set}}$,
- (iv) \mathcal{A} admits a multisolid functor into $\widetilde{\text{Set}}$.

PROOF: (i) \Rightarrow (ii) by 3.7 and 4.1(1). (ii) \Rightarrow (iii) by (2) of Theorem 2.3. (iii) \Rightarrow (iv): \mathcal{A} is always multireflective in $\Pi\mathcal{A}$; hence, composition of the solid functor $\Pi\mathcal{A} \rightarrow \widetilde{\text{Set}}$ with $J_{\mathcal{A}}$ gives a multisolid functor $\mathcal{A} \rightarrow \widetilde{\text{Set}}$. (iv) \Rightarrow (i): \mathcal{A} is multitotal by 3.9(1); furthermore, since $\Pi\mathcal{A}$ is solid over $\Pi\widetilde{\text{Set}}$ and since $\Pi\widetilde{\text{Set}}$ has a generator by 4.1(1), also $\Pi\mathcal{A}$ and then \mathcal{A} has a generator, by 4.1(2). \square

4.3 Corollary. *The following conditions on a category \mathcal{A} are equivalent:*

- (i) \mathcal{A} is multitotal and has a strong generator,
- (ii) $\Pi\mathcal{A}$ is total and has a strong generator,
- (iii) $\Pi\mathcal{A}$ admits a solid, conservative functor into $\widetilde{\text{Set}}$,
- (iv) \mathcal{A} admits a multisolid, conservative functor into $\widetilde{\text{Set}}$.

PROOF: Thanks to the strong generator part of Lemma 4.1, one can mimic the proof of 4.2. \square

4.4. A generator \mathcal{G} of \mathcal{A} is *regular* if the canonical functor $\mathcal{A} \rightarrow \text{Set}^{\mathcal{G}}$ reflects regular epimorphisms; equivalently, $f : A \rightarrow B$ in \mathcal{A} is a regular epimorphism whenever every map $\mathcal{A}(G, f)$, $G \in \mathcal{G}$, is surjective.

Remark. Every cocomplete category with a regular generator is total, see [4]. These are, by [3], precisely the reflective subcategories of monadic categories over $\widetilde{\text{Set}}$.

Lemma.

- (1) *Let \mathcal{A} have a multi-initial object. Then:*
 - a. *A morphism $f : A \rightarrow B$ in $\Pi\mathcal{A}$ is a regular epimorphism if its components are regular epimorphisms in \mathcal{A} and if Sf is injective; these conditions are also necessary whenever \mathcal{A} has a terminal object.*
 - b. *If \mathcal{G} is a regular generator of \mathcal{A} , then $\overline{\mathcal{G}}$ is a regular generator of $\Pi\mathcal{A}$.*
- (2) *If \mathcal{H} is a regular generator of $\Pi\mathcal{A}$, then \mathcal{H}^1 is a regular generator of \mathcal{A} .*

PROOF: (1)a. Put $\varphi = Sf$. By hypothesis, each f_j is a coequalizer of a pair $p_j, q_j : K_j \rightarrow A_{\varphi(j)}$ in \mathcal{A} . Put $I = SA$, $J = SB$, and $K = (K_j)_{j \in J}$ and define $p, q : K \times O \times O \rightarrow A$ by letting Sp map $I - \varphi(J)$ into the first copy of SO and Sq into the second one. Then f is a coequalizer of p, q in $\Pi\mathcal{A}$. Conversely, assuming

f to be the coequalizer of some pair $g, h : L \rightarrow A$ in $\Pi\mathcal{A}$, we first consider $j_1, j_2 \in J$ with $\varphi(j_1) = \varphi(j_2) = i$. With any commutative square

$$\begin{array}{ccc}
 & & B_{j_1} \\
 & \nearrow^{f_{j_1}} & \\
 A_i & & \\
 & \searrow_{f_{j_2}} & \\
 & & B_{j_2} \\
 & & \nearrow \\
 & & C
 \end{array}$$

in \mathcal{A} (which certainly exists when \mathcal{A} has a terminal object) one obtains morphisms $s, t : B \rightarrow C$ in $\Pi\mathcal{A}$ with $sf = tf$, hence $s = t$ and then $j_1 = j_2$. Injectivity of φ now gives that each f_j is a coequalizer of $g_{\varphi(j)}, h_{\varphi(j)}$ in \mathcal{A} .

(1)b. and (2) follow from an easy analysis of the proof of 4.1. \square

Remark. For categories \mathcal{A} which do not have a terminal object the converse implication of (1)a. above is false, in general: suppose that a pair $p, q : A \rightarrow B$ in \mathcal{A} has a multicoequalizer $c_k : B \rightarrow C_k (k \in K)$ in \mathcal{A} . Then the corresponding coequalizer $c : B \rightarrow C = (C_k)_{k \in K}$ in $\Pi\mathcal{A}$ is a regular epimorphism and Sc is a constant function.

4.5 Theorem. *The following conditions on a category \mathcal{A} are equivalent:*

- (i) \mathcal{A} is multitotal and has a regular generator;
- (ii) \mathcal{A} is multicomplete and has a regular generator;
- (iii) $\Pi\mathcal{A}$ is cocomplete and has a regular generator;
- (iv) $\Pi\mathcal{A}$ admits a solid functor into $\widetilde{\mathcal{S}et}$ which reflects regular epimorphisms;
- (v) \mathcal{A} is equivalent to a multireflective subcategory of a monadic category over $\widetilde{\mathcal{S}et}$;
- (vi) \mathcal{A} admits a multisolid functor into $\widetilde{\mathcal{S}et}$ which reflects regular epimorphisms.

PROOF: (i) \Rightarrow (ii) is trivial, and (ii) \Rightarrow (iii) follows from 4.4(2). (iii) \Rightarrow (iv): In the presence of a regular generator, the cocomplete category $\Pi\mathcal{A}$ is actually total (see Remark 4.4), so that 2.3(2) becomes applicable. (iv) \Rightarrow (v): Since $\Pi\mathcal{A}$ has coequalizers, it is equivalent to a full reflective subcategory of a monadic category over $\widetilde{\mathcal{S}et}$ (see also [3]); statement (v) follows since \mathcal{A} is multireflective in $\Pi\mathcal{A}$. (v) \Rightarrow (vi): The forgetful functor of a monadic category over $\widetilde{\mathcal{S}et}$ is solid and reflects regular epimorphisms. One easily shows that the latter statement remains true if we restrict the functor to a multireflective subcategory. (vi) \Rightarrow (i): In order to see that \mathcal{A} has a regular generator, one uses 4.4(1) to show that when the multisolid functor $U : \mathcal{A} \rightarrow \widetilde{\mathcal{S}et}$ reflects regular epimorphisms, the same is true for the solid functor $\Pi U : \Pi\mathcal{A} \rightarrow \Pi\widetilde{\mathcal{S}et}$, so that the existence of a regular generator in $\Pi\widetilde{\mathcal{S}et}$ implies the same for $\Pi\mathcal{A}$; now one applies 4.4(2) again. \square

5. Product completions and dense generators

5.1. Recall that a generator \mathcal{G} of \mathcal{A} is said to be *dense* provided that the full subcategory of \mathcal{A} generated by \mathcal{G} (which we denote also by \mathcal{G}) is dense in \mathcal{A} , that is, every object A of \mathcal{A} is a canonical colimit of the forgetful functor $D_A : (\mathcal{G} \downarrow A) \rightarrow \mathcal{A}$.

In the previous section, we related several types of generators of a category \mathcal{A} to the category $\Pi\mathcal{A}$, and this enabled us to obtain results on multitotal categories analogous to the ones of [3] for total categories. Similarly to (2) of Lemma 4.1, in the case of dense generators, we have the following:

Lemma. *If \mathcal{H} is a dense generator of $\Pi\mathcal{A}$, then \mathcal{H}^1 is a dense generator of \mathcal{A} .*

PROOF: Let A be an object in \mathcal{A} and let $D^1 : (\mathcal{H}^1 \downarrow A) \rightarrow \mathcal{A}$ and $D : (\mathcal{H} \downarrow A) \rightarrow \Pi\mathcal{A}$ be the corresponding canonical diagrams into \mathcal{A} and $\Pi\mathcal{A}$, respectively. If $\gamma : D^1 \rightarrow \Delta B$ is a cocone for D^1 , define $\bar{\gamma} : D \rightarrow \Delta B$ by considering, for each $\Pi\mathcal{A}$ -morphism $h : H \rightarrow A$ with $H \in \mathcal{H}$, the morphism $\bar{\gamma}_h : H \rightarrow B$ such that $\left((\bar{\gamma}_h)_* : H_{(S\bar{\gamma}_h)(*)} \rightarrow B \right) = \left(\gamma_{h_*} : H_{(Sh)(*)} \rightarrow B \right)$. It is easy to show that $\bar{\gamma}$ is a cocone for D . Thus, there exists a unique morphism $w : A \rightarrow B$ such that $w \cdot h = \bar{\gamma}_h$ for every $h : H \rightarrow A$ with $H \in \mathcal{H}$. It is now easily verified that $w : A \rightarrow B$ is also the unique morphism which fulfills the equality $w \cdot g = \gamma_g$ for every $g : G \rightarrow A$ with $G \in \mathcal{H}^1$. \square

5.2. However, for dense generators there is no analogous statement to (1) of Lemma 4.1. More precisely, we shall show next that the existence of a dense generator of $\Pi\mathcal{A}$ in the presence of a dense generator of \mathcal{A} depends on the following large-cardinal axiom:

(M) There do not exist arbitrarily large measurable cardinals.

The statement (M) means that we can find a cardinal ρ such that no cardinal larger or equal to ρ is measurable, or, equivalently, every ultrafilter closed under intersections of less than ρ members contains a singleton set (see A.5 in [2]).

Remark. Recall that a functor $F : \mathcal{A} \rightarrow \mathcal{B}$ is called *coinitial* (the dual of *cofinal*) provided that for every object B of \mathcal{B} the comma-category $(F \downarrow B)$ is connected; this implies that for every diagram $G : \mathcal{B} \rightarrow \mathcal{X}$ we have $\text{colim } G = \text{colim } G \cdot F$ (more precisely, if $(c_B : GB \rightarrow C)_{B \in \text{Ob}(\mathcal{B})}$ is a colimit of G , then $(c_{FA} : GFA \rightarrow C)_{A \in \text{Ob}(\mathcal{A})}$ is a colimit of $G \cdot F$).

Theorem. *The following assertions are equivalent:*

- (i) $\Pi\mathcal{A}$ has a dense generator, for every category \mathcal{A} with a dense generator and a multi-initial object;
- (ii) III has a dense generator (with \mathbb{I} the terminal category);
- (iii) the set-theoretic axiom (M) holds.

PROOF: Since $\text{III} = \text{Set}^{op}$, we have (ii) \Leftrightarrow (iii); this was proved by J. Isbell in [10], by showing that, for any cardinal ρ , the sets of cardinality less than ρ form

a codense cogenerator of $\mathcal{S}et$ if and only if no cardinal $\geq \rho$ is measurable. As (i) \Rightarrow (ii) is trivial, (iii) \Rightarrow (i) remains to be shown. Let \mathcal{G} be a dense generator of \mathcal{A} , and let $O = (O_t)_{t \in T}$ be a multi-initial object of \mathcal{A} . By hypothesis, there is a cardinal number ρ such that $\hat{\rho} = \{I \mid \text{card } I < \rho\}$ is a codense in $\mathcal{S}et$. Without loss of generality, we may assume $O_t \in \mathcal{G}$ for all $t \in T$ and $\rho > \max\{\text{card } T, \aleph_0\}$. We claim that

$$\mathcal{G}^\rho = \{G \in \Pi\mathcal{A} \mid \text{card}(SG) < \rho \text{ and } G_i \in \mathcal{G} \text{ for all } i \in SG\}$$

is dense in $\Pi\mathcal{A}$. Hence, for every $A \in \Pi\mathcal{A}$, we must show that A is a colimit of the canonical diagram $D_A : (\mathcal{G}^\rho \downarrow A) \rightarrow \Pi\mathcal{A}$. For that we use Lemma 2.5 and first show that SD_A^{op} has limit SA in $\mathcal{S}et$.

In fact, we have a commutative diagram

$$\begin{array}{ccc} (\mathcal{G}^\rho \downarrow A)^{op} & \xrightarrow{D_A^{op}} & (\Pi\mathcal{A})^{op} \\ S_A \downarrow & & \downarrow S \\ (SA \downarrow \hat{\rho}) & \xrightarrow{D_{SA}} & \mathcal{S}et \end{array}$$

with S_A induced by S . Since $\lim D_{SA} = SA$ (canonically), it suffices to show that S_A is coinitial. Hence, for every object $(J, \varphi : SA \rightarrow J)$ in $(SA \downarrow \hat{\rho})$ we must show that the comma category $(S_A \downarrow (J, \varphi))$ is connected. In fact, we can define a morphism $c_\varphi : O^J \rightarrow A$ by

$$(c_\varphi)_k : O^J \xrightarrow{p_{\varphi(k)}} O \xrightarrow{!_{A_k}} A_k$$

for every $k \in SA$ (where $p_{\varphi(k)}$ is a projection and $!_{A_k}$ is determined by initiality of O in $\Pi\mathcal{A}$). Hence, we have an object $(O^J, c_\varphi) \in (\mathcal{G}^\rho \downarrow A)$, and the diagram

$$\begin{array}{ccc} SO^J = T \times J & \xrightarrow{\pi_2} & J \\ & \searrow S_{c_\varphi} & \nearrow \varphi \\ & SA & \end{array}$$

commutes. This means that $((O^J, c_\varphi), \pi_2)$ is an object of $(S_A \downarrow (J, \varphi))$; in fact, it is weakly initial in that category, as we shall show next. Given any object $((G, f), \psi)$ in $(S_A \downarrow (J, \varphi))$, so that $\psi : S_A(G, f) \rightarrow (J, \varphi)$ is a morphism in $(SA \downarrow \hat{\rho})$, we claim that the morphism $d_\psi : O^J \rightarrow G$ with

$$(d_\psi)_i : O^J \xrightarrow{p_{\psi(i)}} O \xrightarrow{!_{G_i}} G_i$$

for all $i \in SG$ makes the following diagrams commute:

$$\begin{array}{ccc}
 SG & \xrightarrow{Sd_\psi} & SO^J \\
 \psi \searrow & & \nearrow \pi_2 \\
 Sf \searrow & J & \nearrow Sc_\varphi \\
 \varphi \uparrow & & \\
 SA & &
 \end{array}
 \qquad
 \begin{array}{ccc}
 G & \xleftarrow{d_\psi} & O^J \\
 f \searrow & & \nearrow c_\varphi \\
 & A &
 \end{array}$$

This is obvious for the diagram on the left, while initiality of O in $\Pi\mathcal{A}$ implies commutativity of the right-hand diagram. Consequently we have a morphism $d_\psi : ((O^J, c_\varphi), \pi_2) \rightarrow ((G, f), \psi)$ in $(SA \downarrow (J, \varphi))$. This concludes the proof of $\lim SD_A^{op} = SA$.

According to Lemma 2.5, it now suffices to show that for every $k \in SA$, A_k is the canonical colimit of the diagram $D_k : (\mathcal{G}^\rho \downarrow A) \rightarrow \mathcal{A}$, $(G, f) \mapsto G_{(Sf)(k)}$. But D_k factors as

$$\begin{array}{ccc}
 (\mathcal{G}^\rho \downarrow A) & \xrightarrow{D_k} & \mathcal{A} \\
 F_k \searrow & & \nearrow D_{A_k} \\
 & (\mathcal{G} \downarrow A_k) &
 \end{array}$$

with $F_k : (G, f) \mapsto (G_{(Sf)(k)}, f_k)$. Since, by hypothesis, A_k is the canonical colimit of D_{A_k} , it suffices to show that F_k is cofinal in order to finish the proof. We show that, given any object $(H, g) \in (\mathcal{G} \downarrow A_k)$, the comma category $((H, g) \downarrow F_k)$ is connected. Let $g^{(k)} : H \times O \rightarrow A$ be the $\Pi\mathcal{A}$ -morphism with $(g^{(k)})_k = g : H \rightarrow A_k$ and $(g^{(k)})_l = !_{A_l} : O \rightarrow A_l$ for all $l \neq k$. Then $1_H : (H, g) \rightarrow F_k(H \times O, g^{(k)})$ is a morphism in $(\mathcal{G} \downarrow A_k)$, so $((H \times O, g^{(k)}), 1_H)$ is an object in $((H, g) \downarrow F_k)$. Consider another object $((G, f), h)$ in that category; denote $i = (Sf)(k)$. We analyse two cases:

(a) $(Sf)^{-1}(\{i\}) = \{k\}$. Thus one can define a $\Pi\mathcal{A}$ -morphism $h^{(k)} : H \times O \rightarrow G$ as above, and since $f_k \cdot h = g$ one has $f \cdot h^{(k)} = g^{(k)}$. Consequently, $h^{(k)} : (H \times O, g^{(k)}) \rightarrow (G, f)$ is a morphism in $(\mathcal{G}^\rho \downarrow A)$ with $f \cdot F_k h^{(k)} = g$; hence, we have $h^{(k)} : ((H \times O, g^{(k)}), 1_H) \rightarrow ((G, f), h)$ in $((H, g) \downarrow F_k)$.

(b) $(Sf)^{-1}(\{i\}) \neq \{k\}$. We show that $((G, f), h)$ belongs to the same connected component of the category $((H, g) \downarrow F_k)$ as an object $((\overline{G}, \overline{f}), \overline{h})$ which is of type (a), so the proof will be complete. Put $\overline{G} = G \times G_i$ and let $\overline{f} : G \times G_i \rightarrow A$ be the obvious morphism such that $\overline{f}_l = (f \cdot \pi_1)_l$ for all $l \neq k$, where π_1 is the first projection of $G \times G_i$ and $(S\overline{f})^{-1}(i) = \{k\}$, and put $\overline{h} = h$. Since $\overline{f}_k \cdot \overline{h} = f_k \cdot h = g$, $((\overline{G}, \overline{f}), \overline{h})$ is an object in $((H, g) \downarrow F_k)$. Let $t : G \rightarrow \overline{G}$ be the $\Pi\mathcal{A}$ -morphism such that St identifies the two copies of i of $S\overline{G}$, and all components of t are identities.

Then we have that $\bar{f} \cdot t = f$, so $t : (G, f) \rightarrow (\bar{G}, \bar{f})$ is a morphism in $(\mathcal{G}^p \downarrow A)$; furthermore, $F_k t \cdot h = \bar{h}$, thus we have a morphism $t : ((G, f), h) \rightarrow ((\bar{G}, \bar{f}), \bar{h})$ in $((H, g) \downarrow F_k)$. \square

5.3 Remark. The dense generator of ΠA obtained in 5.2 from a dense generator of a category \mathcal{A} with a multi-initial object (in the presence of the axiom (M)) is distinct from the set $\bar{\mathcal{G}} = \{G \times O \mid G \in \mathcal{G}\} \cup \{O \times O\}$ used in Section 4. (We have seen there that $\bar{\mathcal{G}}$ is a (strong or regular) generator of ΠA whenever \mathcal{A} has a multi-initial object O and a (strong or regular, respectively) generator \mathcal{G} .) Actually, we can prove that the category ΠSet does not have a dense generator of the form $1 \times O$ plus O^J , $J \in \mathcal{J}$ (assuming that \mathcal{J} is dense in Set^{op}). Consider $A = (X_1, X_2)$ where $X_1 = X_2 = 1$. The canonical diagram D of A consists of

- (a) some copies of O^J ;
- (b) four copies of $1 \times O$ indexed by the obvious four morphisms $f : 1 \times O \rightarrow A$ determined by $Sf : \{1, 2\} \rightarrow \{1, 2\}$ uniquely.

Each of the four copies of $1 \times O$ is connected with the objects of type (a) by morphisms $O^J \rightarrow 1 \times O$, but there is no morphism in the opposite direction. The colimit of the canonical diagram is, by Lemma 2.5,

$$\text{colim } D = (\text{colim } D_\varphi)_{\varphi \in \text{lim } SD}.$$

If \mathcal{J} were dense in Set^{op} , we would have, for the subdiagram D' of D of all objects O^J , $\text{lim } SD' = \{1, 2\}$ (canonically). Thanks to the morphisms $O^J \rightarrow 1 \times O$, we have $\text{lim } SD = \text{lim } SD'$. It is easy to see that the choice of index 1 gives a diagram with two distinct copies of 1 without any morphism between them; thus $\text{colim } D_1$ will have two elements. Analogously, $\text{colim } D_2$ has two elements. Hence $(A_1, A_2) \neq (\text{colim } D_1, \text{colim } D_2)$.

5.4. Although the existence of a dense generator of \mathcal{A} does not guarantee the same for ΠA , the relationship between dense generators and multitotal categories is similar to the one for totality:

Theorem. *The following statements are equivalent for a category \mathcal{A} :*

- (i) \mathcal{A} is multitotal with a dense generator;
- (ii) \mathcal{A} is multisolid over $\widetilde{\text{Set}}$ and has a dense generator;
- (iii) \mathcal{A} is equivalent to a full multireflective subcategory of a ranked monadic category over $\widetilde{\text{Set}}$;
- (iv) \mathcal{A} is a full multireflective subcategory of a locally presentable category;
- (v) \mathcal{A} is a full multireflective subcategory of a Grothendieck topos.

PROOF: One proceeds similarly as in the total case, see [3]. \square

Remark. From the Theorem above and from 6.16 of [2], it follows that multitotal categories with dense generators are precisely the locally multipresentable categories, provided that the set-theoretic Vopěnka Principle holds.

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(Received April 28, 1999)