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Commentationes Mathematicae Universitatis Carolinae, Vol. 40 (1999), No. 3, 441--446

Persistent URL: <http://dml.cz/dmlcz/119100>

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On a generalization of QI-rings

J. JIRÁSKO

Abstract. In this paper rings for which every s -torsion quasi-injective module is weakly s -divisible for a hereditary preradical s are characterized in terms of the properties of the corresponding lattice of the (hereditary) preradicals. In case of a stable torsion theory these rings coincide with TQI -rings investigated by J. Ahsan and E. Enochs in [1]. Our aim was to generalize some results concerning QI -rings obtained by J.S. Golan and S.R. López-Permouth in [12]. A characterization of the QI -property in the category $\sigma[M]$ then follows as a consequence.

Keywords: s - QI -rings, s -stable preradicals, weakly s -divisible modules, s -tight modules

Classification: 16D50, 16S90

In what follows, R stands for an associative ring with a unit element and $R\text{-Mod}$ denotes the category of all unitary left R -modules.

First of all we recall some basic definitions from the theory of preradicals (for details see L. Bican, T. Kepka, P. Nĕmec [5] and J.S. Golan [11]).

A preradical r for $R\text{-Mod}$ is any subfunctor of the identity functor, i.e. r assigns to each module M its submodule $r(M)$ in such a way that every homomorphism of M into N induces a homomorphism of $r(M)$ into $r(N)$ by restriction. A preradical r is said to be

- idempotent if $r(r(M)) = r(M)$ for every module M ,
- a radical if $r(M/r(M)) = 0$ for every module M ,
- hereditary if $r(N) = N \cap r(M)$ for every submodule N of a module M ,
- stable if $r(M)$ is a direct summand in M for every injective module M .

A module M is r -torsion if $r(M) = M$ and r -torsionfree if $r(M) = 0$. We shall denote by \mathcal{T}_r (\mathcal{F}_r) the class of all r -torsion (r -torsionfree) modules. If r and s are preradicals then $r \circ s$ is the preradical defined by $(r \circ s)(M) = r(s(M))$, $M \in R\text{-Mod}$ and $r \leq s$ means $r(M) \subseteq s(M)$ for every $M \in R\text{-Mod}$. The idempotent core \bar{r} of a preradical r is defined by $\bar{r}(M) = \sum K$, where K runs through all r -torsion submodules K of M . The injective hull of a module M will be denoted by $E(M)$. The hereditary closure $h(r)$ of a preradical r is defined by $h(r)(M) = M \cap r(E(M))$ for every module M . If \mathcal{A} is a non-empty class of modules then the idempotent preradical $p_{\mathcal{A}}$ is defined by $p_{\mathcal{A}}(M) = \sum \text{Im } f$, $f \in \text{Hom}_R(A, M)$, $A \in \mathcal{A}$ for every $M \in R\text{-Mod}$. The identity functor will be

The author has been supported by GA ČR 2433.

denoted by id . Finally, $\sigma[M]$ denotes the category of all modules subgenerated by a module M .

If M is a module then a module Q is called M -injective if every homomorphism from a submodule of M into Q can be extended to a homomorphism from M into Q . If s is a preradical then a module Q is called weakly s -divisible if it is M -injective for every $M \in \mathcal{T}_s$. Since $\mathcal{T}_s = \mathcal{T}_{\bar{s}}$ we will assume without loss of generality throughout the whole text that s is an idempotent preradical. It is easy to see that a module Q is weakly s -divisible if and only if $s(E(Q)) \subseteq Q$ (see [5]) and that a module Q is M -injective if and only if it is weakly $p_{\{M\}}$ -divisible (see [4]). Further, $E_s(Q) = Q + s(E(Q))$ is the weakly s -divisible hull of a module Q .

If M is a module then a module Q is called M -tight if every homomorphic image of M which is embeddable in $E(Q)$ is also embeddable in Q (see [2]). Now we can define a tight module with respect to a preradical.

Definition 1. *A module Q is said to be s -tight if it is M -tight for every $M \in \mathcal{T}_s$.*

As it is easy to see a module Q is s -tight if and only if $s(E(Q))$ can be embedded in Q and consequently every weakly s -divisible module is s -tight. The converse is true in case of quasi-injective modules.

Lemma 1. *Let Q be a quasi-injective module. Then Q is s -tight if and only if it is weakly s -divisible.*

PROOF: If Q is s -tight then there is a monomorphism $f: s(E(Q)) \rightarrow Q$. Let us consider the following diagram:

$$\begin{array}{ccc} s(E(Q)) & \xrightarrow{f} & Q \\ i \downarrow & & \\ E(Q) & & \end{array},$$

where i is the inclusion map. Then there is a homomorphism $g: Q \rightarrow E(Q)$ such that $fg = i$. Now $Qg \subseteq Q$ since Q is quasi-injective and consequently $s(E(Q)) = \text{Im } i \subseteq Qg \subseteq Q$. Hence Q is weakly s -divisible. \square

The following notion of generalized stability of preradicals plays an important role in our characterization of generalized QI -rings.

Definition 2. *A preradical r is said to be s -stable if $r(M)$ is a direct summand in M for every weakly s -divisible s -torsion module M .*

Proposition 1. *Let r be a preradical. Then*

- (i) *if r is s -stable then $\mathcal{T}_{r \circ s} = \mathcal{T}_r \cap \mathcal{T}_s$ is closed under weakly s -divisible hulls;*
- (ii) *if $r \circ s$ is idempotent (e.g. if r is idempotent and s is hereditary) and $\mathcal{T}_{r \circ s}$ is closed under weakly s -divisible hulls then r is s -stable;*
- (iii) *if $r \circ s$ is stable then r is s -stable. The converse is true if s is stable;*
- (iv) *if r is hereditary and s -stable then $r(M/r(M)) = 0$ for every s -torsion module M .*

PROOF: (i). If $T \in \mathcal{T}_{r \circ s}$ then $E_s(T) = s(E(T))$ and $r(s(E(T))) \oplus A = s(E(T))$ for some module A since r is s -stable. Now $T \subseteq r(s(E(T)))$ and $T \cap A = 0$ yields $A = 0$. Thus $E_s(T) = s(E(T)) \in \mathcal{T}_r$.

(ii). Let Q be a s -torsion weakly s -divisible module. Then $r(Q) \in \mathcal{T}_{r \circ s}$ and consequently $E_s(r(Q)) = s(E(r(Q))) \in \mathcal{T}_{r \circ s}$. Now $E_s(r(Q))$ is a direct summand in Q since Q is s -torsion and weakly s -divisible. But $E_s(r(Q)) \subseteq (r \circ s)(Q) = r(Q) \subseteq E_s(r(Q))$ which yields $E_s(r(Q)) = r(Q)$ and consequently r is s -stable.

(iii). If $r \circ s$ is stable and Q is a weakly s -divisible s -torsion module then $Q = s(E(Q))$. Now $r(Q) = (r \circ s)(E(Q))$ is a direct summand in $E(Q)$ and therefore $r(Q)$ is also a direct summand in Q .

On the other hand, if s is stable, r is s -stable and Q is injective then $s(Q)$ is weakly s -divisible s -torsion. Hence $(r \circ s)(Q)$ is a direct summand in $s(Q)$ and consequently also in Q since s is stable.

(iv). If M is s -torsion then we can consider the following short exact sequence:

$$0 \rightarrow (r \circ s)(E(M))/r(M) \rightarrow s(E(M))/r(M) \rightarrow s(E(M))/(r \circ s)(E(M)) \rightarrow 0.$$

Now $s(E(M))$ is weakly s -divisible s -torsion and therefore $(r \circ s)(E(M))$ is a direct summand in $s(E(M))$. From it follows that $s(E(M))/(r \circ s)(E(M))$ is r -torsionfree. Hence $r(s(E(M))/r(M)) \subseteq (r \circ s)(E(M))/r(M)$ and consequently $r(M/r(M)) = M/r(M) \cap r(s(E(M))/r(M)) \subseteq (M \cap (r \circ s)(E(M)))/r(M) = r(M)/r(M) = 0$, r being hereditary. \square

Corollary 1. *Let M be a module. An idempotent preradical r is $h(p_{\{M\}})$ -stable if and only if $\mathcal{T}_r \cap \sigma[M]$ is closed under M -injective hulls.*

Proposition 2. *Let r and s be hereditary preradicals. Then the following conditions are equivalent:*

- (i) every $M \in \mathcal{T}_s \setminus \mathcal{T}_r$ contains a nonzero r -torsionfree submodule;
- (ii) $\mathcal{T}_r \cap \mathcal{T}_s$ is closed under weakly s -divisible hulls;
- (iii) if $A \subseteq B \subseteq C$ such that $C/A \in \mathcal{T}_s$ and $B/A \in \mathcal{T}_r$ then there is $D \subseteq C$ with $D \cap B = A$ and $C/D \in \mathcal{T}_r$;
- (iv) if $I \subseteq K$ are left ideals with $K/I = r(R/I)$, where $R/I \in \mathcal{T}_s$ then there is a left ideal L with $L \cap K = I$ and $R/L \in \mathcal{T}_r$;
- (v) if $I \subseteq K \neq R$ are left ideals with $K/I = r(R/I)$ and $R/I \in \mathcal{T}_s$ then there is a left ideal $L \neq I$ with $L \cap K = I$;
- (vi) r is s -stable;
- (vii) $\neg(\exists M)(M \in \mathcal{T}_s \ \& \ r(M) \subsetneq {}'M)$;
- (viii) weakly s -divisible hulls of cyclic s -torsion modules split in r .

PROOF: (i) implies (ii). Let $N \in \mathcal{T}_r \cap \mathcal{T}_s$. If $E_s(N) = s(E(N)) \notin \mathcal{T}_r$ then there is $0 \neq K \subseteq s(E(N))$ with $r(K) = 0$. Hence $K \cap N \in \mathcal{T}_r \cap \mathcal{F}_r = 0$ and consequently $K = 0$ since $N \subseteq {}'s(E(N))$, a contradiction.

(ii) implies (iii). By Zorn's lemma there is a submodule D of C maximal with respect to the property $D \cap B = A$. Then $(B + D)/D \cong B/(B \cap D) = B/A \in \mathcal{T}_r \cap$

\mathcal{T}_s . From the maximality of D follows that $(B + D)/D \subseteq {}'C/D$. Now $C/D \in \mathcal{T}_s$ and therefore $C/D \subseteq s(E((B + D)/D))$. Further, $s(E((B + D)/D)) \in \mathcal{T}_r \cap \mathcal{T}_s$ by assumption and consequently $C/D \in \mathcal{T}_r$, r being hereditary.

(iii) implies (iv). Obvious.

(iv) implies (v). Obviously, if $L = I$ then $K = R$, a contradiction.

(vi) is equivalent to (ii). It follows immediately from Proposition 1(i) and (ii).

(v) implies (ii). Let $T \in \mathcal{T}_r \cap \mathcal{T}_s$. If $E_s(T) = s(E(T)) \notin \mathcal{T}_r$ then there is $x \in s(E(T)) \setminus r(s(E(T)))$. Put $I = (0 : x)_l$ and $K = \{t \in R; tx \in r(s(E(T)))\}$. Then $K \neq R$, $K/I = r(R/I)$ and $R/I \in \mathcal{T}_s$. Now by (v) there is a left ideal L of R such that $L \neq I$ and $L \cap K = I$. Let $a \in L \setminus I$. Then $0 \neq ax \in s(E(T))$ and consequently there is $b \in R$ such that $0 \neq bax \in r(s(E(T)))$ by the essentiality of $r(s(E(T)))$ in $s(E(T))$. Hence $ba \in K \cap L = I$, a contradiction.

(iii) implies (vii). Let us suppose on the contrary that there is a module $M \in \mathcal{T}_s$ such that $r(M) \subsetneq {}'M$. Then by (iii) there is a module $D \subseteq M$ with $D \cap r(M) = 0$ and $M/D \in \mathcal{T}_r$, a contradiction.

(vii) implies (i). Let $M \in \mathcal{T}_s \setminus \mathcal{T}_r$. Then $\neg(r(M) \subsetneq {}'M)$ by assumption. Thus there is $0 \neq N \subseteq M$ with $0 = r(M) \cap N = r(N)$.

(vi) implies (viii). Obvious.

(viii) implies (v). Let $I \subseteq K \neq R$ be left ideals of R with $K/I = r(R/I)$ and $R/I \in \mathcal{T}_s$. By assumption $s(E(R/I)) = r(s(E(R/I))) \oplus A$ for some module A . Put $L/I = A \cap (R/I)$. Obviously, $A \cap (K/I) \subseteq A \cap r(s(E(R/I))) = 0$ and therefore $L \cap K = I$. Now, if $L = I$ then $A \cap R/I = 0$. Hence $A = 0$ from the essentiality of R/I in $s(E(R/I))$. Thus $R/I \in \mathcal{T}_r$ since $s(E(R/I)) \in \mathcal{T}_r$, a contradiction. \square

Definition 3. A ring R is said to be a left s - QI -ring if every s -torsion quasi-injective module is weakly s -divisible.

Obviously, if $s \leq t$ are two preradicals then every left t - QI -ring is a left s - QI -ring.

Theorem 1. Let r and s be hereditary preradicals. Then the following conditions are equivalent:

- (i) every $r \circ s$ -torsion quasi-injective module is weakly s -divisible;
- (ii) every $r \circ s$ -torsion quasi-injective module is s -tight;
- (iii) every preradical $t \leq r$ is s -stable;
- (iv) every hereditary preradical $t \leq r$ is s -stable.

PROOF: (i) is equivalent to (ii). It follows immediately from Lemma 1.

(i) implies (iii). Let $t \leq r$. If Q is a weakly s -divisible s -torsion module then $Q = s(E(Q))$ and $t(Q) = (t \circ s)(E(Q))$ is $r \circ s$ -torsion and quasi-injective. Now $t(Q)$ is weakly s -divisible by assumption and consequently $t(Q)$ is a direct summand in Q .

(iii) implies (iv). Obvious.

(iv) implies (i). If Q is $r \circ s$ -torsion and quasi-injective then $Q = p_{\{Q\}}(E(Q)) = h(p_{\{Q\}})(s(E(Q)))$ and $h(p_{\{Q\}}) \leq r$. Hence $h(p_{\{Q\}})(s(E(Q)))$ is a direct summand in $s(E(Q))$ since $h(p_{\{Q\}})$ is s -stable. Thus $Q = h(p_{\{Q\}})(s(E(Q))) = s(E(Q))$ and Q is weakly s -divisible. \square

Corollary 2. *Let s be a hereditary preradical. Then the following conditions are equivalent for a ring R :*

- (i) R is a left s -QI-ring;
- (ii) every s -torsion quasi-injective module is s -tight;
- (iii) every preradical is s -stable;
- (iv) every hereditary preradical is s -stable.

PROOF: It follows immediately from the Theorem 1 if we put $r = id$. \square

Corollary 3. *Let s be a stable hereditary radical. Then the following conditions are equivalent:*

- (i) every s -torsion quasi-injective module is injective;
- (ii) $r \circ s$ is stable for every preradical r ;
- (iii) $r \circ s$ is stable for every hereditary preradical r .

PROOF: It follows immediately from the Corollary 2, Proposition 1(iii) and the fact that if s is stable and Q is s -torsion and weakly s -divisible then Q is injective. \square

Corollary 4. *The following conditions are equivalent for a module M :*

- (i) every quasi-injective module from $\sigma[M]$ is M -injective;
- (ii) every preradical is $h(p_{\{M\}})$ -stable;
- (iii) every hereditary preradical is $h(p_{\{M\}})$ -stable.

PROOF: It follows immediately from the Corollary 2. \square

REFERENCES

- [1] Ahsan J., Enochs E., *Rings all of whose torsion quasi-injective modules are injective*, Glasgow Math. J. **25** (1984), 219–227.
- [2] Al-Huzali, Jain S.K., López-Permouth S.R., *On the weak relative injectivity of rings and modules*, Lecture Notes in Math. 1448, Springer Verlag, 1990, pp. 93–98.
- [3] Azumaya G., Mbuntum F., Varadarajan K., *On M -projective and M -injective modules*, Pacific J. Math. **59** (1975), 9–16.
- [4] Beachy J.A., *A generalization of injectivity*, Pacific J. Math. **41** (1972), 313–328.
- [5] Bican L., Kepka T., Nĕmec P., *Rings, Modules and Preradicals*, Marcel Dekker Inc., New York and Basel, 1982.
- [6] Boyle A.K., *Hereditary QI-rings*, Trans. Amer. Math. Soc. **192** (1974), 115–120.
- [7] Bunu I.D., *On left QI-rings* (in Russian), Mat. Issled. **49** (1979), 23–34.
- [8] Byrd K.A., *When are quasi-injectives injective?*, Canad. Math. Bull. **15** (1972), 599–600.
- [9] Byrd K.A., *Rings, whose quasi-injective modules are injective*, Proc. Amer. Math. Soc. **33** (1972), 235–240.
- [10] Gabriel P., *Des catégories abéliennes*, Bull. Soc. Math. France **90** (1962), 323–448.

- [11] Golan J.S., *Torsion Theories*, Longman Scientific and Technical, Harlow, 1986.
- [12] Golan J.S., López-Permouth S.R., *QI-filters and tight modules*, *Comm. Algebra* **19** (8) (1991), 2217–2229.
- [13] Jain S.K., López-Permouth S.R., Singh S., *On a class of QI-rings*, *Glasgow Math. J.* **34** (1992), 75–81.
- [14] Jirásko J., *Generalized injectivity*, *Comment. Math. Univ. Carolinae* **16** (1975), 621–636.
- [15] Morimoto S., *Weakly divisible and divisible modules*, *Tsukuba J. Math.* **6** (2) (1982), 195–200.

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(Received October 8, 1998)