

Keisuke Hoshino; Norio Kikuchi

Gehring theory for time-discrete hyperbolic differential equations

*Commentationes Mathematicae Universitatis Carolinae*, Vol. 39 (1998), No. 4, 697--707

Persistent URL: <http://dml.cz/dmlcz/119045>

## Terms of use:

© Charles University in Prague, Faculty of Mathematics and Physics, 1998

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.



This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* <http://project.dml.cz>

## Gehring theory for time-discrete hyperbolic differential equations

K. HOSHINO, N. KIKUCHI

*Abstract.* This paper is concerned with extending Gehring theory to be applicable to Rothe's approximate solutions to hyperbolic differential equations.

*Keywords:* Gehring theory, Rothe's approximation, hyperbolic differential equations

*Classification:* 26D15, 35L20, 39A10, 49J40

### 1. Introduction

M. Giaquinta and G. Modica have extended Gehring's result [2] of the higher integrability through reverse Hölder inequalities, which enables to have  $L^p$ -estimates ( $p > 2$ ) for the gradients of weak solutions to nonlinear elliptic-systems and of minimizers to variational problems. In addition, M. Giaquinta and M. Struwe [8] have applied the theory to nonlinear parabolic systems, so that there have been established  $L^p$ -estimates ( $p > 2$ ) for the spatial gradients of weak solutions.

On the other hand, the paper [15] with [9] and [10] has adopted the theory to accomplish the higher integrability of the gradients of solutions to difference partial differential equations of Rothe's approximation to parabolic partial differential equations. In [11], [12] and [13], furthermore, there has been made an attempt to extend such scheme to be applicable to Rothe's approximation systems to hyperbolic partial differential equations.

The objective of this paper is to accomplish Gehring theory due to Giaquinta-Modica in order to have applications for time-discrete difference hyperbolic partial differential equations.

The investigation of Gehring theory follows the idea of the proof due to Giaquinta-Modica. However, when we are concerned with discrete problems, we have to make some careful treatments of local cubes with centers restricted to discrete time levels.

We shall use the following notation: For  $R > 0$  and  $x_0 \in \mathbf{R}^m$ , put

$$D_R(x_0) = \{x \in \mathbf{R}^m : \max_{1 \leq i \leq m} |x^i - x_0^i| < R\},$$

and

$$\Delta = D_{3/2}(0).$$

We denote by  $Q_R(z_0)$  a cube in  $\mathbf{R}^{m+1}$  with center at  $z_0 = (t_0, x_0)$ :

$$Q_R(z_0) = Q_R(t_0, x_0) = \{t \in \mathbf{R}^1 : |t_0 - t| < R\} \times D_R(x_0)$$

and set

$$\begin{aligned} \Sigma &= Q_{3/2}(0) = (-3/2, 3/2) \times \Delta, \\ C_0 &= Q_{1/2}(0), \\ C_k &= \{z \in \Sigma : 2^{-k} < \text{dist}(z, \partial\Sigma) \leq 2^{1-k}\} \quad (k = 1, 2, \dots), \end{aligned}$$

where  $\text{dist}$  is the Euclidean distance and  $\partial\Sigma$  is the boundary of  $\Sigma$ .

The symbol  $\bar{f}_A$  stands for an average over the domain  $A$ :

$$\bar{f}_A = \frac{1}{|A|} \int_A \quad \text{for a measurable subset } A \subset \mathbf{R}^{m+1} \text{ or } A \subset \mathbf{R}^m,$$

where  $|A|$  is the Lebesgue measure of  $A$ . For any real-valued function  $\mathcal{F}$  defined in  $\Sigma$  and any real number  $s$ , we use the notation

$$E(\mathcal{F}, s) = \{z \in \Sigma : \mathcal{F}(z) > s\}.$$

Let  $h > 0$  be a positive number. We set

$$t_n = nh \quad (n \in \mathbf{Z}), \quad L_h = \sum_{n \in \mathbf{Z}} (\{t_n\} \times \Delta).$$

By an  $h$ -time step function, we mean a function defined in  $\mathbf{R}^1 \times \Delta$  and of the form

$$\mathcal{F}(t, x) = \mathcal{F}(nh, x) \quad \text{for } (t, x) \in ((n - 1)h, nh] \times \Delta \quad (n \in \mathbf{Z}).$$

Our main result is stated as follows.

**Theorem.** *Let  $g \in L^q(\Sigma)$  and  $f \in L^r(\Sigma)$ ,  $r > q > 1$ , be nonnegative  $h$ -time step functions. Suppose that there exist two constants  $\theta$  and  $b$  with  $0 \leq \theta < 1$ ,  $b > 1$  such that*

$$(1.1) \quad \int_{Q_{R/2}(z_0)} g^q dz \leq b \left\{ \left( \int_{Q_R(z_0)} g dz \right)^q + \int_{Q_R(z_0)} f^q dz \right\} + \theta \int_{Q_R(z_0)} g^q dz$$

holds for every  $Q_R(z_0) \subset \Sigma$  with  $z_0 \in L_h$ . Then  $g \in L^p_{loc}(\Sigma)$  for  $p \in [q, q + \varepsilon)$  and

$$(1.2) \quad \left( \int_{Q_{R/2}(z_0)} g^p dz \right)^{1/p} \leq C \left\{ \left( \int_{Q_R(z_0)} g^q dz \right)^{1/q} + \left( \int_{Q_R(z_0)} f^p dz \right)^{1/p} \right\}$$

holds for every  $Q_R(z_0) \subset \Sigma$  with  $z_0 \in L_h$ , where  $C$  and  $\varepsilon$  are positive constants depending only on  $b, \theta, q, r, m$ .

*Remark 1.* Note that each domain of integration in inequalities (1.1) and (1.2) is restricted to a cube with its center  $z_0$  on some level of  $L_h$ . To prove the Theorem by using the approach of [3] based on the covering argument due to Calderón-Zygmund ([1], [3]), we have to handle also general cubes with centers not always restricted to  $L_h$ . Thus we must treat the cubes carefully, some of them with centers in  $L_h$  and the others with centers elsewhere.

*Remark 2.* The constants  $C$  and  $\varepsilon$  in the Theorem are independent of  $h$ ; the Theorem is invariant by translations and dilatations. Hence it suffices to verify the inequality (1.2) with  $Q_R(z_0)$  and  $Q_{2R}(z_0)$  replaced, respectively, by  $C_0$  and  $\Sigma$ :

$$(1.3) \quad \left( \int_{C_0} g^p dz \right)^{1/p} \leq C \left\{ \left( \int_{\Sigma} g^q dz \right)^{1/q} + \left( \int_{\Sigma} f^p dz \right)^{1/p} \right\}.$$

Inequality (1.3) is trivially valid if  $g = 0$  a.e. in  $\Sigma$ . Hence by multiplying  $g$  and  $f$  by a suitably chosen constant, we may assume without loss of generality that

$$(1.4) \quad \int_{\Sigma} g^q dz = 1.$$

Following the idea of Giaquinta-Modica ([3]; see also [4]), we introduce a positive-valued function

$$(1.5) \quad \Phi(z) = \sum_{k=0}^{\infty} (2^{m+1})^{-k/q} \chi_{C_k}(z), \quad z \in \Sigma,$$

where  $\chi_{C_k}$  is the characteristic function of  $C_k$ .

According to the definition of  $\Phi$ , it is sufficient to show

$$(1.6) \quad \left( \int_{\Sigma} (g\Phi)^p dz \right)^{1/p} \leq C \left\{ \left( \int_{\Sigma} (g\Phi)^q dz \right)^{1/q} + \left( \int_{\Sigma} (f\Phi)^p dz \right)^{1/p} \right\}$$

instead of proving (1.3).

As in [7, Proposition 5.1], we prove the Theorem in two steps. In the first step, we show that the inequality

$$(1.7) \quad \int_{E(g\Phi, \nu t)} (g\Phi)^q dz \leq a \left\{ t^{q-1} \int_{E(g\Phi, t)} g\Phi dz + \int_{E(f\Phi, t)} (f\Phi)^q dz \right\}$$

holds for any  $t \in [1, \infty)$ , where  $\nu$  and  $a$  are positive constants depending on  $m, q, b, \theta$  (see [3], [7]). The second step, where we derive (1.6) from (1.7), follows entirely the idea of [7, Proposition 5.1].

In the sequel, therefore, we devote ourselves to showing the inequality (1.7).

**2. Proof of the Theorem**

In what follows,  $g, f, \theta$  and  $b$  are the quantities given in the Theorem.

To prove the Theorem, we shall show that (1.7) is valid under the assumption (1.4). We shall state Calderón-Zygmund’s subdivision lemma of the form established in [7].

**Lemma 2.1.** *For every  $s \geq |\Sigma|^{1/q}$ , there exists a family, denoted by  $\Omega$ , of cubes with their sides parallel to the axes and with disjoint interiors such that*

$$(2.1) \quad \begin{aligned} Q &\subset C_k \text{ for some } k = 0, 1, 2, \dots, \\ s^q &< \int_Q (g\Phi)^q dz \leq 3^{m+1} \cdot s^q, \\ g\Phi &\leq s \text{ a.e. on } \Sigma \setminus \bigcup_{Q \in \Omega} Q. \end{aligned}$$

PROOF: We have only to employ the argument expanded in [7], making the first decomposition into  $3 \times 3^m$  cubes and all the others into  $2 \times 2^m$  cubes of the equal size. □

*Remark 3.* The family  $\Omega$  constructed depends on a quantity  $s$ , which will be chosen as  $s = \nu t, t \geq 1$ . In fact, we have by Lemma 2.1 that

$$(2.2) \quad \int_{E(g\Phi, s)} (g\Phi)^q dz \leq 3^{m+1} \cdot s^q \cdot \sum_{Q \in \Omega} |Q|.$$

If  $s = \nu t$ , the left-hand side of (2.2) becomes that of the desired inequality (1.7). In the sequel, we shall estimate the right-hand side of (2.2). The positive constant  $\nu$  will be fixed later.

We estimate the quantity  $\sum_{Q \in \Omega} |Q|$ . According to the location of cubes, we first classify cubes of  $\Omega$ , defined in Lemma 2.1, into three types:

$$\begin{aligned} \Omega_A &= \{Q \in \Omega : Q \cap L_h \neq \emptyset\}, \\ \Omega_B^+ &= \{Q \in \Omega : Q \cap L_h = \emptyset, Q \subset \Sigma^+\}, \\ \Omega_B^- &= \{Q \in \Omega : Q \cap L_h = \emptyset, Q \subset \Sigma^-\}, \end{aligned}$$

where  $\Sigma^+ = \{(t, x) \in \Sigma : t \geq 0\}$  and  $\Sigma^- = \{(t, x) \in \Sigma : t \leq 0\}$ .

Under this setting, we introduce a family of cubes which plays a role for controlling cubes with the centers on  $L_h$ . With each  $Q \in \Omega$ , we associate a cube  $\widehat{Q}$  and a family  $\mathfrak{P}(Q)$  in the following way: For  $Q$  of the form

$$Q = Q_\rho(\tau, \xi) \text{ with } t_{n_\star-1} < \tau \leq t_{n_\star},$$

we put

$$\widehat{Q} = \begin{cases} Q & \text{for } Q \in \Omega_A, \\ Q_\rho(t_{n_*-1} + \rho, \xi) & \text{for } Q \in \Omega_B^+, \\ Q_\rho(t_{n_*} - \rho, \xi) & \text{for } Q \in \Omega_B^- \end{cases}$$

and set

$$\mathfrak{P}(Q) = \left\{ Q_r(z_0) : z_0 \in L_h, \widehat{Q} \subset Q_r(z_0) \subset \Sigma \right\}.$$

**Lemma 2.2.** *For each  $Q \in \Omega$ ,  $Q \subset C_k$  ( $k = 0, 1, 2, \dots$ ), it holds that*

$$(2^{m+1})^{k-1} s^q < \sup_{P \in \mathfrak{P}(Q)} \int_P g^q dz.$$

PROOF: By the assumption on  $Q$ , we have from (2.1) that

$$(2^{m+1})^k s^q < \int_Q g^q dz = \int_{\widehat{Q}} g^q dz.$$

Here, the equality is due to the definition of  $\widehat{Q}$  and the fact that  $g$  is an  $h$ -time step function. Since there exists a cube  $P \in \mathfrak{P}(Q)$  satisfying  $P \supset \widehat{Q}$  with  $|P| \leq 2^{m+1} |\widehat{Q}|$ , we immediately obtain Lemma 2.2.  $\square$

**Lemma 2.3.** *Suppose that  $s \geq (2 \cdot (2^{m+1})^2 \cdot |\Sigma|)^{1/q}$ . Then, for each  $Q \in \Omega$  included in  $C_k$  ( $k = 0, 1, 2, \dots$ ), there exists a cube  $P_Q \in \mathfrak{P}(Q)$  satisfying*

$$(2.3) \quad |P_Q| < (2^{m+1})^{-(k-2)} \cdot s^{-q} \cdot 2|\Sigma|$$

and

$$(2.4) \quad (2^{m+1})^{k-1} s^q < \frac{2b}{1-\theta} \left\{ \left( \int_{P_Q} g dz \right)^q + \int_{P_Q} f^q dz \right\}.$$

PROOF: The condition  $\theta < 1$  allows us to find a cube  $Q_{R/2}(z_0) \in \mathfrak{P}(Q)$ ,  $z_0 \in L_h$ , such that

$$(2.5) \quad \frac{1 + \sqrt{\theta}}{2} \sup_{P \in \mathfrak{P}(Q)} \int_P g^q dz \leq \int_{Q_{R/2}(z_0)} g^q dz.$$

From Lemma 2.2, it follows that

$$(2.6) \quad (2^{m+1})^{k-1} s^q < \frac{2}{1 + \sqrt{\theta}} \int_{Q_{R/2}(z_0)} g^q dz,$$

which implies

$$|Q_R(z_0)| < (2^{m+1})^{-(k-2)} s^{-q} \frac{2|\Sigma|}{1 + \sqrt{\theta}} \int_{\Sigma} g^q dz.$$

Hence, we have by assumption (1.4) that (2.3) holds for  $P_Q = Q_R(z_0)$ .

The estimate (2.3) together with the assumption on  $s$  yield

$$(2.7) \quad |P_Q| < (2^{-k})^{m+1},$$

so that each side of  $P_Q$  is less than  $2^{-k}$  in length. Since  $\widehat{Q} \subset P_Q$  and  $\widehat{Q} \subset C_0 \cup \dots \cup C_k$ , the estimate

$$\text{dist}(P_Q, \partial\Sigma) > \text{dist}(\widehat{Q}, \partial\Sigma) - 2^{-k} \geq 0$$

holds, which implies  $P_Q \subset \Sigma$ . Thereby,  $P_Q = Q_R(z_0) \in \mathfrak{P}(Q)$ . On the other hand, since  $\theta < 1$ , we have by (2.5)

$$\sqrt{\theta} \int_{Q_R(z_0)} g^q dz \leq \frac{1 + \sqrt{\theta}}{2} \sup_{P \in \mathfrak{P}(Q)} \int_P g^q dz \leq \int_{Q_{R/2}(z_0)} g^q dz.$$

By this inequality and (1.1), we obtain

$$(2.8) \quad \int_{Q_{R/2}(z_0)} g^q dz \leq \frac{b}{1 - \sqrt{\theta}} \left\{ \left( \int_{Q_R(z_0)} g dz \right)^q + \int_{Q_R(z_0)} f^q dz \right\}.$$

Consequently, the combination of (2.6) with the estimate (2.8) with  $P_Q = Q_R(z_0)$  yields the estimate (2.4). □

Let  $P_Q \in \mathfrak{P}(Q)$  be a cube in Lemma 2.3. For  $Q \in \Omega_A$ , the cube  $P_Q$  includes  $Q (= \widehat{Q})$ . For  $Q \in \Omega_B^+ \cup \Omega_B^-$ , however,  $P_Q$  does not necessarily include  $Q (\neq \widehat{Q})$ . To surmount the complication, we make another classification of  $\Omega$  in accordance with the size of cubes compared with  $h$ . We introduce two disjoint subclasses of  $\Omega$ :

$$\begin{aligned} \Omega_{[A]} &= \{Q \in \Omega : P_Q \supset Q \text{ or } P_Q \equiv Q_R(z_0) \supset Q_{h/2}(z_0)\}, \\ \Omega_{[B]} &= \{Q \in \Omega : P_Q \not\supset Q \text{ and } P_Q \equiv Q_R(z_0) \subsetneq Q_{h/2}(z_0)\} \end{aligned}$$

and put

$$\Omega_{[B]}^\pm = \Omega_{[B]} \cap \Omega_B^\pm = \{Q \in \Omega_{[B]} : Q \subset \Sigma^\pm\}.$$

Obviously,  $\Omega_A \subset \Omega_{[A]}$  and  $\Omega_{[B]} \subset \Omega_B^+ \cup \Omega_B^-$ .

**Lemma 2.4.** *Suppose that  $s \geq (2 \cdot (2^{m+1})^2 \cdot |\Sigma|)^{1/q}$  and  $Q \in \Omega_{[B]}$ . Let  $P_Q \in \mathfrak{P}(Q)$  be a cube from Lemma 2.3. Then there exists a translation  $P_Q^\dagger$  of  $P_Q$  toward the  $t$ -axis such that*

$$(2.9) \quad Q \subset P_Q^\dagger \subset \Sigma \quad \text{and} \quad P_Q^\dagger \cap L_h = \emptyset.$$

PROOF: Suppose  $Q \subset C_k$  ( $k = 0, 1, 2, \dots$ ). We give here the proof only for  $Q \in \Omega_{[B]}^+$ ; cubes from  $\Omega_{[B]}^-$  can be treated in the same way.

Note that  $Q \equiv Q_\rho(\tau, \xi) \in \Omega_{[B]}^+$  is contained in only one strip  $(t_{n_*-1}, t_{n_*}) \times \Delta$  ( $0 \leq t_{n_*-1} < 3/2$ ). By the definition of  $\Omega_{[B]}$ ,  $P_Q \equiv Q_R(t_{n_0}, x_0) \in \mathfrak{P}(Q)$  satisfies  $R < h/2$ . Hence, the inclusion

$$\widehat{Q} = Q_\rho(t_{n_*-1} + \rho, \xi) \subset P_Q = Q_R(t_{n_0}, x_0)$$

implies  $t_{n_0} = t_{n_*-1}$  and  $\rho \leq R < h/2$ .

If  $t_{n_*} \leq 3/2$ , the relation  $\rho \leq R < h/2$  assures the existence of a translation  $P_Q^\dagger = Q_R(t^\dagger, x_0)$  of  $P_Q$  satisfying

$$Q \subset P_Q^\dagger \subset (t_{n_*-1}, t_{n_*}) \times \Delta \subset \Sigma.$$

In the other case  $t_{n_*} > 3/2$  ( $\geq t_{n_*-1}$ ), we show that there is also a translation  $P_Q^\dagger$  of  $P_Q$  such that

$$(2.10) \quad Q \subset P_Q^\dagger \subset (t_{n_*-1}, 3/2) \times \Delta.$$

Note that  $C_k$  including  $Q$  intersects  $(t_{n_*-1}, 3/2) \times \Delta$  ( $\supset Q$ ) and hence the distance of the  $\Sigma$ 's top surface  $\{3/2\} \times \Delta$  to  $C_k$  is less than to the level  $\{t_{n_*-1}\} \times \Delta$ , that is,  $2^{-k} < 3/2 - t_{n_*-1}$ . Besides, the estimate (2.7) still holds and implies  $2R < 2^{-k}$ . Combining these inequalities, we obtain the relation  $2R < 3/2 - t_{n_*-1}$ , which enables us to find  $P_Q^\dagger$  satisfying (2.10). □

Let  $P_Q$  be a cube from Lemma 2.3 and  $P_Q^\dagger$  a cube from Lemma 2.4. We introduce the following sub-families of  $\Omega_{[B]}$ :

$$\mathfrak{B}_1 = \{Q \in \Omega_{[B]} : \mathfrak{J}(P_Q^\dagger) \geq \mathfrak{J}(P_Q)\},$$

$$\mathfrak{B}_2 = \{Q \in \Omega_{[B]} : \mathfrak{J}(P_Q^\dagger) < \mathfrak{J}(P_Q)\},$$

where

$$\mathfrak{J}(P) = \left( \int_P g \, dz \right)^q + \int_P f^q \, dz.$$



**Lemma 2.5.** *Let  $s \geq (2 \cdot (2^{m+1})^4 \cdot |\Sigma|)^{1/q}$ . Then there exists a family  $\{R_Q : Q \in \Omega\}$  of cubes such that*

$$(2.11) \quad s < C_* \left\{ \int_{R_Q} g\Phi \, dz + \left( \int_{R_Q} (f\Phi)^q \, dz \right)^{1/q} \right\}, \quad Q \in \Omega,$$

and

$$(2.12) \quad \sum_{Q \in \Omega_{[A]} \cup \mathfrak{B}_1} |Q| \leq \left| \bigcup_{Q \in \Omega_{[A]} \cup \mathfrak{B}_1} R_Q \right|, \quad \sum_{Q \in \mathfrak{B}_2^\pm} |Q| \leq \left| \bigcup_{Q \in \mathfrak{B}_2^\pm} R_Q \right|$$

hold, where  $C_* = (2^{m+1})^{1+2/q} (2b/(1-\theta))^{1/q}$  and  $\mathfrak{B}_2^\pm = \mathfrak{B}_2 \cap \Omega_{[B]}^\pm$ .

PROOF: Assume that  $Q \subset C_k$  ( $k = 0, 1, 2, \dots$ ) and that the cube  $P_Q \in \mathfrak{P}(Q)$  from Lemma 2.3 is of the form  $P_Q = Q_R(z_0)$ .

First we shall treat  $Q \in \Omega_{[A]} \cup \mathfrak{B}_1$ . We define

$$R_Q = \begin{cases} Q_{\bar{R}}(z_0) & \text{if } Q \in \Omega_{[A]}, \\ P_Q^\dagger & \text{if } Q \in \mathfrak{B}_1, \end{cases}$$

where  $\bar{R} = \min\{r \geq R : Q_r(z_0) \supset Q\}$ . Then we have  $R_Q \supset Q$ . Since cubes from  $\Omega$  have mutually disjoint interiors, this inclusion implies the first inequality in (2.12).

We shall show that

$$(2.13) \quad |R_Q| \leq 2^{m+1}|P_Q|$$

holds for  $Q \in \Omega_{[A]}$ . In case  $Q \in \Omega_A$ ,  $R_Q = P_Q$  and hence (2.13) is valid. In the other case  $Q \in \Omega_{[A]} \setminus \Omega_A$  ( $\subset \Omega_B^+ \cup \Omega_B^-$ ),  $Q$  and  $\hat{Q}$  are included in one strip  $(t_{n_*-1}, t_{n_*}] \times \Delta$ , and  $P_Q = Q_R(z_0) (\supset \hat{Q})$  satisfies  $R \geq h/2$ . Consequently,  $Q_{2R}(z_0)$  strides across the strip  $(t_{n_*-1}, t_{n_*}] \times \Delta$  and hence includes  $Q$ . This means that  $\bar{R} \leq 2R$ , which results in (2.13).

For  $Q \in \Omega_{[A]}$ , the inclusion  $P_Q \subset R_Q$  and (2.13) gives us

$$(2.14) \quad \mathfrak{J}(P_Q) \leq (2^{m+1})^q \mathfrak{J}(R_Q),$$

which is also true for  $Q \in \mathfrak{B}_1$ , because of  $\mathfrak{J}(P_Q^\dagger) \geq \mathfrak{J}(P_Q)$ .

It follows from (2.4) and (2.14) that

$$(2.15) \quad (2^{m+1})^{k-1} s^q \leq (2^{m+1})^q \frac{2b}{1-\theta} \mathfrak{J}(R_Q).$$

Noting the assumption on  $s$ , we combine (2.3) with (2.13) to get

$$|R_Q| < (2^{-(k+1)})^{m+1}$$

and hence

$$R_Q \subset C_0 \cup \dots \cup C_{k+1},$$

which, together with (2.15), give us the desired estimate (2.11).

Next, we deal with  $Q \in \mathfrak{B}_2^+$  or  $\mathfrak{B}_2^-$ . Note that  $P_Q$  has the center  $z_0 = (t_{n_0}, x_0) \in L_h$  and  $P_Q^\dagger = Q_R(t^\dagger, x_0)$  is included in only one strip  $(t_{n_0}, t_{n_0+1}] \times \Delta$  or  $(t_{n_0-1}, t_{n_0}] \times \Delta$ . We introduce the symmetrization  $P_Q^\ddagger$  of  $P_Q^\dagger$  with respect to the level  $\{t_{n_0}\} \times \Delta$ :

$$P_Q^\ddagger = Q_R(2t_{n_0} - t^\dagger, x_0).$$

Then the property  $\mathfrak{J}(P_Q^\ddagger) < \mathfrak{J}(P_Q)$  for  $Q \in \mathfrak{B}_2^\pm$  implies  $\mathfrak{J}(P_Q^\ddagger) \geq \mathfrak{J}(P_Q)$ . Hence,  $R_Q = P_Q^\ddagger$  satisfies (2.13) and (2.14), from which the estimate (2.11) follows. The second inequality in (2.12) is derived from the relation  $|P_Q^\ddagger| = |P_Q^\dagger|$  and  $Q \subset P_Q^\dagger$ .  $\square$

PROOF OF THE THEOREM: Suppose that  $\nu$  is a constant, not less than  $(2 \cdot (2^{m+1})^4 \cdot |\Sigma|)^{1/q}$ , to be chosen later. Let  $t \geq 1$  and  $s = \nu t$ .

Now we shall estimate  $\sum_{Q \in \Omega} |Q|$  from the right-hand side of (2.2) with  $s = \nu t$ .

This is easily done by means of (2.12):

(2.16)

$$\begin{aligned} \int_{E(g\Phi, \nu t)} (g\Phi)^q dz &\leq 3^{m+1}(\nu t)^q \left( \sum_{Q \in \Omega_{[A] \cup \mathfrak{B}_1}} |Q| + \sum_{Q \in \mathfrak{B}_2^+} |Q| + \sum_{Q \in \mathfrak{B}_2^-} |Q| \right) \\ &\leq 3^{m+1}(\nu t)^q \left( \left| \bigcup_{Q \in \Omega_{[A] \cup \mathfrak{B}_1}} R_Q \right| + \left| \bigcup_{Q \in \mathfrak{B}_2^+} R_Q \right| + \left| \bigcup_{Q \in \mathfrak{B}_2^-} R_Q \right| \right). \end{aligned}$$

On the other hand, for each  $R_Q$ , it follows from (2.11) that

$$\begin{aligned} \nu t |R_Q| &< C_* \left\{ \int_{R_Q} g\Phi dz + |R_Q|^{1-1/q} \left( \int_{R_Q} (f\Phi)^q dz \right)^{1/q} \right\} \\ &\leq C_* \left\{ \int_{R_Q \cap E(g\Phi, t)} g\Phi dz + t |R_Q| \right. \\ &\quad \left. + \frac{q-1}{q} t |R_Q| + \frac{1}{q} t^{1-q} \left( \int_{R_Q \cap E(f\Phi, t)} (f\Phi)^q dz + t^q |R_Q| \right) \right\}. \end{aligned}$$

Thus we obtain

$$(2.17) \quad t|R_Q| \leq \frac{1}{\nu/C_* - 2} \left\{ \int_{R_Q \cap E(g\Phi, t)} g\Phi \, dz + t^{1-q} \int_{R_Q \cap E(f\Phi, t)} (f\Phi)^q \, dz \right\},$$

provided that

$$\nu > 2C_*(m, b, q, \theta) \equiv 2(2^{m+1})^{1+q/2} \left( 2b / (1 - \theta) \right)^{1/q}.$$

By a well-known covering argument, the family  $\{R_Q : Q \in \Omega_{[A]} \cup \mathfrak{B}_1\}$  has a subfamily  $\mathfrak{R}_1$  of mutually disjoint cubes such that, for a positive constant  $C(m)$ ,

$$\left| \bigcup_{Q \in \Omega_{[A]} \cup \mathfrak{B}_1} R_Q \right| \leq C(m) \sum_{R_Q \in \mathfrak{R}_1} |R_Q|.$$

This and (2.17) yield the estimate

$$\left| \bigcup_{Q \in \Omega_{[A]} \cup \mathfrak{B}_1} R_Q \right| \leq \frac{C(m)}{\nu/C_* - 2} \left\{ t^{-1} \int_{E(g\Phi, t)} g\Phi \, dz + t^{-q} \int_{E(f\Phi, t)} (f\Phi)^q \, dz \right\}.$$

The other quantities  $\left| \bigcup_{Q \in \mathfrak{B}_2^+} R_Q \right|$  and  $\left| \bigcup_{Q \in \mathfrak{B}_2^-} R_Q \right|$  from the right-hand side of (2.16)

are estimated in the same way. From these estimates and (2.16), we deduce (1.7).

We thus have completed the proof of the Theorem.  $\square$

#### REFERENCES

- [1] Calderón A.P., Zygmund A., *On the existence of certain singular integrals*, Acta Math. **88** (1952), 85–139.
- [2] Gehring F.W., *The  $L^p$ -integrability of the partial derivatives of a quasi conformal mapping*, Acta Math. **130** (1973), 265–277.
- [3] Giaquinta M., *Multiple integrals in the calculus of variations and non linear elliptic systems*, Vorlesungsreihe des Sonderforschungsbereiches, Universität Bonn, 1981.
- [4] Giaquinta M., *Introduction to regularity theory for nonlinear elliptic systems*, Lecture in Mathematics, Birkhäuser, 1993.
- [5] Giaquinta M., Giusti E., *Partial regularity for the solutions to nonlinear parabolic systems*, Ann. Mat. Pura Appl. **47** (1973), 253–266.
- [6] Giaquinta M., Giusti E., *On the regularity of the minima of variational integrals*, Acta Math. **148** (1982), 31–46.
- [7] Giaquinta M., Modica G., *Regularity results for some classes of higher order non linear elliptic systems*, J. Reine Angew. Math. **311/312** (1979), 145–169.
- [8] Giaquinta M., Struwe M., *On the partial regularity of weak solutions of non-linear parabolic systems*, Math. Z. **142** (1982), 437–451.
- [9] Haga J., Kikuchi N., *On the existence of the harmonic variational flow subject to the two-sided conditions*, Zap. Nauch. Semi. POMI. **243** (1997), 324–337.
- [10] Haga J., Kikuchi N., *On the higher integrability of gradients of the solutions to difference partial differential systems of elliptic-parabolic type*, preprint.
- [11] Hoshino K., *On a method constructing solutions to hyperbolic partial differential equations*, Nonlinear Studies **4** (1997), 133–142.
- [12] Hoshino K., *On a construction of weak solutions to semilinear dissipative hyperbolic systems with the higher integrable gradients*, to appear in Nonlinear Analysis.
- [13] Hoshino K., Kikuchi N., *On a construction of weak solutions to linear hyperbolic partial differential systems with the higher integrable gradients*, Zap. Nauch. Semi. POMI. **233** (1996), 30–52.

- [14] Kikuchi N., *An approach to the construction of Morse flows for variational functionals*, in: *Nematics-Mathematical and Physical Aspects*, ed. by J.-M. Coron, J.-M. Ghidaglia and F. Helein, NATO Adv. Sci. Inst. Ser.C: Math. Phys. Sci. **332**, Kluwer Acad. Publ., Dordrecht-Boston-London, 1991, pp. 195–198.
- [15] Kikuchi N., *A construction method of Morse flows to variational functionals*, *Nonlinear World* **1** (1994), 131–147.
- [16] Rothe E., *Wärmeleitungsgleichung mit nichtkonstanten Koeffizienten*, *Math. Ann.* **104** (1931), 340–362.
- [17] Stein E.M., *Singular integrals and differentiability properties of functions*, Princeton University Press, Princeton, 1970.

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE AND TECHNOLOGY, KEIO UNIVERSITY,  
HIYOSHI, YOKOHAMA 223–8522, JAPAN

*E-mail:* kikuchi@math.keio.ac.jp

(Received September 12, 1997)