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# The Lévy laplacian and differential operators of 2-nd order in Hilbert spaces

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*Abstract.* We shall show that every differential operator of 2-nd order in a real separable Hilbert space can be decomposed into a regular and an irregular operator. Then we shall characterize irregular operators and differential operators satisfying the maximum principle. Results obtained for the Lévy laplacian in [3] will be generalized for irregular differential operators satisfying the maximum principle.

*Keywords:* Lévy laplacian, maximum principle, Dirichlet and Poisson problem

*Classification:* 31C45, 46C99, 47F05

## 0. Preliminaries

First of all, we shall introduce some notation and the Lévy laplacian will be defined. Let $\mathbb{H}$ be a real separable Hilbert space with inner product $(\,.\,,\,.\,)$. The induced norm is denoted by

$$\|x\| = \sqrt{(x,x)},\ x \in \mathbb{H}.$$

A bilinear functional $a : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ is said to be bounded if

$$\|a\| = \sup_{\|x\|=1, \|y\|=1} |a(x,y)| < +\infty, \tag{1}$$

and symmetric if $a(x,y) = a(y,x)$ for all $x, y \in \mathbb{H}$. If $A$ and $B$ are Banach spaces (we shall write B-space instead of Banach space), we denote by $\mathcal{L}(A,B)$ the B-space of all bounded linear operators mapping $A$ into $B$. Let $S^*$ denote the adjoint operator to $S \in \mathcal{L} \equiv \mathcal{L}(\mathbb{H},\mathbb{H})$. An operator $S \in \mathcal{L}$ is called self-adjoint, if $S^* = S$. We denote by $\mathcal{N}_s^2$ the B-space of all symmetric bounded bilinear functionals on $\mathbb{H}$ with the norm (1) and by $\mathcal{L}_s$ the B-space of all self-adjoint operators on $\mathbb{H}$ endowed with the operator norm.

**Remark.** A mapping $\Phi$, defined for each $S \in \mathcal{L}_s$ by

$$\Phi(S)(x,y) = (Sx,y),\ x,y \in \mathbb{H}, \tag{2}$$

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is an isometric isomorphism $\mathcal{L}_s$ onto $\mathcal{N}_s^2$. We may identify elements of these B-spaces.

Let $x \in \mathbb{H}$. For $r > 0$ denote by $B(x,r)$ the ball in $\mathbb{H}$ having its center at $x$ and its radius $r$, i.e.,

$$B(x,r) = \{y \in \mathbb{H};\ \|y - x\| < r\}.$$

Denote by $\mathcal{G}_x$ all real functionals defined on $\mathbb{H}$ having the Fréchet derivatives of 2-nd order at the point $x$. Let $f \in \mathcal{G}_x$. Then the Fréchet derivative of 2-nd order of the functional $f$ at the point $x$ which is denoted by $f''(x)$ is an element of $\mathcal{N}_s^2$. Let $E = \{e_n\}_{n=1}^{\infty}$ be an orthonormal basis of $\mathbb{H}$ (in what follows we shall write ON-basis instead of orthonormal basis). We define $D_{ij}f(x) = f''(x)(e_i, e_j)$, $i, j = 1, 2, 3, \dots$ and the Lévy laplacian of $f$ at the point $x$ by

$$L^E f(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} D_{ii} f(x),$$

whenever the limit exists, as in [5].

**Remark.** If $f''(x) \in \mathcal{L}_\infty \cap \mathcal{L}_s$, then $D_{ii}f(x) = (f''(x)e_i, e_i) \to 0$ and, consequently, $L^E f(x) = 0$.

If $G \subset \mathbb{H}$ is an open set, then we denote by $\mathcal{C}^2(G)$ the set of all functionals $f : G \to \mathbb{R}$ having continuous Fréchet derivatives of 2-nd order on $G$.

## 1. Regular and irregular operators

We shall show that every differential operator of 2-nd order can be decomposed into a regular and an irregular part. First of all, we shall prove a lemma.

**Lemma.** *Let $a$ be a bounded bilinear functional on $\mathbb{H}$ and $b(x) = a(x,x)$, $x \in \mathbb{H}$. Then for each $x \in \mathbb{H}$*

$$b''(x)(h,k) = a(h,k) + a(k,h),\ \ h,k \in \mathbb{H}$$

*and, in particular, $b''(x) = 2a$ if $a$ is symmetric.*

PROOF: It is easy to compute. □

Now we define differential operators which we are going to deal with.

**Definition 1.** *Let $\psi \in \mathcal{L}(\mathcal{N}_s^2, \mathbb{R})$. For each $x \in \mathbb{H}$ define*

$$\boldsymbol{D}_x^\psi f = \psi(f''(x)),\ \ f \in \mathcal{G}_x.$$

*(In what follows we shall write also $\boldsymbol{D}^\psi f(x)$ instead of $\boldsymbol{D}_x^\psi f$.) Then the family $\boldsymbol{D}^\psi = \{\boldsymbol{D}_x^\psi;\ x \in \mathbb{H}\}$ is called a differential operator (of 2-nd order). Denote by $\mathcal{D}$ the set of all such differential operators.*

**Remark.** If $\boldsymbol{D} \in \mathcal{D}$, then there is a unique $\psi \in \mathcal{L}(\mathcal{N}_s^2, \mathbb{R})$ such that $\boldsymbol{D} = \boldsymbol{D}^\psi$. This is a consequence of the previous lemma. If $\boldsymbol{D}^\psi, \boldsymbol{D}^\varphi \in \mathcal{D}$, we define, of course, $\boldsymbol{D}^\psi + \boldsymbol{D}^\varphi = \boldsymbol{D}^{\psi+\varphi}$.

It is apparent that the description of differential operators is equivalent to the description of the dual space $\mathcal{L}_s^*$ of $\mathcal{L}_s$. We denote by $\mathcal{L}_\infty$ the B-space of all compact linear operators on $\mathbb{H}$ with the operator norm. Let $L \in \mathcal{L}_\infty$ and $T = L^*L$. It is well-known that there are an ON-basis $E = \{e_n\}_{n=1}^\infty$ of $\mathbb{H}$ consisting of eigenvectors of $T$ and a sequence of non-negative numbers $\{\mu_n\}$ such that

$$Tx = \sum_{n=1}^\infty \mu_n (x, e_n) e_n, \ x \in \mathbb{H}.$$

Furthermore, $\mu_n$ is an eigenvalue of $T$ corresponding to the eigenvector $e_n$. Define

$$\|L\|_1 = \sum_{n=1}^\infty (\mu_n)^{\frac{1}{2}}. \tag{3}$$

A compact operator $L$ is said to be nuclear if $\|L\|_1 < \infty$. We denote by $\mathcal{L}_1$ the B-space of all nuclear operators on $\mathbb{H}$ with the norm (3).

Notice that for every $A \in \mathcal{L}_1 \cap \mathcal{L}_s$ there are an ON-basis $E = \{e_n\}_{n=1}^\infty$ of $\mathbb{H}$ and a sequence $\{\lambda_n\} \in \ell^1$, i.e. $\sum_{n=1}^\infty |\lambda_n| < +\infty$, such that

$$Ax = \sum_{n=1}^\infty \lambda_n (x, e_n) e_n, \ x \in \mathbb{H}.$$

It is well-known that $AB \in \mathcal{L}_1$ if $A \in \mathcal{L}_1$ and $B \in \mathcal{L}$, see [4, p. 121].

Let $E = \{e_n\}_{n=1}^\infty$ be an ON-basis of $\mathbb{H}$ and $L \in \mathcal{L}_1$. Define the trace of $L$ by

$$tr(L) = \sum_{n=1}^\infty (Le_n, e_n). \tag{4}$$

The sum in (4) is absolutely convergent and independent of the choice of the ON-basis $E$, as shown in [4, Theorem 8.1, p. 127].

**Example 1.** Now we describe a differential operator of the above type in $\mathbb{R}^m$, see Definition 1. Let $\mathbb{H} = \mathbb{R}^m$, $E = \{e_i\}_{i=1}^m$ be the standard basis of $\mathbb{R}^m$ and $\psi \in \mathcal{L}(\mathcal{N}_s^2, \mathbb{R})$. Then there is a symmetric matrix $A = [a_{ij}]_{i,j=1,2,\dots,m}$ such that

$$\psi(a) = \sum_{i,j=1}^m a_{ij} a(e_i, e_j), \ a \in \mathcal{N}_s^2.$$

Consequently,

$$\boldsymbol{D}^\psi f(x) = \sum_{i,j=1}^m a_{ij} D_{ij} f(x) = tr(A f''(x))$$

for each $x \in \mathbb{H}$ and $f \in \mathcal{G}_x$. Denote by $\Delta$ the Laplace operator, i.e.,

$$\Delta = \sum_{i=1}^{m} \frac{\partial^2}{\partial x_i^2}.$$

Then $\Delta = \boldsymbol{D}^{tr}$.

**Definition 2.** *Let $\psi \in \mathcal{L}_s^*$. If there is an $A \in \mathcal{L}_1 \cap \mathcal{L}_s$ such that*

$$\psi(B) = tr(AB), \ B \in \mathcal{L}_s,$$

*then $\psi$ is said to be regular. If $\psi(B) = 0$ for each $B \in \mathcal{L}_\infty \cap \mathcal{L}_s$, then $\psi$ is said to be irregular.*

**Theorem 1.** *For each $\psi \in \mathcal{L}_s^*$, there are a unique regular functional $\psi_r$ and a unique irregular functional $\psi_i$ such that $\psi = \psi_r + \psi_i$.*

Proof: By the Hahn-Banach theorem there is a $\tau \in \mathcal{L}_\infty^*$ such that $\tau = \psi$ on $\mathcal{L}_\infty \cap \mathcal{L}_s$. Moreover, for each $\tau \in \mathcal{L}_\infty^*$ there exists a unique operator $A \in \mathcal{L}_1$ such that $\|\tau\| = \|A\|_1$ and

$$\tau(E) = tr(AE), \ E \in \mathcal{L}_\infty,$$

see [4, Theorem 12.3, p. 170]. Take such an $A \in \mathcal{L}_1$ and define $B = \frac{1}{2}(A + A^*)$, $C = \frac{1}{2}(A - A^*)$, $\psi_r(D) = tr(BD)$, $D \in \mathcal{L}_s$ and $\psi_i(D) = tr(CD) + \psi(D) - tr(AD)$, $D \in \mathcal{L}_s$.

Since $B \in \mathcal{L}_1 \cap \mathcal{L}_s$, $\psi_r$ is regular. In order to prove that $\psi_i$ is irregular it is sufficient to show that

$$tr(CD) = 0, \ D \in \mathcal{L}_\infty \cap \mathcal{L}_s.$$

Let $D \in \mathcal{L}_\infty \cap \mathcal{L}_s$. Then there are an ON-basis $E = \{e_n\}_{n=1}^\infty$ of $\mathbb{H}$ and a sequence of real numbers $\{\lambda_n\}$ such that

$$Dx = \sum_{n=1}^{\infty} \lambda_n (x, e_n) e_n, \ x \in \mathbb{H}.$$

Since $C^* = -C$ we have $(Ch, h) = 0$ for each $h \in \mathbb{H}$. Trivially,

$$tr(CD) = \sum_{n=1}^{\infty} \lambda_n (Ce_n, e_n) = 0,$$

and the assertion is proved. Obviously, $\psi_r + \psi_i = \psi$. It only remains to show uniqueness of such a decomposition. We prove that there is no non-trivial $\psi \in \mathcal{L}_s^*$ which is both regular and irregular. Let $A \in \mathcal{L}_1 \cap \mathcal{L}_s$ satisfy that $tr(AD) = 0$ for each $D \in \mathcal{L}_\infty \cap \mathcal{L}_s$. Fix an $h \in \mathbb{H}$, $\|h\| = 1$ and consider $D \in \mathcal{L}_\infty \cap \mathcal{L}_s$ defined by $Dk = (k, h)h$, $k \in \mathbb{H}$. By assumptions we get

$$(Ah, h) = tr(AD) = 0.$$

Since $A$ is a self-adjoint operator we conclude $A = 0$. □

**Definition 3.** *Let $\psi \in \mathcal{L}(\mathcal{N}_s^2, \mathbb{R})$. Then $\boldsymbol{D}^\psi$ is said to be a regular or irregular operator if $\psi$ is regular or irregular, respectively.*

**Theorem 2.** *Each differential operator can be uniquely written as a sum of a regular and an irregular operators.*

PROOF: This follows directly from Theorem 1 and Remark following Definition 1. □

**Remark.** Definition 1 is a special case of the definition of differential operator of n-th order on functionals defined in a topological linear space in [1]. The concept of regular functional introduced there is slightly different from that defined above. In [1] a functional $\psi \in \mathcal{L}^*$ is said to be regular provided $\|\psi\| = \|\psi|_{\mathcal{L}_\infty}\|$, i.e.,

$$\sup\{|\psi(f)|;\ f \in \mathcal{L}, \|f\| = 1\} = \sup\{|\psi(f)|;\ f \in \mathcal{L}_\infty, \|f\| = 1\}.$$

A functional $\psi \in \mathcal{L}^*$ was shown to be regular if and only if there was an $A \in \mathcal{L}_1$ such that

$$\psi(B) = tr(AB),\ B \in \mathcal{L}.$$

Furthermore, it was proved that each $\psi \in \mathcal{L}^*$ could be uniquely written as a sum of a regular functional and a functional vanishing on $\mathcal{L}_\infty$. A decomposition of a differential operator, which is a consequence of the decomposition of $\psi$, is not, however, unique in general.

J.L. Daleckij and S.V. Fomin developed the theory of regular differential operators in [2]. They dealt with parabolic equation arising from these differential operators there.

Now we give some examples of differential operators.

**Example 2.** Let $\boldsymbol{D}^\psi$ be a regular operator. Then there is an $A \in \mathcal{L}_1 \cap \mathcal{L}_s$ such that

$$\psi(B) = tr(AB),\ B \in \mathcal{L}_s.$$

There exist an ON-basis $E = \{e_n\}_{n=1}^\infty$ of $\mathbb{H}$ and a sequence $\{\lambda_n\} \in \ell^1$ such that

$$Ax = \sum_{n=1}^\infty \lambda_n (x, e_n) e_n,\ x \in \mathbb{H}.$$

By easy computation we get for each $x \in \mathbb{H}$ and $f \in \mathcal{G}_x$

$$\boldsymbol{D}^\psi f(x) = tr(Af''(x)) = \sum_{n=1}^\infty (Af''(x)e_n, e_n) = \\ = \sum_{n=1}^\infty (f''(x)e_n, Ae_n) = \sum_{n=1}^\infty \lambda_n D_{nn} f(x).$$

**Example 3.** By Example 1, in $\mathbb{R}^m$ each differential operator of 2-nd order of the above type is regular and only trivial differential operator is irregular.

**Example 4.** The Lévy laplacian can be extended to a differential operator of the above type and each such extension gives an irregular operator. In fact, define

$$\tau(b) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} b(e_i, e_i)$$

for each $b \in P = \{c \in \mathcal{N}_s^2;\ \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} c(e_i, e_i) \text{ exists}\}$. Obviously,

$$|\tau(b) - \tau(c)| \le \|b - c\|$$

for each $b, c \in P$, which means that $\tau$ is a bounded linear functional on $P$. By the Hahn-Banach theorem there is a $\psi \in \mathcal{L}(\mathcal{N}_s^2, \mathbb{R})$ such that $\psi|_P = \tau$. Then the differential operator $\boldsymbol{D}^\psi$ is an extension of $L^E$ and each such extension is an irregular operator because $\tau = 0$ on $\mathcal{L}_\infty \cap \mathcal{L}_s$ by the remark at the end of Preliminaries.

## 2. Characteristic properties of irregular operator

**Theorem 3.** *Let $\boldsymbol{D} \in \mathcal{D}$ and $x \in \mathbb{H}$.*

A. *If $\boldsymbol{D}$ is an irregular operator, then the following assertions hold.*
   (1) *For each $u, v \in \mathcal{G}_x$, the formula $\boldsymbol{D}(uv)(x) = u(x)\boldsymbol{D}v(x) + v(x)\boldsymbol{D}u(x)$ holds.*
   (2) *Let $u = (u_1, \dots, u_m)$ have components from $\mathcal{G}_x$ and a real function $F$ defined on $\mathbb{R}^m$ have the second Fréchet derivative at $u(x)$. Then $\boldsymbol{D}(F \circ u)(x) = \sum_{n=1}^{m} \partial_n F(u(x))\boldsymbol{D}u_n(x)$.*

B. *If $\boldsymbol{D}$ is a differential operator satisfying the above condition (1) or the condition (2) for each $u \in \mathcal{G}_x$ and $F(y) = y^2$, $y \in \mathbb{R}$, then $\boldsymbol{D}$ is an irregular operator.*

Proof: Let $\boldsymbol{D} = \boldsymbol{D}^\psi$ be an irregular operator, $x \in \mathbb{H}$ and $u, v \in \mathcal{G}_x$. Then it is easy to show that

$$\begin{aligned}(uv)''(x)(h,k) = v(x)u''(x)(h,k) + v'(x)(h)u'(x)(k) + \\ + u'(x)(h)v'(x)(k) + u(x)v''(x)(h,k),\ h,k \in \mathbb{H}.\end{aligned}$$

A linear operator $S$ on $\mathbb{H}$ is said to be finite-dimensional if $S(\mathbb{H})$ is a finite dimensional subspace of $\mathbb{H}$. We denote by $\mathcal{L}_f$ the space of all finite-dimensional operators on $\mathbb{H}$. Obviously, $\mathcal{L}_f \subset \mathcal{L}_\infty$. Define $t \in \mathcal{N}_s^2$ by

$$t(h,k) = u'(x)(h)v'(x)(k) + v'(x)(h)u'(x)(k),\ h,k \in \mathbb{H}.$$

The bilinear functional $t \in \mathcal{N}_s^2$ is identified with an operator $S \in \mathcal{L}_s \cap \mathcal{L}_f$ with $\dim S(\mathbb{H}) \leq 2$. Recalling that $\psi$ is irregular we have $\psi(t) = 0$ and

$$\boldsymbol{D}^{\psi}(uv)(x) = \psi((uv)''(x)) = \psi(u(x)v''(x) + v(x)u''(x)) = \\ = u(x)\boldsymbol{D}^{\psi}v(x) + v(x)\boldsymbol{D}^{\psi}u(x),$$

which completes the proof of (1).

Let $u$ and $F$ satisfy the assumptions of (2). A simple calculation shows that for each $h, k \in \mathbb{H}$

$$(F \circ u)''(x)(h,k) = \sum_{n=1}^{m} \partial_n F(u(x))u_n''(x)(h,k) + \sum_{i,p=1}^{m} \partial_{ip} F(u(x))u_i'(x)(h)u_p'(x)(k).$$

Since the second sum of the above equality for a fixed $x$ is a member of $\mathcal{N}_s^2$ identified with an operator of $\mathcal{L}_s \cap \mathcal{L}_f$ we conclude the proof of (2) as above. It remains to prove the part B.

Let $x \in \mathbb{H}$ and $\boldsymbol{D} = \boldsymbol{D}^{\psi}$ satisfy the assumptions of B. Let $d \in \mathbb{H}^*$, and put $e(y,z) = d(y)d(z)$, $y, z \in \mathbb{H}$. Since $(d^2)''(x) = 2e$ and $d''(x) = 0$, we get

$$2\psi(e) = \boldsymbol{D}^{\psi}(d^2)(x) = 2d(x)\boldsymbol{D}^{\psi}d(x) = 0.$$

Hence, $\psi(e) = 0$. If $a, b \in \mathbb{H}^*$ and $c \in \mathcal{N}_s^2$ is defined by

$$c(y,z) = a(y)b(z) + a(z)b(y), \; y, z \in \mathbb{H} \tag{3}$$

we have $\psi(c) = 0$ for the following equality holds:

$$c(y,z) = (a+b)(y)(a+b)(z) - a(y)a(z) - b(y)b(z), \; y, z \in \mathbb{H}.$$

Noting that each bilinear functional of $\mathcal{N}_s^2$ which is identified with an operator of $\mathcal{L}_s \cap \mathcal{L}_f$ can be written as a finite sum of bilinear functionals of the type (3) and using the density of $\mathcal{L}_f \cap \mathcal{L}_s$ in $\mathcal{L}_\infty \cap \mathcal{L}_s$ with respect to the operator norm and the continuity of $\psi$, we conclude that $\psi$ is irregular. □

It is well-known that the Laplace operator in $\mathbb{R}^m$ is independent of the choice of an ON-basis in $\mathbb{R}^m$. We now show that the independence of a differential operator of functionals of a certain class defined in the following definition on the choice of an ON-basis in $\mathbb{H}$ is equivalent to the irregularity of the differential operator if $\mathbb{H}$ is an infinite dimensional space.

**Definition 4.** *We denote by $\mathcal{F}$ the set of all $S \in \mathcal{L}_s$ such that there are a real number $\lambda$ and a linear operator $T \in \mathcal{L}_s \cap \mathcal{L}_\infty$ for which $S = \lambda I + T$. Here $I$ is the identity operator on $\mathbb{H}$. If $a \in \mathbb{H}$ define*

$$\Omega_a = \{f \in \mathcal{G}_a; \; f''(a) \in \mathcal{F}\}.$$

*If $U$ is a unitary operator on $\mathbb{H}$, i.e., $U$ is a linear operator satisfying*

$$(Ux, Uy) = (x, y), \; x, y \in \mathbb{H},$$

*and an $a \in \mathbb{H}$, then we define a rotation around the point $a$ by*

$$U_a(x) = a + U(x - a), \; x \in \mathbb{H}.$$

**Theorem 4.** *Let $\mathbb{H}$ be an infinite dimensional space and $a \in \mathbb{H}$. If $\boldsymbol{D} = \boldsymbol{D}^{\psi} \in \mathcal{D}$, then the following statements are equivalent each to other.*

(i) *The differential operator $\boldsymbol{D}$ is irregular.*
(ii) *If $S \in \mathcal{F}$, then $\psi(U^*SU) = \psi(S)$ for each unitary operator $U$ on $\mathbb{H}$.*
(iii) *If $f \in \Omega_a$, then the following condition of invariance $(CI)$ holds:*
*$(CI)$ If $U$ is a unitary operator on $\mathbb{H}$, then $\boldsymbol{D}(f \circ U_a)(a) = \boldsymbol{D}f(a)$.*

Proof: We now show that (i) implies (ii). Let $S \in \mathcal{F}$. Then there are $\lambda \in \mathbb{R}$ and $T \in \mathcal{L}_s \cap \mathcal{L}_\infty$ such that $S = \lambda I + T$. Recalling that $\psi$ is irregular if $\psi(C) = 0$ for each $C \in \mathcal{L}_s \cap \mathcal{L}_\infty$, we get for each unitary operator $U$ that

$$\psi(U^*SU) = \psi(\lambda I + U^*TU) = \psi(\lambda I) = \psi(S)$$

because $U^*U = I$ and $U^*TU$ is a compact operator on $\mathbb{H}$.

It is easy to show that for each $f \in \mathcal{G}_a$ and each unitary operator $U$ on $\mathbb{H}$ the equality

$$(f \circ U_a)''(a) = U^* f''(a) U$$

holds and hence

$$\boldsymbol{D}^{\psi}(f \circ U_a)(a) = \psi(U^* f''(a) U). \tag{4}$$

By (4), (iii) follows from (ii). Now we prove that (ii) follows from (iii). Fix a linear operator $S \in \mathcal{F}$. Consider a functional $f$ defined by

$$f(x) = \frac{1}{2}(Sx, x), \ \ x \in \mathbb{H}.$$

Since $f''(a) = S$ we have that $f \in \Omega_a$ and according to (iii) and (4)

$$\psi(U^*SU) = \boldsymbol{D}^{\psi}(f \circ U_a)(a) = \boldsymbol{D}^{\psi} f(a) = \psi(S).$$

It only remains to prove that (ii) implies (i). Fix a linear functional $\psi \in \mathcal{L}_s^*$ satisfying (ii). According to Theorem 1, there are a unique regular functional $\psi_r$ and a unique irregular functional $\psi_i$ such that $\psi = \psi_r + \psi_i$. We shall show that $\psi_r \equiv 0$. As we know there are an ON-basis $E = \{e_n\}_{n=1}^{\infty}$ and a sequence $\{\lambda_n\} \in \ell^1$ such that for each $T \in \mathcal{L}_s$

$$\psi_r(T) = \sum_{n=1}^{\infty} \lambda_n (Te_n, e_n). \tag{5}$$

By the hypothesis and the definition of an irregular operator we obtain for each $T \in \mathcal{L}_s \cap \mathcal{L}_\infty$ and unitary operator $U$ on $\mathbb{H}$

$$\psi_r(U^*TU) = \psi(U^*TU) = \psi(T) = \psi_r(T).$$

Now we show that $\{\lambda_n\}$ is a constant sequence. Assume on the contrary that $\lambda_i \neq \lambda_j$ for some $i, j$. Defining

$$Tx = (x, e_i)e_i, \; x \in \mathbb{H},$$

and considering a unitary operator $U$ on $\mathbb{H}$ such that $Ue_j = e_i$, $Ue_i = e_j$ and $Ue_n = e_n$ for $n \neq i, j$ we get

$$\lambda_i = \psi_r(T) = \psi_r(U^*TU) = \lambda_j,$$

a contradiction. For $\mu = \lambda_1 = \lambda_2 = \dots$ we have $\psi_r = \mu tr$. For the first time we shall use the assumption that $\mathbb{H}$ is an infinite dimensional space. Since every constant infinite sequence of $\ell^1$ is trivial $\mu = 0$ and the proof is complete. $\square$

Let $\mathbb{H} = \mathbb{R}^m$, $a \in \mathbb{H}$ and $\boldsymbol{D}^\psi \in \mathcal{D}$. Then $\mathcal{F} = \mathcal{L}_s$, $\Omega_a = \mathcal{G}_a$ and $\psi = \psi_r$. Notice that in this case we actually proved that if the condition (CI) holds for each $f \in \mathcal{G}_a$, then there is a $\mu \in \mathbb{R}$ such that

$$\psi(A) = \mu tr(A), \; A \in \mathcal{L}_s.$$

Since the trace $tr$ satisfies (ii) in Theorem 4 we verified the following remark.

**Remark.** Let $\mathbb{H} = \mathbb{R}^m$, $a \in \mathbb{H}$ and $\boldsymbol{D} \in \mathcal{D}$. Then the condition (CI) holds for each $f \in \mathcal{G}_a$ if and only if there is a $\mu \in \mathbb{R}$ such that $\boldsymbol{D} = \mu\Delta$.

**Remark.** By the previous theorem, if $a \in \mathbb{H}$ and $\boldsymbol{D}^\psi$ is an irregular operator for which $\psi(I) = 1$, then $\boldsymbol{D}^\psi$ restricted to $\Omega_a$ coincides with the invariant Laplace operator at the point $a$ introduced in [7]. In the paper it is shown that

$$\Omega_a = \{f \in \mathcal{G}_a;\; f \text{ satisfies (CI)}\},$$

which means that $\Omega_a$ is a set of all functionals $f \in \mathcal{G}_a$ the differential operator of which is independent of the choice of an ON-basis.

## 3. Differential operators satisfying maximum principle

We now characterize differential operators satisfying the maximum principle on the class of functionals defined below. In this paragraph $G \subset \mathbb{H}$ will be a nonempty bounded open set unless otherwise specified.

**Definition 5.** *We denote by $\mathcal{C}_b(G)$ the set of all functionals $f : \overline{G} \to \mathbb{R}$ which are bounded and uniformly continuous on $\overline{G}$. For each $g \in \mathcal{C}_b(G)$ define*

$$\|g\|_0 = \sup_{x \in \overline{G}} |g(x)|.$$

*Moreover, we denote by $\mathcal{C}_b^2(G)$ the set of all $f \in \mathcal{C}_b(G)$ for which the mapping*

$$f'' : G \to \mathcal{N}_s^2$$

*is bounded and uniformly continuous.*

**Remark.** The linear space $\mathcal{C}_b(G)$ with the norm $\| \|_0$ is a B-space because uniform convergence preserves both boundedness and uniform continuity. If $G \subset \mathbb{H}$ and $f$ is a uniformly continuous functional on $G$, then there exists exactly one uniformly continuous functional $F$ on $\overline{G}$ such that $F|_G = f$.

**Example 5.** Quadratic forms of $\mathcal{L}$ restricted to $\overline{G}$ belong to $\mathcal{C}_b^2(G)$. Let $S \in \mathcal{L}$ and $f(x) = (Sx, x),\ x \in \overline{G}$. For each $x \in \mathbb{H}$

$$f'(x)h = (Sx, h) + (S^*x, h),\ h \in \mathbb{H} \text{ and } f''(x) = S + S^*.$$

Then it is easy to verify that $f$, $f'$ and $f''$ are bounded and uniformly continuous on $G$.

**Definition 6.** *A differential operator $\boldsymbol{D} \in \mathcal{D}$ is said to have the property (MP) on $G$ if for every $f \in \mathcal{C}_b^2(G)$ with $\boldsymbol{D}f \geq 0$ on $G$ we have*

$$\sup f(\overline{G}) = \sup f(\partial G).$$

**Remark.** A functional $f \in \mathcal{C}_b^2(G)$ does not necessarily attain its maximum on $\overline{G}$. Consider e.g. a quadratic form

$$f(x) = \sum_{n=1}^{\infty} \left(1 - \frac{1}{n+1}\right) x_n^2,\ x \in G = B(0,1),$$

where $E = \{e_n\}_{n=1}^{\infty}$ is an ON-basis of $\mathbb{H}$ and $x_n = (x, e_n)$ for $n \in \mathbb{N}$ and $x \in \mathbb{H}$.

If a $B \in \mathcal{L}_s$ is positive (i.e., $(Bh, h) \geq 0$ for each $h \in \mathbb{H}$), we write $B \geq 0$. For each $x_0 \in \mathbb{H}$ define

$$\tau_{x_0}(x) = \frac{1}{2}\|x - x_0\|^2,\ x \in \mathbb{H}.$$

Then $\tau''_{x_0}(x) = I$ for each $x \in \mathbb{H}$.

**Theorem 5.** *A differential operator $\boldsymbol{D}^{\psi} \in \mathcal{D}$ has the property (MP) on $G$ if and only if $\psi$ satisfies both the following conditions:*

(P1) $\psi(I) > 0$;
(P2) $(B \in \mathcal{L}_s, B \geq 0) \Rightarrow \psi(B) \geq 0$.

Proof: Firstly, we suppose that $\boldsymbol{D}^{\psi}$ have the property (MP) on $G$. Then we notice that $\psi(I) > 0$. Indeed, if this were not this case, we would get a contradiction by considering $-\tau_{x_0}$ on $G$ for a point $x_0 \in G$. Suppose now that the property (P2) does not hold. Let $B \in \mathcal{L}_s$ satisfy $\psi(B) < 0$ and $(Bh, h) \geq 0$ for each $h \in \mathbb{H}$. There is a $c > 0$ such that

$$-c\psi(I) - \psi(B) > 0.$$

Let $x_0 \in G$. Define $f(x) = -c\tau_{x_0}(x) - \frac{1}{2}(B(x-x_0), x-x_0)$, $x \in G$. We have $\boldsymbol{D}f \geq 0$ on $G$ and

$$0 = f(x_0) = \sup f(G) > \sup f(\partial G),$$

a contradiction. It remains to show the converse assertion.

This is analogous to the proof of the maximum principle for the Lévy laplacian in [3]. Let $\psi$ have both properties (P1) and (P2). Without loss of generality, we may suppose that $\psi(I) = 1$. Fix an $f \in \mathcal{C}_b^2(G)$. We shall proceed in two steps.
(i) Suppose first that there exists an $a > 0$ such that $\boldsymbol{D}f \geq a$ on $G$ and

$$u_0 = \sup f(G) > \sup f(\partial G).$$

We shall deduce a contradiction. There is a sequence $\{z_n\} \subset G$ such that $\lim_{n\to\infty} f(z_n) = u_0$. For each $x \in \mathbb{H}$ and $F \subset \mathbb{H}$ denote

$$d(x, F) = \inf\{\|x - y\|;\ y \in F\}.$$

Observe that $d(z_n, \partial G)$ does not approach zero as $n \to \infty$. In fact, if this were not the case there would be a sequence $\{y_n\} \subset \partial G$ such that

$$\lim_{n\to\infty} \|y_n - z_n\| = 0$$

and therefore

$$\lim_{n\to\infty} f(y_n) = u_0$$

by the uniform continuity of $f$ on $\overline{G}$, which is a contradiction. Since $d(z_n, \partial G)$ does not approach zero there exist $\delta_1 > 0$ and a subsequence $\{x_n\}$ of the sequence $\{z_n\}$ such that $d(x_n, \partial G) > \delta_1$ for all $n \in \mathbb{N}$. The sequence $\{x_n\}$ can be chosen so that for each $n \in \mathbb{N}$

$$f(x_n) > u_0 - \frac{1}{n} \geq f(x) - \frac{1}{n},\ x \in G. \tag{1}$$

Using the uniform continuity of $f''$ on $G$ we can fix a $\delta_2 > 0$ such that for all $x, y \in G$, $\|x - y\| < \delta_2$,

$$\|f''(x) - f''(y)\| = \sup_{\|h\|=1, \|k\|=1} |f''(x)(h,k) - f''(y)(h,k)| < \frac{a}{4}. \tag{2}$$

Put $r = \min(\delta_1, \delta_2)$ and fix $m \in \mathbb{N}$ such that

$$\frac{1}{m} < \frac{a}{8} r^2. \tag{3}$$

Since $\psi(f''(x_m) - \frac{a}{2}I) > 0$ there is an $h \in \mathbb{H}$ such that $\|h\| = 1$ and

$$D_{hh}f(x_m) = f''(x_m)(h,h) > \frac{a}{2}$$

by the second property of $\psi$.
By (2), $D_{hh}f(x) > \frac{a}{4}$ for each $x \in I = \{x_m + th;\ t \in (-r, r)\}$. Now we define

$$w(x) = f(x) - \frac{a}{4}\tau_{x_m}(x),\ \ x \in \overline{I}.$$

Since $D_{hh}w > 0$ on $I$, $w$ attains its maximum on $\overline{I}$ at a point $x^* \in \{x_m \pm rh\}$. By (1) and (3) we have

$$f(x_m) = w(x_m) \le w(x^*) = f(x^*) - \frac{a}{8}r^2 < f(x_m),$$

a contradiction.
(ii) Going back to the general case we suppose $\boldsymbol{D}f \ge 0$ on $G$. For a fixed $\delta > 0$ consider the function $v = v_\delta$ defined by

$$v(x) = f(x) + \delta\tau_0(x),\ \ x \in \overline{G}.$$

Since $\boldsymbol{D}v \ge \delta$ on $G$ we get by (i)

$$\sup v(G) = \sup v(\partial G).$$

Denote $c = \sup \tau_0(\partial G) < +\infty$. Hence

$$\sup f(G) \le \sup v(G) = \sup v(\partial G) \le \sup f(\partial G) + \delta c.$$

Since $\delta$ is an arbitrary positive number we get $\sup f(G) \le \sup f(\partial G)$. The proof is complete. □

We now consider a differential operator $\boldsymbol{D} \in \mathcal{D}$ having the property (MP) on $G$. We shall prove several lemmas which allow us to establish the maximum principle for a larger class of functionals.

**Lemma.** *Assume that $u_n \in \mathcal{C}_b^2(G)$ and $q \in \mathcal{C}_b(G)$ such that $u_n \to 0$ and $\boldsymbol{D}u_n \to q$ uniformly on $G$, i.e., with respect to the norm $\|\ \|_0$. Then $q \equiv 0$.*

Proof: It is the same as the proof of Theorem 3.1 in [3]. For the sake of completeness we will reproduce it here. We show this assertion by contradiction. Without loss of generality, we may suppose that $\boldsymbol{D} = \boldsymbol{D}^\psi$ with $\psi(I) = 1$ and there is a point $x_0 \in G$ such that $q(x_0) > 0$. Denote $c = \frac{1}{2}q(x_0)$. By continuity of $q$, there is a $\delta > 0$ such that $q \ge c$ on a ball $B = B(x_0, \delta) \subset G$. By hypotheses, there is an $m \in \mathbb{N}$ such that for each $n \ge m$ we have

$$\boldsymbol{D}u_n \ge \frac{c}{2} \text{ on } B \text{ and } \|u_n\|_0 \le \frac{c}{8}\delta^2.$$

Fix $n \ge m$. Since $\boldsymbol{D}(u_n - \frac{c}{2}\tau_{x_0}) \ge 0$ on $B$ we get, by Theorem 5,

$$u_n(x_0) \le \sup u_n(\partial B) - \frac{c}{4}\delta^2 \le -\frac{c}{8}\delta^2,$$

which is a contradiction. □

**Definition 7.** *We denote $\mathcal{A}(G) \equiv \mathcal{A}(\boldsymbol{D}; G)$ the set of all $u \in \mathcal{C}_b(G)$ for which there are a sequence $\{u_n\} \subset \mathcal{C}_b^2(G)$ and a functional $q \in \mathcal{C}_b(G)$ such that both $u_n \to u$ and $\boldsymbol{D}u_n \to q$ uniformly on $G$. We define $\bar{\boldsymbol{D}}u = q$.*

**Remark.** By the previous lemma, for each $u \in \mathcal{A}(G)$, such a functional $q$ is uniquely determined and, moreover, it is clear that $\bar{\boldsymbol{D}} = \boldsymbol{D}$ on $\mathcal{C}_b^2(G)$. There is a question of relationship between $\boldsymbol{D}$ and $\bar{\boldsymbol{D}}$ on $\mathcal{A}(G) \cap (\mathcal{C}^2(G) \setminus \mathcal{C}_b^2(G))$.

**Example 6.** If $u_n \in \mathcal{A}(G)$, $u_n \to u$ uniformly on $G$ and $\bar{\boldsymbol{D}}u_n = 0$ on $G$, then $u \in \mathcal{A}(G)$ and $\bar{\boldsymbol{D}}u = 0$ on $G$.

**Theorem 6** (Maximum principle for $\bar{\boldsymbol{D}}$). *If $u \in \mathcal{A}(G)$ and $\bar{\boldsymbol{D}}u \geq 0$ on $G$, then $\sup u(G) = \sup u(\partial G)$.*

Proof: This follows from Theorem 5. We may assume that $\boldsymbol{D} = \boldsymbol{D}^\psi$ with $\psi(I) = 1$. Let $u$ satisfy the assumptions of the theorem. There is a sequence $\{u_n\} \subset \mathcal{C}_b^2(G)$ such that $u_n \to u$ and $\boldsymbol{D}u_n \to \bar{\boldsymbol{D}}u$ uniformly on $G$. Fix an $\varepsilon > 0$ and $x_0 \in G$. By hypotheses, there is an $m \in \mathbb{N}$ such that for each $n \geq m$

$$\boldsymbol{D}u_n \geq -\varepsilon \text{ and } \|u_n - u\|_0 \leq \varepsilon.$$

Fix $n \geq m$. Since $\boldsymbol{D}(u_n + \varepsilon\tau_{x_0}) \geq 0$ we get

$$u_n(x_0) \leq \sup u_n(\partial G) + \varepsilon \sup \tau_{x_0}(\partial G) \leq \sup u(\partial G) + \varepsilon(1 + \sup \tau_{x_0}(\partial G)).$$

Letting $n \to +\infty$ and $\varepsilon \to 0+$, $u(x_0) \leq \sup u(\partial G)$. Since $x_0$ is an arbitrary point of $G$, we have

$$\sup u(G) \leq \sup u(\partial G).$$

The proof is complete. □

## 4. The Dirichlet and Poisson problems

In this paragraph $G \subset \mathbb{H}$ will be a nonempty bounded open set unless otherwise specified and we shall consider an irregular differential operator $\boldsymbol{D} \in \mathcal{D}$ having the property (MP) on $G$, see Definition 6. According to Definition 7 we have $\bar{\boldsymbol{D}}$ defined on $\mathcal{A}(G)$. We will solve the Dirichlet and Poisson problems for $\bar{\boldsymbol{D}}$ on a certain class of functionals defined in what follows by generalizing results obtained for the Lévy laplacian in [3]. First of all, we shall show the following lemma which is a generalization of the assertion (2) in Theorem 3 for an irregular operator. Put $\mathbb{N} = \{1, 2, 3, \dots\}$ and $\mathbb{N}_0 = \{0, 1, 2, \dots\}$.

**Lemma.** *Let $p \in \mathbb{N}$, $I \subset \{1, \dots, p\}$, $J = \{1, \dots, p\} \setminus I$, $F : \mathbb{R}^p \to \mathbb{R}$ be a continuous function such that $\partial_j F$ is a continuous function for each $j \in J$ and $s_i \in \mathcal{A}(G)$ for $i = 1, \dots, p$ such that $\bar{\boldsymbol{D}}s_i = 0$ on $G$ for each $i \in I$. Denote $u = (s_1, \dots, s_p)$ on $G$. Then $F \circ u \in \mathcal{A}(G)$ and*

$$\bar{\boldsymbol{D}}(F \circ u) = \sum_{j \in J} (\partial_j F) \circ u \bar{\boldsymbol{D}} s_j \text{ on } G.$$

Proof: We shall prove only a special case of the lemma we need later on. The proof of the general case is analogous. Assume that $p = m+1$ and $I = \{1, \ldots, m\}$. Then there are sequences $\{P_n\} \subset \mathcal{C}^\infty(\mathbb{R}^{m+1})$ and $\{Q_n\} \subset \mathcal{C}^\infty(\mathbb{R}^{m+1})$ such that $P_n \to F$ and $Q_n \to \partial_{m+1} F$ uniformly on each compact set of $\mathbb{R}^{m+1}$. Define for each $n \in \mathbb{N}$ the function $F_n : \mathbb{R}^{m+1} \to \mathbb{R}$ by

$$F_n(t_1, \ldots, t_{m+1}) = P_n(t_1, \ldots, t_m, 0) + \int_0^{t_{m+1}} Q_n(t_1, \ldots, t_m, t)\, dt.$$

Obviously, $\partial_{m+1} F_n = Q_n$ and hence $F_n \to F$ and $\partial_{m+1} F_n \to \partial_{m+1} F$ uniformly on each compact set of $\mathbb{R}^{m+1}$ as $n \to \infty$. Since each $s_i$ is an element of $\mathcal{A}(G)$, there is a sequence

$$\{s_i^n\}_{n=1}^\infty \subset \mathcal{C}_b^2(G)$$

such that $s_i^n \to s_i$ and $\boldsymbol{D} s_i^n \to \bar{\boldsymbol{D}} s_i$ uniformly on $G$ as $n \to \infty$. Define for each $n \in \mathbb{N}$

$$u_n(x) = (s_1^n(x), \ldots, s_m^n(x), s_{m+1}^n(x)),\ \ x \in G.$$

Now we show that

$$F_n \circ u_n \to F \circ u \text{ and } (\partial_{m+1} F_n) \circ u_n \to (\partial_{m+1} F) \circ u \tag{1}$$

uniformly on $G$ as $n \to \infty$. We have to prove that $H_n \circ u_n \to H \circ u$ uniformly on $G$ if $H_n \to H$ uniformly on each compact set of $\mathbb{R}^{m+1}$. We may write

$$H_n \circ u_n - H \circ u = H_n \circ u_n - H \circ u_n + H \circ u_n - H \circ u.$$

Since $H_n \to H$ uniformly on each compact set of $\mathbb{R}^{m+1}$ and $\{u_n\}$ is a uniformly bounded sequence of functionals on $G$ it is seen that $H_n \circ u_n - H \circ u_n \to 0$ uniformly on $G$. In conclusion, $H \circ u_n \to H \circ u$ uniformly on $G$ because $u_n \to u$ uniformly on $G$ and $H$ is a uniformly continuous function on each compact set of $\mathbb{R}^{m+1}$.

Fix $n \in \mathbb{N}$. According to (2) in Theorem 3 we have

$$\boldsymbol{D}(F_n \circ u_n) = \sum_{i=1}^m (\partial_i F_n) \circ u_n \boldsymbol{D} s_i^n + (\partial_{m+1} F_n) \circ u_n \boldsymbol{D} s_{m+1}^n \text{ on } G.$$

Since for each $i = 1, \ldots, m$ the function $\partial_i F_n$ is bounded on each compact subset of $\mathbb{R}^{m+1}$ and $\bar{\boldsymbol{D}} s_i = 0$ on $G$ we may assume that $\{s_i^n\}_{i=1}^m$ are, moreover, chosen so that the absolute value of the first term on the right-hand side of the above equality is less than $\frac{1}{n}$. Hence by (1) we get

$$F_n \circ u_n \to F \circ u \text{ and } \boldsymbol{D}(F_n \circ u_n) \to (\partial_{m+1} F) \circ u \bar{\boldsymbol{D}} s_{m+1}$$

uniformly on $G$. Thus $F \circ u \in \mathcal{A}(G)$ and $\bar{\boldsymbol{D}}(F \circ u) = (\partial_{m+1} F) \circ u \bar{\boldsymbol{D}} s_{m+1}$ on $G$.

□

**Definition 8.** *We denote by $\mathcal{F}(G)$ the set of all functionals $f : \overline{G} \to \mathbb{R}$ for which there are an $m \in \mathbb{N}_0$, a continuous function $F : \mathbb{R}^{m+1} \to \mathbb{R}$ having continuous partial derivative with respect to the last variable (i.e., $\partial_{m+1} F : \mathbb{R}^{m+1} \to \mathbb{R}$ is a continuous function) and $s_i \in \mathcal{A}(G)$ satisfying $\bar{\boldsymbol{D}} s_i = 0$ on $G$ for $i = 1, 2, \ldots, m$ such that*

$$f(x) = F(s_1(x), \ldots, s_m(x), \|x\|^2), \; x \in \overline{G}. \tag{2}$$

*If $f \in \mathcal{F}(G)$, then $f$ is said to be a fundamental functional on $G$. Furthermore, define*

$$\mathcal{H}(G) = \{f \in \mathcal{A}(G); \; \bar{\boldsymbol{D}} f = 0 \text{ on } G\}.$$

**Remark.** By the previous lemma and Definition 8, $\mathcal{H}(G) \subset \mathcal{F}(G) \subset \mathcal{A}(G)$. In fact, if $s \in \mathcal{H}(G)$ and $F(t_1, t_2) = t_1, \; (t_1, t_2) \in \mathbb{R}^2$, then

$$s(x) = F(s(x), \|x\|^2), \; x \in \overline{G},$$

which means that $s \in \mathcal{F}(G)$. The second inclusion is a direct consequence of the lemma.

**Remark.** In [3], $f$ is said to be fundamental if $F$ is, moreover, an element of $\mathcal{C}^2(\mathbb{R}^m)$ in the above definition.

We will solve the Dirichlet and Poisson problems on sets defined as follows.

**Definition 9.** *A bounded open set $G \subset \mathbb{H}$ is said to be fundamental if there is a functional $s \in \mathcal{A}(G)$ such that*

$$\bar{\boldsymbol{D}} s = 0 \text{ on } G \text{ and } s(x) = \|x\|^2, \; x \in \partial G. \tag{3}$$

Examples of fundamental functionals and fundamental sets are given later on.

**Remark.** By the maximum principle for $\bar{\boldsymbol{D}}$ (see Theorem 6) for each fundamental set $G$ there exists exactly one $s$ satisfying (3); such $s$ is called the representation of $G$.

**Remark.** For every $M \subset \mathbb{H}$ and $a \in \mathbb{H}$ define

$$M_a = \{a + m; \; m \in M\}.$$

If $G$ is a fundamental set and $a \in \mathbb{H}$, then $G_a$ is a fundamental set. In fact, if $s$ is the representation of $G$, then functional $s_a$ defined by

$$s_a(x) = s(x - a) + 2(a, x) - \|a\|^2, \; x \in \overline{G_a}$$

is the representation of $G_a$.

**Remark.** In [3] it is shown that the regularity of $G \subset \mathbb{R}^m$ which is a necessary and sufficient condition for the solvability of the classical Dirichlet problem is equivalent to the fundamentality of $G$, i.e., the existence of a solution of the following boundary value problem

$$\Delta u = 0 \text{ on } G \text{ and } u(x) = \|x\|^2,\ x \in \partial G.$$

**Theorem 7.** *Let $G \subset \mathbb{H}$ be a fundamental set and $f \in \mathcal{F}(G)$. Then there exists exactly one $u \in \mathcal{F}(G)$ such that*

$$\text{(DP)} \qquad \bar{\boldsymbol{D}} u = 0 \text{ on } G \text{ and } u = f \text{ on } \partial G,$$

*and there exists exactly one $v \in \mathcal{F}(G)$ such that*

$$\text{(PP)} \qquad \bar{\boldsymbol{D}} v = f \text{ on } G \text{ and } v = 0 \text{ on } \partial G.$$

**Remark.** If for every $f \in \mathcal{F}(G)$ there is a solution of (DP), then $G$ is fundamental. In fact, a solution of (DP) for $f(x) = \|x\|^2,\ x \in \overline{G}$ is the representation of $G$. Consequently, the fundamentality of $G$ is a necessary and sufficient condition for the solvability of (DP) on $\mathcal{F}(G)$. (See Theorem 4.3 in [3].)

Proof: Recalling that $\mathcal{F}(G) \subset \mathcal{A}(G)$ and using the maximum principle for $\bar{\boldsymbol{D}}$ on $\mathcal{A}(G)$ (see Theorem 6), we get the uniqueness of solutions of problems (DP) and (PP).

We have to show the existence of solutions. Let $s$ be the representation of $G$ and $f$ be as in (2). Then

$$u(x) = F(s_1(x), \dots, s_m(x), s(x)),\ x \in \overline{G}$$

is a solution of the problem (DP). In fact, by the previous lemma we have that $\bar{\boldsymbol{D}} u = 0$ on $G$ and, obviously, the boundary condition holds.

It only remains to solve the problem (PP). Let $\boldsymbol{D} = \boldsymbol{D}^\psi$ and $c = \psi(I)$. Define a function $H$ on $\mathbb{R}^{m+2}$ by

$$H(t_1, \dots, t_{m+2}) = -\frac{1}{2c} \int_{t_{m+2}}^{t_{m+1}} F(t_1, \dots, t_m, r)\, dr.$$

Then $v(x) = H(s_1(x), \dots, s_m(x), s(x), \|x\|^2),\ x \in \overline{G}$ is a solution of the problem (PP). Trivially, $v = 0$ on $\partial G$. By the previous lemma, we get

$$\bar{\boldsymbol{D}} v(x) = 2c\partial_{m+2} H(s_1(x), \dots, \|x\|^2) = f(x),\ x \in G.$$

The proof is complete. □

The so-called simple functionals were introduced by G.E. Šilov in [6]. A.B. Mingarelli and S. Wang developed Šilov's ideas in [3].

**Definition 10.** *A functional $s : \overline{G} \to \mathbb{R}$ is said to be simple if there are an $m \in \mathbb{N}$, a continuous function $h : \mathbb{R}^m \to \mathbb{R}$ and an orthonormal set $\{a_1, \ldots, a_m\} \subset \mathbb{H}$ such that*

$$s(x) = h((a_1, x), \ldots, (a_m, x)), \ x \in \overline{G}. \tag{4}$$

*Denote by $\mathcal{S}(G)$ the set of all simple functionals on $G$. Furthermore, denote by $\mathcal{H}$ the set of all functionals $f : \mathbb{H} \to \mathbb{R}$ such that $f|_G \in \mathcal{H}(G)$ for each bounded open set $G \subset \mathbb{H}$.*

**Remark.** If an $n \in \mathbb{N}$, $\{b_1, \ldots, b_n\} \subset \mathbb{H}$ (not necessarily orthonormal set) and $g : \mathbb{R}^n \to \mathbb{R}$ is continuous, then the functional

$$s(x) = g((b_1, x), \ldots, (b_n, x)), \ x \in \overline{G}, \tag{5}$$

is simple. In fact, there are $m \in \mathbb{N}$, $m \leq n$ and an orthonormal set $\{a_1, \ldots, a_m\} \subset \mathbb{H}$ such that each $b_i$ can be expressed as a linear combination of $a_1, \ldots, a_m$, i.e., for $i = 1, \ldots, n$ there is a sequence $\{c_{ij}\}_{j=1}^m \subset \mathbb{R}$ such that $b_i = \sum_{j=1}^m c_{ij} a_j$. Define a linear function $S : \mathbb{R}^m \to \mathbb{R}^n$ by

$$S(t_1, \ldots, t_m) = (\sum_{j=1}^m c_{1j} t_j, \ldots, \sum_{j=1}^m c_{nj} t_j)$$

and $h = g \circ S$. Obviously, $S((a_1, x), \ldots, (a_m, x)) = ((b_1, x), \ldots, (b_n, x))$, $x \in \mathbb{H}$ and

$$s(x) = h((a_1, x), \ldots, (a_m, x)), \ x \in \overline{G},$$

which concludes the proof. Moreover, $S(G)$ is an algebra because a sum and a product of two functionals of $S(G)$ are of the form (5).

**Example 7.** Let $a \in \mathbb{H}$ and $t(x) = (a, x)$, $x \in \overline{G}$. Then $t \in \mathcal{C}_b^2(G)$, $t'' = 0$ on $G$ and $\bar{\boldsymbol{D}} t = \boldsymbol{D} t = 0$ on $G$. Thus $t \in \mathcal{H}(G)$. Therefore, by the lemma at the beginning of the paragraph, $\mathcal{S}(G) \subset \mathcal{H}(G)$.

**Example 8.** Quadratic forms of compact operators restricted to $\overline{G}$ belong to $\mathcal{H}(G)$. Let $S \in \mathcal{L}_\infty$ and $f(x) = (Sx, x)$, $x \in \overline{G}$. Then $f \in \mathcal{C}_b^2(G)$,

$$f''(x) = S + S^* \in \mathcal{L}_\infty \cap \mathcal{L}_s, \ x \in G.$$

Since $\boldsymbol{D}$ is irregular $\bar{\boldsymbol{D}} f = \boldsymbol{D} f = 0$ on $G$.

Sets introduced in the following definition present examples of fundamental sets. They were considered by G.E. Šilov in [6].

**Definition 11.** *A set $G \subset \mathbb{H}$ is said to be s-fundamental if there are an $m \in \mathbb{N}$, a continuous function $h : \mathbb{R}^m \to \mathbb{R}$ such that*

$$\{t \in \mathbb{R}^m;\ h(t_1, \dots, t_m) > \sum_{i=1}^{m} t_i^2\}$$

*is a nonempty bounded set of $\mathbb{R}^m$ and an orthornormal set $\{a_1, \dots, a_m\} \subset \mathbb{H}$ such that*

$$G = \{x \in \mathbb{H};\ h((a_1, x), \dots, (a_m, x)) > \|x\|^2\}. \tag{6}$$

**Remark.** Every s-fundamental set is really fundamental. Let $G$ be as in (6). We shall show the boundedness of $G$. Since

$$g(t_1, \dots, t_m) = h(t_1, \dots, t_m) - \sum_{i=1}^{m} t_i^2$$

is a continuous function on $\mathbb{R}^m$ and is positive on a bounded set, i.e., there is an $R > 0$ such that $\sum_{i=1}^{m} t_i^2 \le R$ if $g(t_1, \dots, t_m) > 0$, there is a $K > 0$ such that $K \ge g$ on $\mathbb{R}^m$. Suppose that $x \in G$. Then

$$0 \le \|x\|^2 - \sum_{i=1}^{m} |(a_i, x)|^2 < g((a_1, x), \dots, (a_m, x)).$$

Consequently, $\sum_{i=1}^{m} |(a_i, x)|^2 \le R$ and $\|x\|^2 \le K + R$. Hence $G$ is bounded. Obviously, $s(x) = h((a_1, x), \dots, (a_m, x)),\ x \in \overline{G}$ is the representation of $G$.

**Example 9.** Each ball in $\mathbb{H}$ is s-fundamental. In fact, for each $r > 0$

$$B(0, r) = \{x \in \mathbb{H};\ \|x\|^2 < r^2\}$$

is an s-fundamental set and it is obvious that if $G$ is s-fundamental and $a \in \mathbb{H}$, then $G_a$ is also s-fundamental.

**Example 10.** Let $\{\lambda_n\} \subset (0, 1)$, $\lambda_n \to 1$, $r > 0$ and $E = \{e_n\}_{n=1}^{\infty}$ be an ON-basis of $\mathbb{H}$. For $x \in \mathbb{H}$ and $n \in \mathbb{N}$ put $x_n = (x, e_n)$. Then

$$G = \{x \in \mathbb{H};\ r^2 > \sum_{n=1}^{\infty} \lambda_n x_n^2\}$$

is fundamental but not s-fundamental. In fact, $G$ is bounded and

$$s(x) = r^2 + \sum_{n=1}^{\infty} (1 - \lambda_n) x_n^2,\ x \in \overline{G}$$

is the representation of $G$ because $s$ is up to a constant a quadratic form of a compact operator and hence $s \in \mathcal{H}(G)$, see Example 8. Let $m \in \mathbb{N}$, $h : \mathbb{R}^m \to \mathbb{R}$ be a continuous function, $\{a_1, \dots, a_m\} \subset \mathbb{H}$ be an orthonormal set and

$$s(x) = h((a_1, x), \dots, (a_m, x)), \ x \in \overline{G}.$$

There is $0 \neq x \in G$ such that $(a_i, x) = 0$ for $i = 1, \dots, m$. For such a point $x$ we get

$$\sum_{n=1}^{\infty} (1 - \lambda_n) x_n^2 = 0$$

and hence $x_n = 0$ for $n = 1, 2, 3 \dots$, which is impossible. The set $G$ is not s-fundamental.

If $K \subset \mathbb{H}$, then we denote by $\mathcal{C}_w(K)$ the set consisting of all continuous functions $f : (K, w) \to \mathbb{R}$ where $w$ is the weak topology on $\mathbb{H}$. Let us emphasize that, in the following theorem, $\overline{G}$ means the closure with respect to the norm topology.

**Theorem 8.** *If $G$ is an s-fundamental set, then $\overline{G}$ is a w-compact set. If a bounded open set $G \subset \mathbb{H}$ is such that $\overline{G}$ is a w-compact set, then $\mathcal{C}_w(\overline{G}) \subset \mathcal{H}(G)$. Moreover, $\mathcal{C}_w(\mathbb{H}) \subset \mathcal{H}$.*

**Corollary.** *If $G$ is a bounded convex open set, then $\mathcal{C}_w(\overline{G}) \subset \mathcal{H}(G)$.*

Proof: Assume that $G$ is as in (6), i.e.,

$$G = \{x \in \mathbb{H};\ h(x_1, \dots, x_m) > \|x\|^2\}$$

where $x_i = (a_i, x)$ for each $x \in \mathbb{H}$ and $i = 1, \dots, m$. Denote

$$Q = \{t \in \mathbb{R}^m;\ \text{there is a point } x \in G \text{ such that } t = (x_1, \dots, x_m)\}.$$

Then we prove that

$$\overline{G} = \{x \in \mathbb{H};\ (x_1, \dots, x_m) \in \overline{Q} \text{ and } h(x_1, \dots, x_m) \geq \|x\|^2\}.$$

Since $\{a_1, \dots, a_m\} \subset \mathbb{H}$ is an orthonormal set there is an ON-basis $E = \{e_n\}_{n=1}^{\infty}$ of $\mathbb{H}$ such that $a_i = e_i$ for $i = 1, \dots, m$. We write $x_i = (x, e_i)$ for each $x \in \mathbb{H}$ and $i \in \mathbb{N}$. Assume that a point $x \in \mathbb{H}$ is an element of the set on the right-hand side. If there is a number $i \in \mathbb{N}$, $i > m$ such that $x_i \neq 0$, then for each $p \in \mathbb{N}$ $x^p = x - \frac{x_i}{p} e_i \in G$ and $x^p \to x$, which implies that $x \in \overline{G}$. If this is not the case and we assume that $h(x_1, \dots, x_m) = \|x\|^2$, there is a sequence $\{y^p\}_{p=1}^{\infty} \subset G$ such that $y_i^p \to x_i$ for each $i = 1, 2, 3, \dots, m$ as $p \to \infty$ and we may argue as follows. For each $p \in \mathbb{N}$ we have

$$\sum_{i=m+1}^{\infty} (y_i^p)^2 < h(y_1^p, \dots, y_m^p) - \sum_{i=1}^{m} (y_i^p)^2.$$

Since the right-hand term tends to zero as $p \to +\infty$ we have that $y^p \to x$ and $x \in \overline{G}$. The converse inclusion holds trivially. Now we are going to show that $\overline{G}$ is a $w$-compact set. Since $\overline{G}$ is a bounded set we must prove only that $\overline{G}$ is a $w$-closed set. Obviously, the functional

$$s(x) = h(x_1, \dots, x_m),\ x \in \mathbb{H},$$

belongs to $\mathcal{C}_w(\mathbb{H})$. Then this is an immediate consequence of the equality

$$\overline{G} = \bigcap_{n=1}^{\infty} \{x \in \mathbb{H};\ h(x_1, \dots, x_m) \geq \sum_{i=1}^{n} x_i^2\} \cap \{x \in \mathbb{H};\ (x_1, \dots, x_m) \in \overline{Q}\}$$

because each set on the right-hand side is $w$-closed.

Let $G \subset \mathbb{H}$ be a bounded open set such that $\overline{G}$ is a $w$-compact set. As G.E. Šilov in [6] we prove that $\mathcal{S}(G)$ is a dense subset of $\mathcal{C}_w(\overline{G})$ with respect to the supremum norm $\|\ \|_0$. By the remark following Definition 10, $\mathcal{S}(G)$ is an algebra containing all constant functionals on $\overline{G}$ contained in $\mathcal{C}_w(\overline{G})$. Since the restriction of $\mathbb{H}^*$ to $\overline{G}$ is contained in $\mathcal{S}(G)$, $\mathcal{S}(G)$ separates points of $\overline{G}$, i.e., for each two different points $x, y \in \overline{G}$ there is an $f \in \mathcal{S}(G)$ such that $f(x) \neq f(y)$. Now the assertion follows by the Stone-Weierstrass approximation theorem. The remainder of the assertion follows directly from the definition $\mathcal{A}(G)$ and $\bar{\boldsymbol{D}}$ (see Definition 7).

It only remains to verify that $\mathcal{C}_w(\mathbb{H}) \subset \mathcal{H}$. Let $G_1$ and $G_2$ be bounded open subsets of $\mathbb{H}$, $G_1 \subset G_2$ and $f \in \mathcal{H}(G_2)$. Then we get that

$$f|_{G_1} \in \mathcal{H}(G_1),$$

see Definition 7. Consequently, $\mathcal{C}_w(\mathbb{H}) \subset \mathcal{H}$ because every bounded set is contained in a ball.

□

**Example 11.** Obviously, if $s \in \mathcal{C}_w(\mathbb{H})$ and $G = \{x \in \mathbb{H};\ s(x) > \|x\|^2\}$ is a nonempty bounded set, then $G$ is fundamental.

**Remark.** At the end of the second paragraph we mentioned that the invariant Laplace operator at a point $a \in \mathbb{H}$ coincides up to a positive multiple with the irregular operator $\boldsymbol{D}$ restricted to $\Omega_a$. Moreover, if $G \subset \mathbb{H}$ is a bounded open set, then V.Ja. Sikirjavyj proved the maximum principle for the invariant Laplace operator on

$$\Omega_b(\overline{G}) = \{f : \overline{G} \to \mathbb{R};\ f \text{ is bounded and continuous on } \overline{G}, f \in \Omega_a \forall a \in G\}.$$

Because of the maximum principle we may extend the invariant Laplace operator and solve the Dirichlet and Poisson problems for the extension just like in the fourth and this paragraphs we have done for $\boldsymbol{D}|_{\mathcal{C}_b^2(G)}$ (see [8]).

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