The nonseparability of simply presented mixed groups

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Abstract. It is demonstrated that an isotype subgroup of a simply presented abelian group can be simply presented without being a separable subgroup. In particular, the conjecture based on a variety of special cases that Warfield groups are absolutely separable is disproved.

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A group is said to be simply presented if it can be presented by generators and relations where each relation involves at most two generators. For example, a free group is simply presented, and so is a free abelian group since commutativity conforms to a simple presentation. In the context of the fact that any group can be presented with relations that involve at most three generators and the fact that in the category of abelian groups the important class (containing all finite abelian groups) of direct sums of cyclic groups consists precisely of those groups that can be presented with relations that involve only one generator, it is not surprising that simply presented groups form an important class of abelian groups. Indeed, in a certain sense, they constitute the middle ground between direct sums of cyclic groups and all abelian groups.

For convenience of terminology, we shall henceforth assume that all groups considered are abelian. For notation and terminology not defined herein, we refer to [F]. However, we prefer and will use the notation $|x|$ for the height (sequence) of the element x in the group G, and likewise $|x|_p$ denotes the height of x at the prime p. If the containing group G is not clear from the context, we write $|x|^G$ or $|x|_p^G$ for clarity. A subgroup H of G is said to be *isotype* if $|x|^H = |x|^G$ for all $x \in H$, and our attention will be focused primarily on isotype subgroups of simply presented groups.

Torsion simply presented groups were first determined by numerical invariants in [H1], and they can be described structurally in a number of ways [F]. For the structure of torsion groups, it is of course enough to determine the structure of p-groups, and suffice it here to say that, for p-groups, it turns out that simply presented, totally projective, Axiom 3, and balanced projective groups are all the same. The torsion-free simply presented groups are probably even better known

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than the torsion ones, for they are simply direct sums of subgroups of $\mathbf{Q}$. These groups were classified by Reinhold Baer [B] in 1937; they are now generally called completely decomposable torsion-free groups.

Simply presented groups that are truly mixed (neither torsion nor torsion-free) can be more complicated than the torsion or torsion-free simply presented groups that we have just described. For example, summands of torsion and torsion-free simply presented groups, respectively, are of like kind. (In the latter case, this is the Baer-Kulikov-Kaplansky Theorem.) However, summands of mixed simply presented groups need not again be simply presented. In fact, summands of mixed simply presented groups constitute another important class of groups known as *Warfield groups*. Since a summand is always an isotype subgroup, we know that isotype subgroups of simply presented groups (upon which we are focusing) are not necessarily again simply presented. Indeed, this is true even for torsion or torsion-free simply presented groups, although summands are again simply presented in this case.

A basic question is the following: when is an isotype subgroup of a simply presented group again simply presented? For torsion groups (equivalently, for p-groups), a complete answer to this question was given in [H2], and partial answers have been given for torsion-free groups; see, for example, [DR]. For both torsion and torsion-free groups, a necessary condition that an isotype subgroup be simply presented is that it be separable in the following sense.

Definition. *A subgroup H of G is said to be* **separable** *if, for every $g \in G$, there exists a countable sequence of elements $h_1, h_2, \ldots, h_n, \ldots$ in H with the property that for each $h \in H$*

$$|g + h| \leq |g + h_n|$$

for some n (depending on the choice of h). A group H is **absolutely separable** *if it is separable in every group G whenever H appears as an isotype subgroup.*

The following conjecture has been part of the folklore of simply presented groups.

Conjecture. *Any isotype subgroup H of a simply presented group G must be separable in G in order to be, itself, simply presented.*

The conjecture was proved for the torsion case in [H2]. Proofs for the torsion-free case appear in [AH] and [HM3]. In addition, the conjecture was established in [HM2] for p-local mixed groups, that is, for groups that are modules over the integers localized at p. We should mention that, in all of these cases, the conjecture has played an important role in the structure theory of certain classes of groups beyond simply presented groups, see, for example, [HM1], [HM2], [HM3], and [DR]. The main purpose of this note is to show, however, that the conjecture in general is false.

Theorem. *There exist isotype subgroups of simply presented groups that are not separable subgroups, but yet they are themselves simply presented.*

Proof: To begin, we choose a simply presented p-group T of length ω_1 whose α-th Ulm invariant is $\aleph_1$ for each $\alpha < \omega_1$. Clearly, such a group T exists; see, for example, [F]. Note that T is actually a direct sum of countable groups. Further, we can appeal either to the general existence theorem of [HR] or we can use an *ad hoc* construction to conclude that there exists a simply presented group H containing an element x of infinite order for which the following three conditions are satisfied.

(1) $H/\langle x\rangle \cong T$.

(2) The height matrix M of x in H has for each row, except the row associated with the distinguished prime p, the sequence $0, 1, 2, \ldots$; but the row associated with the prime p is $\omega_1, \omega_1+1, \omega_1+2, \ldots$.

(3) For each ordinal $\alpha < \omega_1$, we have that $H = H_t + p^\alpha H$ where H_t denotes the maximal torsion subgroup of H.

It follows that $p^{\omega_1}H = \langle x\rangle$ and that H_t is an ω_1-elementary S-group in the sense of Warfield [War].

As a companion to the simply presented group H defined in the preceding paragraph, we introduce the p-local group $B = \mathbf{Z}_p \otimes H$ where $\mathbf{Z}_p$ is the localization of the ring of rationals integers $\mathbf{Z}$ at the prime ideal (p). Since H_t is a p-primary group, the canonical map $\pi : H \to B$ is a monomorphism that preserves p-heights and maps H_t isomorphically onto the maximal torsion subgroup B_t of B. In fact, $p^{\omega_1}B$ is the $\mathbf{Z}_p$-submodule generated by $y = \pi(x)$, and we have that $B/p^{\omega_1}B \cong T$. By condition (3), $B = B_t + p^\alpha B$ for all $\alpha < \omega_1$.

Using the map π to identify B_t with H_t, we write $H_t = S = B_t$. Although this equality is initially in the sense of isomorphism, it becomes absolute in G if we define the group G by means of the pushout diagram

$$\begin{array}{ccc} S & \longrightarrow & H \\ \big\downarrow & & \big\downarrow \\ B & \longrightarrow & G \end{array}$$

where $S \to H$ is the identity map and $S \to B$ is the restriction of π to $S = H_t$. Thereby, H and B become subgroups of G with $H + B = G$ and $H \cap B = S$. Moreover, H and B are isotype in G since both $G/H \cong B/B_t$ and $G/B \cong H/H_t$ are torsion-free. What we would like to do is show that G is simply presented and that the simply presented isotype subgroup H fails to be separable in G.

First choose $y_1 \in B$ so that $py_1 = y = \pi(x)$. From what we have observed above, there exists an element $b_\lambda \in B_t = S$, for each $\lambda < \omega_1$, for which $y_1 + b_\lambda \in p^\lambda G$. By way of contradiction, assume that H is separable in G and choose a fixed sequence $h_0,\ h_1, \ldots, h_n, \ldots$ of elements in H so that, for each $h \in H$,

$$|y_1 + h| \le |y_1 + h_n|$$

for some $n < \omega$. Since $b_\lambda \in S \subseteq H$, this yields

$$(*) \qquad |y_1 + b_\lambda| \le |y_1 + h_{n(\lambda)}|$$

for some $n(\lambda) < \omega$. Condition (*) means that $|y_1 + b_\lambda|_q \leq |y_1 + h_{n(\lambda)}|_q$ for every prime q. But since B is p-local and $y_1 + b_\lambda \in B$, we have that $|y_1 + b_\lambda|_q = \infty$ whenever $q \neq p$. This, in the presence of (*), implies that $|y_1 + h_{n(\lambda)}|_q = \infty$ for $q \neq p$ and hence that $y_1 + h_{n(\lambda)} \in B$; recall that $G/B \cong H/H_t$ has no element of infinite q-height when $q \neq p$ due to the fact that the height matrix of x is M. We therefore conclude that $h_{n(\lambda)} \in B$, which requires that $h_{n(\lambda)} \in H \cap B = H_t$. Moreover, it is obviously impossible for the inequality $|y_1 + h_{n(\lambda)}|_p < \omega_1$ to hold for all λ since $|y_1 + h_{n(\lambda)}|_p < \mu < \omega_1$ for all λ precludes $|y_1 + b_\mu|_p \leq |y_1 + h_{n(\mu)}|_p$, but the latter is a consequence of (*). Therefore, we can conclude that $|y_1 + h_{n(\mu)}|_p \geq \omega_1$ for some μ. Since $h_{n(\mu)} \in H_t$, we know that $p^k h_{n(\mu)} = 0$ for some positive integer k. This yields

$$|p^k y|_p = |p^{k+1} y_1|_p \geq \omega_1 + k + 1,$$

which contradicts the fact that $|p^k y|_p = |p^k x|_p = \omega_1 + k$.

To complete the proof of the theorem, it remains only to show that G is simply presented. Actually it is enough to show that G is a Warfield group (= direct summand of a simply presented group) because H remains isotype in any simply presented group G' containing G as a summand. Hence, we will show that G is a Warfield group. Toward this end, we rely on the characterization of Warfield groups given in [HR].

First, we claim that the two-element set $\{x - y, y\}$ is a decomposition basis for G. The crucial fact here is that $|h - \pi(h)|_p \geq \omega_1$ for all $h \in H$, but this follows from condition (3) since $h - \pi(h) = (h + z) - \pi(h + z)$ for all $z \in S = H_t$. In view of the fact that $|x|_p = \omega_1$, there is a sequence $x_0, x_1, \ldots, x_n, \ldots$ in H such that $p^n x_n = x$ for each $n < \omega$. Therefore, $x - y = p^n(x_n - \pi(x_n)) \in p^{\omega_1 + n}G$ by what we have just observed. Thus $x - y$ is in $p^{\omega_1 + \omega}G$, and consequently $|x - y|_p = \infty$ since $G_t = S$ has length ω_1 (see [F, p. 200]). Then, noting that $|y|_q = \infty$ for $q \neq p$, we see that $C = \langle x - y\rangle \oplus \langle y\rangle$ is a *valuated* coproduct in the sense that $|m(x - y) + ny|_q = min\{|m(x - y)|_q, |ny|_q\}$ for all $m, n \in \mathbf{Z}$ and all primes q. We have shown that $\{x - y, y\}$ is a decomposition basis for G. Further, it is of course clear that G/C is torsion, and we know that C is a nice subgroup of G by Theorem 1.8 in [HM4]. The only thing left for us to prove in order to conclude that G is a Warfield group is to demonstrate that the torsion group G/C is simply presented. However, this follows from Theorem 1 in [Wal] since $(H + C)/C \cong H/\langle x\rangle \cong T$ is simply presented and since $G/(H + C)$ is countable. □

The following corollary is an immediate consequence of the preceding theorem. Recall that a group H is absolutely separable if it is a separable subgroup of every group that contains it as an isotype subgroups, so we have shown that the conjecture is false. As we mentioned earlier, both torsion and torsion-free simply presented groups are absolutely separable and so are p-local simply presented groups. However, we have proved that global ones need not be.

Corollary. *Global Warfield groups are not necessarily absolutely separable; in fact, simply presented groups are not always absolutely separable.*

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