Splitting ω-covers

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Abstract. The authors give a ZFC example for a space with *Split*(Ω, Ω) but not *Split*(Λ, Λ).

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In this paper, we give a ZFC example of a space satisfying property *Split*(Ω, Ω) but not *Split*(Λ, Λ). This solves Problem 6 of [1]. Finally we show that it is consistent with ZF not to have any space without *Split*(Ω, Ω).

Let us first review the relevant definitions. We start with defining two special classes of open covers.

Definition 1. *Let H be a topological space. An open cover $\mathcal{U} = \{U_\alpha : \alpha < \kappa\}$ is called a* large cover *or* λ-cover, *if $\{\alpha < \kappa : x \in U_\alpha\}$ is infinite for every point $x \in H$.*

$\mathcal{U}$ is called an ω-cover, *if $U \neq H$ for all $U \in \mathcal{U}$ and for every finite subset $F \subseteq H$, there exists some open set U in the cover $\mathcal{U}$ which contains F.*

Definition 2. *A topological space H is said to* satisfy property *Split*(Λ, Λ) *(resp. Split(Ω, Ω)), if one can split every large cover (resp. ω-cover) $\mathcal{U}$ into two disjoint large covers (resp. ω-covers) $\mathcal{U}_1, \mathcal{U}_2$.*

We say H satisfies *Split*$(\Lambda, \mathcal{O})$, *if one can split every large cover $\mathcal{U}$ into two disjoint open covers $\mathcal{U}_1, \mathcal{U}_2$.*

In the following, let H denote the space

$$H = \{(x_i : i < \omega) \in 2^\omega : x_i = 1 \text{ for infinitely many } i \in \omega\},$$

carrying the product topology, where $2 = \{0, 1\}$ is discrete.

We note that H is Lindelöf in every finite power.

This space is homeomorphic to the space ${}^\omega\omega$ with the product topology, where ω is discrete.

We will use the following two well-known lemmas. We write χ_M for the characteristic function of M.

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Lemma 3. *Let $\mathcal{G}$ be a nonprincipal ultrafilter over ω. Then $\mathcal{G}^\bullet = \{\chi_M : M \in \mathcal{G}\} \subseteq 2^\omega$ does not have the Baire property. In particular, $\mathcal{G}^\bullet$ is not Π^1_1 in 2^ω.*[1]

Lemma 4. *Every continuous image of H is Σ^1_1 in H.*[2]

Lemma 5. *Let $\mathcal{U}$ be an ω-cover of a topological space X. Then, whenever $\mathcal{U}$ is the union of $\mathcal{U}_1, \ldots, \mathcal{U}_n$, at least one of the $\mathcal{U}_i$'s is an ω-cover of X.*

Proof: Suppose that none of the $\mathcal{U}_i$'s is an ω-cover. Fix for each $\mathcal{U}_i$ some finite $F_i \subseteq \omega$ which is not covered by a set in $\mathcal{U}_i$. But then $F_1 \cup \ldots \cup F_n$ is not covered by any set in $\mathcal{U} = \mathcal{U}_1 \cup \ldots \cup \mathcal{U}_n$, a contradiction to the assumption that $\mathcal{U}$ is an ω-cover. □

Fact 6. *H does not satisfy Split$(\Lambda, \mathcal{O})$ and thus not Split(Λ, Λ).*

Proof: Consider the following "canonical" cover of H:

For $n \in \omega$, let $U(n) = \{p \in 2^\omega : p(n) = 1\}$.

Then $\mathcal{U} = \{U(n) : n \in \omega\}$ is a λ-cover of H. Suppose $\mathcal{U} = \mathcal{U}_1 \cup \mathcal{U}_2$, where $\mathcal{U}_1$ and $\mathcal{U}_2$ are disjoint. This partition defines a partition of ω into disjoint A_1 and A_2 by $A_i = \{n \in \omega : U(n) \in \mathcal{U}_i\}$ for $i = 1, 2$. Assume, without loss of generality, that A_1 is infinite. But then the function $p \in H$ defined by $p(n) = \chi_{A_1}(n)$ is not covered by $\mathcal{U}_2$, a contradiction.

Thus $\mathcal{U}$ does not split into two disjoint open covers. □

Lemma 7. *H satisfies Split(Ω, Ω).*

In order to prove this lemma, we need a definition and an easy fact.

Definition 8. *A space X is called* ω-Lindelöf, *if every ω-cover has a countable ω-subcover.*

Fact 9. *Suppose a space X is Lindelöf in every finite power. Then X is ω-Lindelöf.*

Proof: Suppose X^m is Lindelöf for each $m \in \omega$, and let $\mathcal{U}$ be an ω-cover. Let $F = \{f_1, \ldots, f_n\} \subseteq X$ be finite of cardinality n. Now F is contained in some $U \in \mathcal{U}$. Thus, the point $(f_1, \ldots, f_n) \in H^n$ is contained in $U \times \ldots \times U$. It follows that $\{U \times \ldots \times U : U \in \mathcal{U}\}$ is a cover of X^n and has a countable subcover $\mathcal{V}$. Let $\mathcal{U}_n = \{U : U \times \ldots \times U \in \mathcal{V}\}$. Then $\mathcal{U}_n$ is a cover of X with the property that every subset of X with cardinality n is contained in some element of $\mathcal{U}_n$. Thus $\bigcup_{n\in\omega} \mathcal{U}_n$ is a countable ω-subcover of $\mathcal{U}$.

Proof of Lemma 7: Suppose that the cover $\mathcal{U}$ does not split into two ω-covers. By Fact 9, we can assume without loss of generality that $\mathcal{U}$ is countable, say $\mathcal{U} = \{U_n : n \in \omega\}$. By Lemma 5, this means that whenever we split $\mathcal{U}$ into $\mathcal{U}_1$ and $\mathcal{U}_2$, then exactly one of $\mathcal{U}_1$ and $\mathcal{U}_2$ is an ω-cover for H. Now every

[1] See e.g. [2, Exercise 2H.5, p. 110], and [3, Theorem 4.1.1, p. 205].

[2] See [3, Exercise 1E.6, p. 43].

such partition yields a corresponding partition of ω into A_1 and A_2 defined by $A_i = \{n \in \omega : U_n \in \mathcal{U}_i\}$ for $i = 1, 2$. For a subset A of ω, let U_A denote the set $\{U_n : n \in A\}$.

Claim 10. *$\mathcal{G} = \{A \subseteq \omega : U_A$ is an ω-cover$\}$ is a nonprincipal ultrafilter over ω.*

Proof: By the choice of $\mathcal{U}$, we have for $A \subseteq \omega$ either A or $\omega \setminus A$ is in $\mathcal{G}$. Also, it is clear that $\mathcal{G}$ is closed under supersets.

Thus the only thing we have to prove is that $\mathcal{G}$ has the finite intersection property.

So suppose $A_1, \dots, A_n \in \mathcal{G}$ and $A_1 \cap \dots \cap A_n \notin \mathcal{G}$. Then the complement of the left hand side, $(\omega \setminus A_1) \cup \dots \cup (\omega \setminus A_n)$, is in $\mathcal{G}$.

By Lemma 5 one of the $\omega \setminus A_i$'s must be in $\mathcal{G}$, a contradiction to the choice of $\mathcal{U}$.

This proves that $\mathcal{G}$ has the fip and therefore $\mathcal{G}$ is an ultrafilter. Furthermore, an ω-cover cannot consist of one single open set. Thus our ultrafilter is nonprincipal. □

Now let us return to the proof of Lemma 7.

Using Lemma 3, we will reach a contradiction by proving that $\mathcal{G}^\bullet$ is a Π^1_1 set.

Claim 11. *$\mathcal{G}^\bullet$ is Π^1_1 in the space 2^ω.*

Proof: A set M is in $\mathcal{G}$ iff χ_M is in $\mathcal{G}^\bullet$ iff

(1) for all finite subsets F of H, there is an $m \in M$ such that $F \subseteq U_m$,

where $\mathcal{U} = \{U_m : m \in \omega\}$ is our open cover which does not split into two ω-covers.

Fix any linear order $\leq$ on H, e.g. the lexicographical order.

Consider a set $F = \{f_1, \dots, f_n\}$ of cardinality n and suppose, without loss of generality, that $f_1 \leq \dots \leq f_n$. Then we can view F as a point $(f_1, \dots, f_n)$ in the product space H^n, and (1) becomes

$$(\forall\, n)(\forall\, F \in H^n)(\exists\, m \in M)(F \in (U_m)^n).$$

But this formula is clearly Π^1_1.

This proves the claim and hence Lemma 7 is proved. □

The following two observations are due to A. Arhangel'skii, who kindly permitted us to include them in this paper.

Lemma 12. *The property Split(Ω, Ω) is preserved under continuous surjections.*

Proof: Let $f : X \to Y$ be a continuous surjection from a topological space X onto a topological space Y. Suppose X satisfies *Split*(Ω, Ω). Let $\mathcal{V}$ be an ω-cover of Y. Then clearly $\mathcal{U} = \{f^{-1}[V] : V \in \mathcal{V}\}$ is an ω-cover of X. Split $\mathcal{U}$ into two ω-covers $\mathcal{U}_1$ and $\mathcal{U}_2$. Define $\mathcal{V}_i = \{f[U] : U \in \mathcal{U}_i\}$ for $i = 1, 2$. We claim that $\mathcal{V}_1$ and $\mathcal{V}_2$ are ω-covers of Y. Consider without loss of generality $i = 1$.

First note that $\mathcal{V}_1$ is an open cover because of its very definition and because f is onto.

Now let $G \subseteq Y$ be finite. Choose some finite $F \subseteq X$ whose image under f is G.

Now F is covered by some $U \in \mathcal{U}_1$. But then $G = f[F]$ is covered by $V = f[U] \in \mathcal{V}_1$. Thus $\mathcal{V}_1$ is an ω-cover and similarly $\mathcal{V}_2$ is.

We conclude that Y satisfies *Split*(Ω, Ω). □

Note that since H and ${}^{\omega}\omega$ are homeomorphic, the continuous images of H are exactly the analytic spaces. Thus we get as an immediate consequence of Lemmas 4 and 12.

Corollary 13. *Every analytic space satisfies Split*(Ω, Ω).

Let us note that the proof of Lemma 7 implies that

Lemma 14 ([ZF])**.** *If X is a space and $\mathcal{U}$ is an ω-cover of $\mathcal{U}$ that cannot be split, then there exists a nonprincipal ultrafilter on $\mathcal{U}$.*

Given the existence of an ultrafilter over ω, we can construct an example for a space not satisfying *Split*(Ω, Ω).

Example 15. *Let $\mathcal{F} \subseteq \wp(\omega)$ be a nonprincipal ultrafilter over ω. Then $F^{\bullet} = \{\chi_M : M \in \mathcal{F}\} \subseteq 2^{\omega}$ does not satisfy Split*(Ω, Ω).

Proof: Let $U_n = \{b \in \mathcal{F} : n \in b\}$ and $\mathcal{U} = \{U_n : n \in \omega\}$.

Let $A = \{a_1, \dots, a_k\}$ be a finite subset of $\mathcal{F}$. Then $a = a_1 \cap \dots \cap a_k$ is nonempty. Pick $n \in a$. Then $a_1, \dots, a_k$ are in U_n. Thus $\mathcal{U}$ is an ω-cover of $\mathcal{F}$.

Now suppose $\mathcal{U} = \mathcal{U}_1 \cup \mathcal{U}_2$ is a disjoint partition of $\mathcal{U}$. Let $a = \{n \in \omega : U_n \in \mathcal{U}_1\}$. Either a or $\omega \setminus a$ is in $\mathcal{F}$. Without loss of generality suppose that a is. But a is not covered by $\mathcal{U}_2$. This proves that $\mathcal{U}$ does not split into two disjoint ω-covers and hence $\mathcal{F}$ does not satisfy *Split*(Ω, Ω). □

Thus the Axiom of Choice implies the existence of a space without *Split*(Ω, Ω).

On the other hand, Andreas Blass constructed in [4] a model of ZF without any nonprincipal ultrafilter. In this model, every topological space will satisfy *Split*(Ω, Ω).

References

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