Commentationes Mathematicae Universitatis Carolinae

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Commentationes Mathematicae Universitatis Carolinae, Vol. 38 (1997), No. 2, 263--272

Persistent URL: http://dml.cz/dmlcz/118924

A note on lattice renormings

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Abstract. It is shown that every strongly lattice norm on $c_0(\Gamma)$ can be approximated by C^∞ smooth norms. We also show that there is no lattice and Gâteaux differentiable norm on $C_0[0,\omega_1]$.

Keywords: smooth norms, approximation, lattice norms, $c_0(\Gamma)$, $C_0[0,\omega_1]$

Classification: 46B03, 46B20

It has been recently shown in [1] and [2] that every equivalent norm on the classical separable Banach spaces c_0 or ℓ_p, p even, (as well as on many other spaces) can be uniformly approximated on bounded sets by a sequence of C^∞-Fréchet smooth norms.

Although the method of construction requires some technical conditions on the space to be satisfied (in particular the existence of a Schauder basis), it seems to suggest that perhaps the following statement should be valid:

Suppose X is a separable Banach space that admits an equivalent C^k-Fréchet smooth norm. Then every equivalent norm on X can be approximated uniformly on bounded sets by a sequence of C^k-Fréchet smooth norms.

On the other hand, we do not know of any example of a nonseparable Banach space where a similar statement would be valid for $k \geq 2$.

In the present note we give a partial solution to this problem for the space $c_0(\Gamma)$ and $k=\infty$. More precisely we show that on $c_0(\Gamma)$, Γ uncountable, every equivalent strongly lattice norm can be approximated by a sequence of C^∞-Fréchet smooth norms.

In the second part of our paper, we show that there exists no lattice Gâteaux differentiable norm on $C_0([0,\omega_1])$, the space of continuous functions on the ordinal segment $[0,\omega_1]$ that vanish at ω_1 (where ω_1 is the first uncountable ordinal and $[0,\omega_1]$ is in its normal topology as in [4]). More information on the space $C_0([0,\omega_1])$ can be found e.g. [3, p. 259]. Proposition 2 of this paper is of interest when compared with some results of Haydon [5]–[6]. In [5], a lattice norm on $C_0[0,\omega_1] \oplus c_0[0,\omega_1]$ is constructed, which is C^∞-Fréchet differentiable and locally dependent on finitely many coordinates when restricted to a rather large open

In part supported by NSERC (Canada).

* Supported by Grants AV 101–95–02 and GAČR 201–94–0069

subset of $C_0[0,\omega_1] \oplus c_0[0,\omega_1]$. This norm is then used to obtain C^∞-Fréchet smooth (necessarily non-lattice) renormings of $C_0[0,\omega_1]$.

The notation and terminology we use are mostly standard, as in [3].

By a strongly lattice norm on $c_0(\Gamma)$ we mean an equivalent norm $\|\cdot\|$ such that $\|\sum\limits_{\gamma\in\Gamma} y_\gamma e_\gamma\| \geq \|\sum\limits_{\gamma\in\Gamma} x_\gamma e_\gamma\|$ whenever $\sum\limits_{\gamma\in\Gamma} y_\gamma e_\gamma$, $\sum\limits_{\gamma\in\Gamma} x_\gamma e_\gamma \in c_0(\Gamma)$ are such that for every $\gamma \in \Gamma$ $|y_\gamma| \geq |x_\gamma|$ is satisfied.

Theorem 1. *Every equivalent strongly lattice norm on $c_0(\Gamma)$ can be approximated (uniformly on bounded sets) by C^∞-Fréchet smooth norms.*

Proof: Denote the given strongly lattice norm by $\|\cdot\|$. We first introduce an auxiliary function f_Δ. For arbitrary $1 > \Delta > 0$ and $\sum\limits_{\gamma\in\Gamma} x_\gamma e_\gamma \in c_0(\Gamma)$ denote by

$$f_\Delta\Big(\sum_{\gamma\in\Gamma} x_\gamma e_\gamma\Big) = \sup\Big\{\|\sum_{\gamma\in\Gamma} y_\gamma e_\gamma\|, \text{ where } y_\gamma = x_\gamma \text{ if } |x_\gamma| > \Delta \text{ and } |y_\gamma| \leq \Delta \text{ if } |x_\gamma| \leq \Delta\Big\}.$$

Clearly, $f_\Delta(\cdot) \geq \|\cdot\|$ on $c_0(\Gamma)$.

In fact, $f_\Delta(\cdot)$ is a Lipschitz function on $(c_0(\Gamma), \|\cdot\|_\infty)$ with the Lipschitz constant less than or equal to the Lipschitz constant of $\|\cdot\|$ (on $(c_0(\Gamma), \|\cdot\|_\infty)$).

It is standard to check the following elementary properties of $f_\Delta(\cdot)$:

(i) $f_\Delta(\sum\limits_{\gamma\in\Gamma} x_\gamma e_\gamma) = f_\Delta(\sum\limits_{\gamma\in\{\alpha\in\Gamma, |x_\alpha|>\Delta\}} x_\gamma e_\gamma)$. In other words, the value of $f_\Delta(x)$ depends only on those coordinates of x that are in absolute value larger than Δ.

(ii) $f_\Delta(\sum\limits_{\gamma\in\Gamma} x_\gamma e_\gamma) \leq f_\Delta(\sum\limits_{\gamma\in\Gamma} y_\gamma e_\gamma)$ whenever we have $\|y_\gamma\| \geq \|x_\gamma\|$ for every $\gamma \in \Gamma$.

The property (ii) is a "strongly lattice" property of $f_\Delta(\cdot)$ and follows directly from the strongly lattice property of $\|\cdot\|$.

We now proceed with our construction of approximating C^∞-norm.

Given $\varepsilon > 0$, from the equivalence of $\|\cdot\|$ and $\|\cdot\|_\infty$ it follows that there exists $1 > \Delta > 0$ such that

$$\|\cdot\| \leq f_\Delta(\cdot) \leq \|\cdot\| + \varepsilon$$

for every $x \in c_0(\Gamma)$.

Put $F_\Delta(x) = f_\Delta^2(x)$.

Then $F_\Delta(\cdot)$ shares properties (i), (ii) and satisfies:

$$\|\cdot\|^2 \leq F_\Delta(\cdot) \leq (\|\cdot\| + \varepsilon)^2 = \|\cdot\|^2 + 2\varepsilon\|\cdot\| + \varepsilon^2.$$

Thus the convex function $C_\Delta(\cdot)$ defined by:

$$C_\Delta(x) = \inf\left\{ \sum_{i=1}^{n} \lambda_i F_\Delta(x_i),\ \ x = \sum_{i=1}^{n} \lambda_i x_i,\ \sum_{i=1}^{n} \lambda_i = 1,\ \lambda_i > 0 \right\}$$

also satisfies $\|\cdot\|^2 \le C_\Delta(\cdot) \le (\|\cdot\|+\varepsilon)^2$, because $\|\cdot\|^2$ is convex and $C_\Delta(\cdot) \le F_\Delta 9(\cdot)$. It is straightforward to show that also the strongly lattice property for $C_\Delta(\cdot)$ is preserved, i.e. $C_\Delta(x) \ge C_\Delta(y)$ for $x, y \in c_0(\Gamma)$, such that for every $\gamma \in \Gamma$ either $\|y_\gamma\| \ge \|x_\gamma\|$. We will now show that for $1 > \varepsilon > 0$ we have

$$C_\Delta(x) = \inf\left\{ \sum_{i=1}^{n} \lambda_i F_\Delta(x_i),\ \ x = \sum_{i=1}^{n} \lambda_i x_i,\ \sum_{i=1}^{n} \lambda_i = 1,\ \lambda_i > 0 \text{ and } \|x_i\| \le 100 \right\}$$

for every $x \in c_0(\Gamma)$ with $\|x\| \le 2$.

To this end, it is enough to find for every $\{x_i\}_{i=1}^n$, $\lambda_i > 0$, $\sum\limits_{i=1}^{n} \lambda_i = 1$, $x = \sum\limits_{i=1}^{n} \lambda_i x_i$ another system $\{y_i\}_{i=1}^m$, $\lambda_i' > 0$, $\sum\limits_{i=1}^{m} \lambda_i' = 1$, $x = \sum\limits_{i=1}^{m} \lambda_i' y_i$, where $\|y_i\| \le 100$ and such that

$$\sum_{i=1}^{m} \lambda_i' F_\Delta(y_i) \le \sum_{i=1}^{n} \lambda_i F_\Delta(x_i).$$

Suppose without loss of generality that $\|x_i\| \le 100$ for $1 \le i \le j$ and $\|x_i\| > 100$ for $j < i \le n$. We may assume that $j \ge 1$, since otherwise $F_\Delta(x_i) \ge 100^2$ for every $1 \le i \le n$, and then $F_\Delta(x) \le 3^2 < 100^2$ would give us a better estimate.

Put

$$v_1 = \frac{\sum\limits_{i=1}^{j} \lambda_i x_i}{\sum\limits_{i=1}^{j} \lambda_i}, \quad v_2 = \frac{\sum\limits_{i=j+1}^{n} \lambda_i x_i}{\sum\limits_{i=j+1}^{n} \lambda_i},$$

$$\xi_1 = \sum_{i=1}^{j} \lambda_i, \quad \xi_2 = 1 - \xi_1.$$

Clearly, $x = \xi_1 v_1 + \xi_2 v_2$.

We may assume that $F_\Delta(v_1) \ge \frac{1}{\xi_1} \sum\limits_{i=1}^{j} \lambda_i F_\Delta(x_i)$ and

$F_\Delta(v_2) \ge \frac{1}{\xi_2} \sum\limits_{i=j+1}^{n} \lambda_i F_\Delta(x_i)$.

Indeed, if for example $F_\Delta(v_1) < \frac{1}{\xi_1}\sum_{i=1}^{j}\lambda_i F_\Delta(x_i)$, we obtain that $x = \xi_1 v_1 + \sum_{i=j+1}^{n}\lambda_i x_i$, $\xi_1 + \sum_{i=j+1}^{n}\lambda_i = 1$, $\xi_1 \ge 0$, $\lambda_i \ge 0$ and

$$\xi_1 F_\Delta(v_1) + \sum_{i=j+1}^{n} F_\Delta(x_i) < \sum_{i=1}^{n}\lambda_i F_\Delta(x_i)$$

gives us even a better estimate of $C_\Delta(x)$.

By assumption, $F_\Delta(x_i) \ge 100^2$ for $j+1 \le i \le n$. Thus $\frac{1}{\xi_2}\sum_{i=j+1}^{n}\lambda_i F_\Delta(x_i) \ge 100^2$. The trivial estimate for $C_\Delta(x)$ is $F_\Delta(x) \le 3^2 = 9$. Thus $\frac{1}{\xi_1}\sum_{i=1}^{j}\lambda_i F_\Delta(x_i) \le 9$ (otherwise the trivial estimate would give us a smaller value than $\sum_{i=1}^{n}\lambda_i F_\Delta(x_i) = \xi_1\big(\frac{1}{\xi_1}\sum_{i=1}^{j}\lambda_i F_\Delta(x_i)\big) + \xi_2\big(\frac{1}{\xi_2}\sum_{i=j+1}^{n}\lambda_i F_\Delta(x_i)\big)$.

Consequently, $\|v_1\|^2 \le C_\Delta(v_1) \le 9$ and we have $\|v_1\| \le 3$. Similarly, $\big(\|v_2\| + \varepsilon\big)^2 \ge F_\Delta(v_2) \ge 100^2$ and we have $\|v_2\| \ge 99$.

Thus there exists $v_3 \in c_0(\Gamma)$, $\|v_3\| = 50$, $v_3 = \alpha_1 v_1 + \alpha_2 v_2$ where $\alpha_1 + \alpha_2 = 1$, $\alpha_i \ge 0$. Since $v_3 - \alpha_1 v_1 = \alpha_2 v_2$, we have $47 \le \alpha_2\|v_2\|$. Thus

$$\alpha_1\frac{1}{\xi_1}\sum_{i=1}^{j}\lambda_i F_\Delta(x_i) + \alpha_2\frac{1}{\xi_2}\sum_{i=j+1}^{n}\lambda_i F_\Delta(x_i) \ge \alpha_2\|v_2\|^2 \ge 47\|v_2\| \ge 47\cdot 99.$$

Moreover the trivial estimate gives us

$$F_\Delta(v_3) \le \big(\|v_3\| + \varepsilon\big)^2 \le 51^2 < 47\cdot 99.$$

Therefore

$$F_\Delta(v_3) \le \alpha_1\frac{1}{\xi_1}\sum_{i=1}^{j}\lambda_i F_\Delta(x_i) + \alpha_2\frac{1}{\xi_2}\sum_{i=j+1}^{n}\lambda_i F_\Delta(x_i),$$

$$\frac{\xi_2}{\alpha_2}F_\Delta(v_3) \le \frac{\xi_2}{\alpha_2}\frac{\alpha_1}{\xi_1}\sum_{i=1}^{j}\lambda_i F_\Delta(x_i) + \sum_{i=j+1}^{n}\lambda_i F_\Delta(x_i),$$

$$\sum_{i=1}^{j}\lambda_i F_\Delta(x_i) - \frac{\xi_2}{\alpha_2}\frac{\alpha_1}{\xi_1}\sum_{i=1}^{j}\lambda_i F_\Delta(x_i) + \frac{\xi_2}{\alpha_2}F_\Delta(v_3) \le \sum_{i=1}^{n}\lambda_i F_\Delta(x_i),$$

$$\sum_{i=1}^{j}\big(1 - \frac{\xi_2}{\alpha_2}\frac{\alpha_1}{\xi_1}\big)\lambda_i F_\Delta(x_i) + \frac{\xi_2}{\alpha_2}F_\Delta(v_3) \le \sum_{i=1}^{n}\lambda_i F_\Delta(x_i).$$

However,

$$(1-\frac{\xi_2}{\alpha_2}\frac{\alpha_1}{\xi_1})\sum_{i=1}^{j}\lambda_i x_i+\frac{\xi_2}{\alpha_2}v_3=\xi_1 v_1+\xi_2\big(\frac{v_3}{\alpha_2}-\frac{\alpha_1}{\alpha_2}v_1\big)=\xi_1 v_1+\xi_2 v_2=x.$$

It is easy to verify that $\sum_{i=1}^{j}\big(1-\frac{\xi_2}{\alpha_2}\frac{\alpha_1}{\xi_1}\big)\lambda_i+\frac{\xi_2}{\alpha_2}=1$. It follows that $\alpha_2>\xi_2$, since $\|v_3\|=50$ while $\|x\|\le 2$. Therefore $(1-\frac{\xi_2}{\alpha_2}\frac{\alpha_1}{\xi_1})\lambda_i\ge 0$ for every $1\le i\le j$.

Thus the system $\{x_i\}_{i=1}^{j}\cup\{v_3\}$, $\{\big(1-\frac{\xi_2}{\alpha_2}\frac{\alpha_1}{\xi_1}\big)\lambda_i\}_{i=1}^{j}\cup\{\frac{\xi_2}{\alpha_2}\}$ gives us a smaller estimate of $C_\Delta(x)$ than the original one $\{x_i\}_{i=1}^{n}$, $\{\lambda_i\}$. Clearly, all $\|x_i\|\le 100$, $1\le i\le j$, $\|v_3\|\le 100$.

Since $\|\cdot\|$ and $\|\cdot\|_\infty$ are equivalent norms on $c_0(\Gamma)$, it follows from our previous considerations that there exists a constant k such that

$$C_\Delta(x)=\inf\left\{\sum_{i=1}^{j}\lambda_i F_\Delta(x_i),\ \ x=\sum_{i=1}^{j}\lambda_i x_i,\ \sum_{i=1}^{j}\lambda_i=1,\ \lambda_i>0 \text{ and } \|x_i\|_\infty\le k\right\}$$

for every $\|x\|\le 2$.

We proceed by proving that there exists $\delta>0$ such that

$$C_\Delta\big(\sum_{\gamma\in\Gamma}x_\gamma e_\gamma\big)=C_\Delta\big(\sum_{\gamma\in\{\alpha,|x_\alpha|>\delta\}}x_\gamma e_\gamma\big)$$

for every $x=\sum_{\gamma\in\Gamma}x_\gamma e_\gamma\in c_0$ such that $\|x\|\le 2$.

In fact, we will show that choosing $\delta<\frac{\Delta^2}{2k+2+\Delta}$ is sufficient.

Since C_Δ is upper semi-continuous (as the infimum of a family of continuous functions - F_Δ is continuous as the square of a Lipschitz function f_Δ), and, moreover, from the strongly lattice property of C_Δ it is enough to prove that

$$C_\Delta(\sum_{\gamma\in\Gamma}x_\gamma e_\gamma)=C_\Delta(\sum_{\substack{\gamma\in\Gamma\\ \gamma\ne\gamma_0}}x_\gamma e_\gamma),$$

whenever $|x_{\gamma_0}|\le\delta$.

We will proceed as follows. Given $x=\sum_{\gamma\in\Gamma}x_\gamma e_\gamma$, for arbitrary $\{y_i\}_{i=1}^{n}\subset c_0(\Gamma)$, $\{\lambda_i\}_{i=1}^{n}$, $\lambda_i>0$, $\sum_{i=1}^{n}\lambda_i=1$, $\|y_i\|_\infty\le k$ such that $\sum_{i=1}^{n}\lambda_i y_i=\sum_{\substack{\gamma\in\Gamma\\ \gamma\ne\gamma_0}}x_\gamma e_\gamma$, we will construct $\{x_i\}_{i=1}^{n}\subset c_0(\Gamma)$ such that $(x_i)_\gamma=(y_i)_\gamma$ for $1\le i\le n$, $\gamma\ne\gamma_0$, $\sum_{i=1}^{n}\lambda_i x_i=x$ and in addition

$$\sum_{i=1}^{n}\lambda_i F_\Delta(x_i)\le\sum_{i=1}^{n}\lambda_i F_\Delta(y_i).$$

Consequently,

$$C_\Delta\big(\sum_{\gamma\in\Gamma} x_\gamma e_\gamma\big) \le C_\Delta\big(\sum_{\substack{\gamma\in\Gamma\\ \gamma\neq\gamma_0}} x_\gamma e_\gamma\big).$$

This implies our claim, since $C_\Delta(\cdot)$ shares the strongly lattice property, so the opposite inequality is satisfied.

Without loss of generality assume that, $\delta \ge x_{\gamma_0} > 0$ and

$$\begin{aligned} k \ge (y_i)_{\gamma_0} &> \Delta && \text{for } 1 \le i \le j_1,\\ \Delta \ge (y_i)_{\gamma_0} &\ge 0 && \text{for } j_1 < i \le j_2,\\ 0 > (y_i)_{\gamma_0} &\ge -\Delta && \text{for } j_2 < i \le j_3,\\ -\Delta > (y_i)_{\gamma_0} &\ge -k && \text{for } j_3 < i \le n. \end{aligned}$$

Put $s_1 = \sum_{i=1}^{j_1} \lambda_i$, $s_2 = \sum_{i=j_1+1}^{j_2} \lambda_i$, $s_3 = \sum_{i=j_2+1}^{j_3} \lambda_i$, $s_4 = \sum_{i=j_3+1}^{n} \lambda_i$.

If $(s_3+s_4)\Delta \ge \delta$, then

$$\sum_{i=1}^{j_2} \lambda_i (y_i)_{\gamma_0} + \sum_{i=j_2+1}^{n} \lambda_i \Delta \ge \sum_{i=j_2+1}^{n} \lambda_i \Delta \ge (s_3+s_4)\Delta \ge \delta.$$

Therefore for every $j_2 < i \le n$ we can find numbers $\tilde y_i$, such that $\Delta \ge \tilde y_i \ge (y_i)_{\gamma_0}$ and

$$\sum_{i=1}^{j_2} \lambda_i (y_i)_{\gamma_0} + \sum_{i=j_2+1}^{n} \lambda_i \tilde y_i = x_{\gamma_0}.$$

We define $x_i = y_i$ for $1 \le i \le j_2$, and $x_i = \sum_{\substack{\gamma\in\Gamma\\ \gamma\neq\gamma_0}} (y_i)_\gamma e_\gamma + \tilde y_i e_{\gamma_0}$ for $j_2 < i \le n$. It follows that

$$F_\Delta(x_i) = F_\Delta\big(\sum_{\substack{\gamma\in\Gamma\\ \gamma\neq\gamma_0}} (y_i)_\gamma e_\gamma\big) \le F_\Delta(y_i).$$

Thus $\sum_{i=1}^{n} \lambda_i F_\Delta(x_i) \le \sum_{i=1}^{n} \lambda_i F_\Delta(y_i)$ and the claim is established.

If $(s_3+s_4)\Delta < \delta$, we obtain $0 = \big(\sum_{i=1}^{n} \lambda_i (y_i)\big)_{\gamma_0} \ge s_1\Delta - (s_3+s_4)k$. Therefore $s_1 \le \frac{\delta k}{\Delta^2}$. Thus $s_2 = 1 - s_1 - s_3 - s_4 \ge 1 - \frac{\delta(k+1)}{\Delta^2}$. We can find numbers $\tilde y_i$ for $j_1 < i \le j_2$, such that $(y_i)_{\gamma_0} \le \tilde y_i \le \Delta$ and

$$\sum_{i=1}^{j_1} \lambda_i (y_i)_{\gamma_0} + \sum_{i=j_2+1}^{n} \lambda_i (y_i)_{\gamma_0} + \sum_{i=j_1+1}^{j_2} \lambda_i \tilde y_i = x_{\gamma_0}.$$

Indeed, $\bigl|\sum\limits_{i=j_2+1}^{n}\lambda_i(y_i)_{\gamma_0}\bigr| \le (s_3+s_4)k \le \frac{\delta k}{\Delta}$. Consequently, $s_2\Delta - \frac{\delta k}{\Delta} \ge \Delta - \frac{\delta(k+1)}{\Delta} - \frac{\delta k}{\Delta} > \delta$ by our choice of δ.

Putting $(x_i)_\gamma = \tilde{y}_i$ for $j_1 < i \le j_2$, $\gamma = \gamma_0$ and $(x_i)_\gamma = (y_i)_\gamma$ for any other choices of i and γ, we obtain again

$$\sum_{i=1}^{n}\lambda_i F_\Delta(x_i) = \sum_{i=1}^{n}\lambda_i F_\Delta(y_i).$$

Hence we proved that $C_\Delta(\cdot)$ is a convex function on $c_0(\Gamma)$, $\|\cdot\|^2 \le C_\Delta(\cdot) \le \bigl(\|\cdot\|+\varepsilon\bigr)^2$ and, for $\|x\| \le 2$, $C_\Delta(x)$ depends only on those coordinates x_γ of x for which $|x_\gamma| \ge \delta$. More precisely,

$$C_\Delta\Bigl(\sum_{\gamma\in\Gamma} x_\gamma e_\gamma\Bigr) = C_\Delta\Bigl(\sum_{\gamma\in\Gamma_1} x_\gamma e_\gamma\Bigr),$$

where $\Gamma_1 = \{\gamma\in\Gamma,\ \ |x_\gamma| \ge \delta\}$.

We will now construct a C^∞-Fréchet smooth convex function on the set $\{x \in c_0(\Gamma),\ \|x\| < 2\}$, which uniformly approximates $C_\Delta(\cdot)$. To this end, choose a C^∞-smooth bump function $b(t)$ on $\mathbb{R}$, $0 \le b(t) = b(-t)$, $\operatorname{supp} b \subset [-\frac{\delta}{4}, \frac{\delta}{4}]$, $\int\limits_{-\infty}^{\infty} b(t)\,dt = 1$.

It is elementary to check that from the symmetry condition on b and the convexity of f it follows that

$$f(r) \le \int_{-\infty}^{\infty} f(t)b(r-t)\,dt$$

for arbitrary convex continuous function defined on $\mathbb{R}$.

It is standard to check that for arbitrary $\gamma_0 \in \Gamma$, the function

$$C_\Delta^{\gamma_0}\Bigl(\sum_{\gamma\in\Gamma} x_\gamma e_\gamma\Bigr) = \int_{-\infty}^{\infty} C_\Delta\Bigl(\sum_{\substack{\gamma\in\Gamma\\ \gamma\ne\gamma_0}} x_\gamma e_\gamma + te_{\gamma_0}\Bigr) b(x_{\gamma_0}-t)\,dt$$

is convex and $C_\Delta^{\gamma_0}(\cdot) \ge C_\Delta(\cdot)$.

Put $\Pi = \bigl\{\pi = \{\gamma_1,\dots,\gamma_n\},\ n\in\mathbb{N},\ \gamma_i\in\Gamma\bigr\}$ to be the set of all finite subsets of Γ. For $\pi = \{\gamma_1,\dots,\gamma_n\} \in \Pi$ define

$$C_\Delta^{\pi}\Bigl(\sum_{\gamma\in\Gamma} x_\gamma e_\gamma\Bigr) =$$

$$= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} C_\Delta\Bigl(\sum_{\substack{\gamma\in\Gamma\\ \gamma\notin\pi}} x_\gamma e_\gamma + \sum_{i=1}^{n} t_i e_{\gamma_i}\Bigr) b(x_{\gamma_1}-t_1)\dots b(x_{\gamma_n}-t_n)\,dt_1\dots dt_n.$$

For every $\pi \in \Pi$, C_Δ^π is a convex function satisfying $C_\Delta^{\pi_2}(\cdot) \geq C_\Delta^{\pi_1}(\cdot)$ whenever $\pi_1 \subset \pi_2$.

Define $\tilde{C}_\Delta(x) = \sup\{C_\Delta^\pi(x),\ \ \pi \in \Pi\}$.

Suppose $x = \sum_{\gamma\in\Gamma} x_\gamma e_\gamma$, $\|x\| \leq 2 - \frac{\delta}{2}$, $\Gamma_1 = \{\gamma \in \Gamma,\ |x_\gamma| \leq \frac{\delta}{4}\}$, $\Gamma_2 = \Gamma \setminus \Gamma_1$. Clearly $\Gamma_2 \in \Pi$. For every $y \in c_0(\Gamma)$ such that $\|y - x\|_\infty < \frac{\delta}{4}$, we have $|y_\gamma| \leq \frac{\delta}{2}$ for $\gamma \in \Gamma_1$. For such y the following formula is satisfied:

$$\tilde{C}_\Delta(y) = C_\Delta^{\Gamma_2}(y) =$$
$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} C_\Delta\Big(\sum_{\substack{\gamma\in\Gamma\\ \gamma\notin\Gamma_2}} y_\gamma e_\gamma + \sum_{i=1}^{n} t_i e_{\gamma_i}\Big) b(y_{\gamma_1} - t_1)\dots b(y_{\gamma_n} - t_n)\, dt_1 \dots dt_n,$$

where $\Gamma_2 = \{\gamma_1, \dots, \gamma_n\}$.

Indeed, for every $\Gamma_3 = \{\gamma_1, \dots, \gamma_m\}$, $\Gamma_2 \subset \Gamma_3$ we have

$$C_\Delta^{\Gamma_3}(y) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} C_\Delta\Big(\sum_{\substack{\gamma\in\Gamma\\ \gamma\notin\Gamma_3}} y_\gamma e_\gamma + \sum_{i=1}^{m} t_i e_{\gamma_i}\Big) b(y_{\gamma_1} - t_1)\dots b(y_{\gamma_n} - t_n)\, dt_1 \dots dt_m,$$

and thus

$$C_\Delta^{\Gamma_3}(y)$$
$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} C_\Delta\Big(\sum_{\substack{\gamma\in\Gamma\\ \gamma\notin\Gamma_2}} y_\gamma e_\gamma + \sum_{i=1}^{n} t_i e_{\gamma_i}\Big) b(y_{\gamma_1} - t_1)\dots b(y_{\gamma_n} - t_n)\, dt_1 \dots dt_n$$
$$= C^{\Gamma_2}(y),$$

because the function $\phi(t_1, \dots, t_m) = C_\Delta\big(\sum_{\substack{\gamma\in\Gamma\\ \gamma\notin\Gamma_3}} y_\gamma e_\gamma + \sum_{i=1}^{m} t_i e_{\gamma_i}\big)$ is for any given $t_1, \dots, t_n$ constant in variables $t_{n+1}, \dots, t_m$ satisfying $|t_{n+1} - y_{\gamma_{n+1}}| \leq \frac{\delta}{4}, \dots,$ $|t_m - y_{\gamma_m}| \leq \frac{\delta}{4}$. The function $\tilde{C}_\Delta(\cdot)$ restricted to $B_{\|\cdot\|_\infty}\big(x, \frac{\delta}{4}\big)$ thus depends only on the coordinates $\{y_{\gamma_1}, \dots, y_{\gamma_n}\}$ of y and is easily observed to be C^∞-Fréchet smooth. The trivial estimate gives us

$$\|x\|^2 \leq C_\Delta(x) \leq \tilde{C}_\Delta(x) \leq \sup\{C_\Delta(x+v), \|v\|_\infty < \frac{\delta}{2}\}$$
$$\leq \sup\{\big(\|x+v\| + \varepsilon\big)^2,\ \|v\|_\infty < \frac{\delta}{2}\}.$$

By the standard argument of choosing ε and δ small enough, we obtain, via the implicit function theorem, that the C^∞-Fréchet smooth norm defined as the Minkowski functional of the set $\{x,\ \tilde{C}_\Delta(x) \le 1\}$ approximates arbitrary well (on bounded sets) the original norm $\|\cdot\|$.

We say that a norm $|||\cdot|||$ defined on a $C(K)$ space depends locally on finitely many coordinates if for every $f \in C(K)$ there exist a finite set $\{k_1, \dots, k_n\} \subset K, \varepsilon > 0$ and $\phi : \mathbb{R}^n \to \mathbb{R}$ such that

$$|||g||| = \phi\big(g(k_1), \dots, g(k_n)\big),$$

whenever $\|g - f\| < \varepsilon$. $\square$

Proposition 2. *There exists no lattice and Gâteaux differentiable* (*not necessarily equivalent*) *norm* $C_0([0, \omega_1])$. *There exists no lattice* (*not necessarily equivalent*) *norm on* $C_0([0, \omega_1])$ *that depends locally on finitely many coordinates.*

Proof: Assume that $\|\cdot\|$ is a given norm on $C_0([0, \omega_1])$. Let us first define, for a given non-limit ordinal $\alpha < \omega_1$, φ_α on $[\alpha, \omega_1)$ by

$$\varphi_\alpha(\beta) = \|\chi_{[\alpha,\beta]}\| \ \text{ for } \beta \text{ a nonlimit ordinal,}$$
$$\varphi_\alpha(\beta) = \sup\{\varphi_\alpha(\gamma),\ \gamma < \beta,\ \gamma \text{ nonlimit}\} \text{ for } \beta \text{ a limit ordinal.}$$

The function φ_α is well defined since $\chi_{[\alpha,\beta]} \in C_0[0, \omega_1]$ whenever α, β are nonlimit ordinals. By the lattice condition on $\|\cdot\|$, φ_α is a nondecreasing function defined on $[0, \omega_1)$. Thus for some nonlimit $\beta_\alpha > \alpha$ we have

$$\varphi_\alpha(\beta_\alpha) = \varphi_\alpha(\gamma) \text{ for every } \gamma \in [\beta_\alpha, \omega_1].$$

Similarly, by the lattice assumption, whenever $\alpha_1 < \alpha_2$ are nonlimit ordinals, $\varphi_{\alpha_1}(\beta_{\alpha_1}) \le \varphi_{\alpha_2}(\beta_{\alpha_2})$. Therefore, there exists $\alpha_0 \in \omega_1$ such that

$$\varphi_{\alpha_0}(\beta_{\alpha_0}) \ge \varphi_\alpha(\beta) \text{ whenever } \beta \ge \alpha \ge \alpha_0.$$

Let us define, by induction, a sequence $\{\alpha_i\}_{i=0}^\infty$ as follows: α_0 comes from the above consideration, $\alpha_{i+1} = \beta_{\alpha_i} + 1$.

Choose a closed and open countable interval $[\alpha_0, \beta] \subset [0, \omega_1)$ such that $\beta \ge \alpha_i$ for every $i \in \mathbb{N}$. Clearly, $\chi_{[\alpha_0,\beta]} \in C_0([0, \omega_1])$ and

$$0 < \|\chi_{[\alpha_0,\beta]}\| = \|\chi_{[\alpha_i,\beta_{\alpha_i}]}\| \ \text{ for every } i \in \mathbb{N}.$$

Also,

$$\|\chi_{[\alpha_0,\beta]} + t\,\chi_{[\alpha_i,\beta_{\alpha_i}]}\| \ge \|(1+t)\,\chi_{[\alpha_i,\beta_{\alpha_i}]}\| = (1+t)\|\chi_{[\alpha_0,\beta]}\| \ \text{ for every } t \ge 0.$$

Thus, the directional derivative of $\|\cdot\|$ at $\chi_{[\alpha_0,\beta]}$ in direction of $v_i = \chi_{[\alpha_i,\beta_{\alpha_i}]}$ satisfies:

$$\frac{\partial\|\chi_{[\alpha_0,\beta]}\|}{\partial v_i} \geq \frac{\partial\|\chi_{[\alpha_i,\beta_{\alpha_i}]}\|}{\partial v_i} \geq \|\chi_{[\alpha_i,\beta_{\alpha_i}]}\| = \|\chi_{[\alpha_0,\beta]}\|.$$

However, assuming the existence of the Gâteaux derivative $\|\chi_{[\alpha_0,\beta]}\|'$, we estimate

$$\Big\|\|\chi_{[\alpha_0,\beta]}\|'\Big\|_1 \geq \frac{\langle\|\chi_{[\alpha_0,\beta]}\|', \sum_{i=0}^{n} v_i\rangle}{\sum_{i=0}^{n} v_i} = \frac{\sum_{i=0}^{n} \frac{\partial\|\chi_{[\alpha_0,\beta]}\|}{\partial v_i}}{\|\chi_{[\alpha_0,\beta]}\|} \geq n$$

for all $n \in \mathbb{N}$. ($\|\sum_{i=0}^{n} v_i\| = \|\chi_{[\alpha_0,\beta]}\|$ by the lattice property of $\|\cdot\|$.) This is a contradiction. □

This proves the first half of Proposition 2. The proof for the second part requires only minor modifications.

Acknowledgment. We would like to thank the referee for suggesting some improvements and for finding an error in the original version. The second author would also like to thank the Department of Mathematics, University of Alberta, for hospitality and support during the preparation of this note.

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(*Received* March 22, 1996)