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Dimension and ε -translations

TATSUO GOTO

Dedicated to Professor Akihiro Okuyama on his 60th birthday

Abstract. Some theorems characterizing the metric and covering dimension of arbitrary subspaces in a Euclidean space will be obtained in terms of ε -translations; some of them were proved in our previous paper [G1] under the additional assumption of the boundedness of subspaces.

Keywords: metric dimension, covering dimension, ε -translation, uniformly 0-dimensional mappings

Classification: Primary 55M10

1. Introduction

In the previous paper [G1] we proved some theorems which characterize the metric dimension μdim for bounded subspaces in a Euclidean space in terms of ε -translations. In this paper, these results will be extended for arbitrary (unbounded) subspaces and also, we will obtain some results characterizing the covering dimension dim in terms of some classes of ε -translations such as \mathcal{U} - 0 -dimensional mappings in the sense of [Z-S] or uniformly 0-dimensional mappings of Katětov [Ka1].

Throughout this paper, all spaces are assumed to be *metric* and mappings are *continuous*.

2. Metric dimension and ε -translations

Let $X \subseteq \mathbb{R}^n$ and $\varepsilon > 0$. Then a mapping $f : X \rightarrow \mathbb{R}^n$ is called an ε -translation if $\|x - f(x)\| < \varepsilon$ for every $x \in X$. The *metric dimension* μdim of X is defined to be the least integer m for which X admits open covers of order $\leq m + 1$ with arbitrarily small meshes [Sm1]. Suppose \mathcal{U} is a locally finite open cover of X and $\mathcal{P} = \{p_U : U \in \mathcal{U}\}$ is an arbitrary set in \mathbb{R}^n . Consider the rectilinear closed (degenerate in general) simplex $(p_{U_0}, \dots, p_{U_r})$ with vertices p_{U_0}, \dots, p_{U_r} for every finite number of elements $U_0, \dots, U_r \in \mathcal{U}$ with $U_0 \cap \dots \cap U_r \neq \emptyset$. Let \mathcal{N} be the family of all of these simplexes and we call \mathcal{N} the *complex determined by \mathcal{U} and \mathcal{P}* . Then the κ -mapping $f : X \rightarrow \cup \mathcal{N}$ relative to \mathcal{U} and \mathcal{P} is defined by

$$f(x) = \sum_{U \in \mathcal{U}} f_U(x) p_U \quad \text{where} \quad f_U(x) = \frac{d(x, X - U)}{\sum_{V \in \mathcal{U}} d(x, X - V)} \quad \text{for } x \in X.$$

If for some $\varepsilon > 0$, $\delta(U \cup \{p_U\}) < \varepsilon$ for every $U \in \mathcal{U}$, then f is an ε -translation. By a *simplicial complex* \mathcal{K} in \mathbb{R}^n , we mean a geometric (not necessarily finite) simplicial complex which is locally finite in \mathbb{R}^n at every point in $\cup \mathcal{K}$. Also, a *polyhedron* means an underlying space of a simplicial complex. If $P = \cup \mathcal{K}$ and \mathcal{K} is a uniform complex in the sense of Smirnov [Sm2], then we call \mathcal{K} a *uniform triangulation* of P . We note that if a polyhedron P in \mathbb{R}^n admits a uniform triangulation, then P is closed in \mathbb{R}^n [Sm2]. The following lemma is an extension of [Eg, Theorem 3].

Lemma 1. *Let X be an arbitrary subspace in \mathbb{R}^n with $\mu\dim X \leq m$, $0 \leq m \leq n - 1$. Then for every $\varepsilon > 0$ and every sequence $\{H_i\}$ of $(n - m - 1)$ -dimensional planes in \mathbb{R}^n , there exists an ε -translation of $f : X \rightarrow P \subseteq \mathbb{R}^n - \cup H_i$ where P is an m -dimensional polyhedron with a uniform triangulation.*

PROOF: Take a $\delta > 0$ with $4\sqrt{n}\delta/3 < \varepsilon$. For every integer k , we denote by $E(k)$ the open interval $((k - \frac{2}{3})\delta, (k + \frac{2}{3})\delta)$ and set

$$\mathcal{E} = \{E(k_1, \dots, k_n) : k_1, \dots, k_n \in \mathbb{Z}\} \text{ where } E(k_1, \dots, k_n) = E(k_1) \times \dots \times E(k_n).$$

Then \mathcal{E} is an open cover of \mathbb{R}^n by open n -cubes with mesh $< \varepsilon$. For $E \in \mathcal{E}$, we denote by p_E the center of E and set $\mathcal{P} = \{p_E : E \in \mathcal{E}\}$. Let \mathcal{N} be the complex determined by \mathcal{E} and \mathcal{P} . Denote by $\tau(k_1, \dots, k_n)$ the closed n -cube $\{x \in \mathbb{R}^n : k_i\delta \leq x_i \leq (k_i + 1)\delta, 1 \leq i \leq n\}$, and we set $\mathcal{T} = \{\tau(k_1, \dots, k_n) : k_1, \dots, k_n \in \mathbb{Z}\}$. Then for every simplex $\sigma \in \mathcal{N}$ there exists $\tau \in \mathcal{T}$ such that all vertices of σ are those of τ . For $\tau \in \mathcal{T}$, let V_τ be the set of vertices of τ . Then the family of all $(n - 1)$ -dimensional planes determined by n points from V_τ defines a cellular decomposition of τ , and applying the barycentric decomposition [AH], we obtain a simplicial decomposition \mathcal{K}_τ of τ . Then $\mathcal{K} = \cup\{\mathcal{K}_\tau : \tau \in \mathcal{T}\}$ defines a uniform triangulation of \mathbb{R}^n since every \mathcal{K}_τ is finite and congruent to each other.

Now let \mathcal{U} be an open cover of X with mesh $\mathcal{U} < \delta/3$ and $\text{ord } \mathcal{U} \leq m + 1$; such a cover exists since $\mu\dim X \leq m$ by assumption. Since $\delta/3$ is a Lebesgue number of \mathcal{E} there exists $i : \mathcal{U} \rightarrow \mathcal{E}$ such that $U \subseteq i(U)$ for every $U \in \mathcal{U}$. Define an open cover \mathcal{V} of X by

$$\mathcal{V} = \{V_E : E \in \mathcal{E}\} \text{ where } V_E = \cup\{U \in \mathcal{U} : i(U) = E\}.$$

Then \mathcal{V} is a star-finite open cover of X and $\text{ord } \mathcal{V} \leq m + 1$. Let \mathcal{L} be the complex determined by \mathcal{V} and \mathcal{P} and $g : X \rightarrow \cup \mathcal{L}$ the κ -mapping relative to \mathcal{V} and \mathcal{P} . Note that $\cup \mathcal{L} \subseteq \cup \mathcal{K}^{(m)}$ where $\mathcal{K}^{(m)}$ is the m -skeleton of \mathcal{K} and that g is a λ -translation where $\lambda = \text{mesh } \mathcal{E}$, because $\delta(V_E \cup \{p_E\}) \leq \delta(E)$ for $E \in \mathcal{E}$. Let $V_0 = \{p_i\}$ be the set of vertices in $\mathcal{K}^{(m)}$. Since \mathcal{K} is uniform, so is $\mathcal{K}^{(m)}$. Hence by [Sm2, Corollary to Theorem 2] there exists an $\varepsilon' > 0$ satisfying the condition:

if $\{q_i\} \subseteq \mathbb{R}^n$ and $\|p_i - q_i\| < \varepsilon'$ for every i , then there exist a uniform complex \mathcal{K}' with vertices in $\{q_i\}$ and an isomorphism $\varphi : \mathcal{K}^{(m)} \rightarrow \mathcal{K}'$ sending each simplex $(p_{i_0}, \dots, p_{i_r})$ to $(q_{i_0}, \dots, q_{i_r})$.

We may assume that $\lambda + \varepsilon' < \varepsilon$. Moreover, by [Ku, p. 307] we can choose $\{q_i\}$ so that $\{q_i\}$ is in general position relative to $\{H_i\}$ i.e., $\sigma \cap (\cup H_i) = \emptyset$ for every simplex σ whose vertices are in $\{q_i\}$ and $\dim \sigma \leq m$. Then the polyhedron $P = \cup \mathcal{K}'$ is disjoint from $\cup H_i$ and the homeomorphism $h : \cup \mathcal{K}^{(m)} \rightarrow P$ induced from φ , which is linear on each simplex, is an ε' -translation. Then $f = h \circ g : X \rightarrow P$ is a desired ε -translation since $\|x - f(x)\| \leq \|x - g(x)\| + \|g(x) - h(g(x))\| < \lambda + \varepsilon' < \varepsilon$ for every $x \in X$. \square

Let m, n be integers with $0 \leq m \leq n - 1$. The space N_m^n is defined to be the set of points in R^n at most m of whose coordinates are rationals. Then we have $\dim N_m^n = \mu \dim N_m^n = m[E]$. The space S_m^n , which was defined in [G2] by modifying the space $S_{n,m}$ in [G1], satisfies the relations:

$$N_m^n \subseteq S_m^n, \mu \dim S_m^n = m \text{ and } \dim S_m^n = \min\{2m, n - 1\}.$$

Note that $\dim X \leq 2\mu \dim X$ for every X by [Ka2]. Hence, among those subspaces in R^n of metric dimension m , S_m^n is of the maximal difference with its covering dimension.

The following theorem is an extension of [G1, Theorem 1] which was proved under the additional condition of the boundedness of X .

Theorem 2. *Let X be an arbitrary subspace in R^n and m an integer with $0 \leq m \leq n - 1$. Then the following conditions are equivalent.*

- (a) $\mu \dim X \leq m$.
- (b) For every $\varepsilon > 0$ and every polyhedron P in R^n of dimension $\leq n - m - 1$, there exists an ε -translation $f : X \rightarrow R^n$ with $f(X) \cap P$ (or $\text{Cl}(f(X)) \cap P$) = \emptyset .
- (c) For every $\varepsilon > 0$ and every polyhedron P with a uniform triangulation in R^n of dimension $\leq n - m - 1$, there exists an ε -translation $f : X \rightarrow R^n$ with $f(X) \cap P$ (or $\text{Cl}(f(X)) \cap P$) = \emptyset .

PROOF: Since every polyhedron admits a triangulation consisting of countably many simplexes, (a) implies (b) by Lemma 1. Obviously (b) implies (c).

Assume that the condition (c) is satisfied. Then for every $\varepsilon > 0$, as was proved essentially in [G1, Theorem 1], there exists an ε -translation of X into an m -dimensional polyhedron; it needs only to observe that the polyhedron $B_{i,n-m-1}$ in [G1] allows a uniform triangulation. Hence by [Sm1, Corollary 2] we have $\mu \dim X \leq m$. \square

The following theorem which extends [G1, Theorem 2], can be proved similarly by use of Lemma 1 and its proof is omitted.

Theorem 3. *For every subspace X in R^n and every integer m with $0 \leq m \leq n - 1$, the following conditions are equivalent.*

- (a) $\mu \dim X \leq m$.
- (b) For every $\varepsilon > 0$ there exists an ε -translation f of X into an m -dimensional polyhedron P (with a uniform triangulation) such that $P \subseteq N_m^n$.

- (c) For every $\varepsilon > 0$ there exists an ε -translation f of X into an m -dimensional polyhedron P (with a uniform triangulation) such that $P \subseteq S_m^n$.

3. Covering dimension and ε -translations

Let \mathcal{U} be an open cover of a space X and $A \subseteq X$. Then we write \mathcal{U} -dim $A \leq 0$ if there exists a pairwise disjoint open collection \mathcal{U}_0 in X such that $\cup \mathcal{U}_0 \supseteq A$ and \mathcal{U}_0 refines \mathcal{U} . A mapping $f : X \rightarrow Y$ is called \mathcal{U} -0-dimensional (or \mathcal{U} -dim $f \leq 0$) if for some open cover \mathcal{V} of Y , \mathcal{U} -dim $f^{-1}(V) \leq 0$ for every $V \in \mathcal{V}$ [Z-S].

Lemma 4. *Let \mathcal{U} be a countable star-finite open cover of a space X with ord $\mathcal{U} \leq k + 1$ and \mathcal{N} the complex determined by \mathcal{U} and $\mathcal{P} = \{p_U : U \in \mathcal{U}\} \subseteq \mathbb{R}^n$. If \mathcal{N} consists of non-degenerate simplexes and is locally finite in \mathbb{R}^n at every point in $\cup \mathcal{N}$, then \mathcal{U} -dim $f \leq 0$ for the κ -mapping f determined by \mathcal{U} and \mathcal{P} .*

PROOF: By [Ku, p. 239], there exists a geometric realization \mathcal{K} of the nerve of \mathcal{U} in \mathbb{R}^{2k+1} . Let $\mathcal{Q} = \{q_U : U \in \mathcal{U}\}$ where q_U is the vertex of \mathcal{K} corresponding to $U \in \mathcal{U}$, and let $\pi : \mathcal{K} \rightarrow \mathcal{N}$ be the mapping sending each simplex $(q_{U_0}, \dots, q_{U_r})$ to $(p_{U_0}, \dots, p_{U_r})$. Since \mathcal{K} is locally finite, π induces a mapping $p : \cup \mathcal{K} \rightarrow \cup \mathcal{N}$ uniquely which is linear on each simplex in \mathcal{K} . Clearly we have $f = p \circ g$ for the κ -mapping g relative to \mathcal{U} and \mathcal{Q} .

Let $y \in \cup \mathcal{N}$. Then y is contained in the interior of only finitely many simplexes in \mathcal{N} , say $\sigma_1, \dots, \sigma_s$. Since p is homeomorphic on each simplex, $p^{-1}(y)$ consists of exactly s points. For every $z_i \in p^{-1}(y)$, we choose a simplex $\tau_i \in \mathcal{K}$ such that z_i is in the interior of τ_i , $1 \leq i \leq s$. Let W_i be the open star of τ_i in \mathcal{K} , and then $\{W_i : 1 \leq i \leq s\}$ is pairwise disjoint. For, if otherwise, there would be a simplex $\tau \in \mathcal{K}$ with distinct faces τ_i and τ_j . But this contradicts that p is homeomorphic on τ . Let \mathcal{L} be the subcomplex of \mathcal{K} such that $\cup \mathcal{L} = \cup \mathcal{K} - \cup \{W_i : 1 \leq i \leq s\}$. Since \mathcal{N} is locally finite by assumption, $V_y = \cup \mathcal{N} - p(\cup \mathcal{L})$ is an open neighborhood of y such that $f^{-1}(V_y) = g^{-1}p^{-1}(V_y) \subseteq \cup \{g^{-1}(W_i) : 1 \leq i \leq s\}$. Since g is a κ -mapping, $\{g^{-1}(W_i) : 1 \leq i \leq s\}$ refines \mathcal{U} . This means \mathcal{U} -dim $f^{-1}(V_y) \leq 0$ and hence \mathcal{U} -dim $f \leq 0$. □

Theorem 5. *Let X be an arbitrary subspace in \mathbb{R}^n and k an integer with $0 \leq k \leq n$. Then $\dim X \leq k$ iff for every finite open cover \mathcal{U} of X , there exists an ε -translation $f : X \rightarrow \mathbb{R}^n$ such that \mathcal{U} -dim $f \leq 0$ and $f(X)$ (or $\text{Cl}(f(X))$) $\subseteq N_k^n$.*

PROOF: *Necessity.* Let $\varepsilon > 0$ and $\mathcal{U} = \{U_1, \dots, U_r\}$ be an open cover of X . Let \mathcal{E} be the cover of \mathbb{R}^n by open n -cubes with mesh $< \varepsilon$ in the proof of Lemma 1. Since $\dim X \leq k$, there exists an open cover $\mathcal{V} = \{V(k_1, \dots, k_n; j) : k_1, \dots, k_n \in \mathbb{Z}, 1 \leq j \leq r\}$ such that ord $\mathcal{V} \leq k + 1$ and $V(k_1, \dots, k_n; j) \subseteq E(k_1, \dots, k_n) \cap U_j$ for every k_i and j . As in the proof of Lemma 1, we can take $\mathcal{P} = \{p_V : V \in \mathcal{V}\}$ in \mathbb{R}^n such that

- \mathcal{P} is in general position in \mathbb{R}^n ,
- $p_V \in E(k_1, \dots, k_n)$ for $V = V(k_1, \dots, k_n; j)$, and
- $\cup \mathcal{N} \subseteq N_k^n$ where \mathcal{N} is the complex determined by \mathcal{V} and \mathcal{P} .

Then \mathcal{N} consists of non-degenerate simplexes and is locally finite in \mathbb{R}^n . Hence the κ -mapping f relative to \mathcal{V} and \mathcal{P} is \mathcal{U} -0-dimensional by Lemma 4 and is a desired ε -translation since $\delta(V \cup \{p_V\}) < \text{mesh } \mathcal{E} < \varepsilon$. The proof of the sufficiency is almost evident. \square

Let $X \subseteq \mathbb{R}^n$ and $\varepsilon > 0$. We denote by $T_\varepsilon(X)$ the collection of all ε -translations of X into \mathbb{R}^n and set $T(X) = \cup\{T_\varepsilon(X) : \varepsilon > 0\}$. Then $T(X)$ is complete relative to the metric defined by $d(f, g) = \sup\{\|f(x) - g(x)\| : x \in X\}$.

Theorem 6. *Let X be a bounded subspace in \mathbb{R}^n with $0 \leq k \leq n$. Then $\dim X \leq k$ iff for every $\varepsilon > 0$ there exists a uniformly 0-dimensional ε -translation $f : X \rightarrow \mathbb{R}^n$ such that $f(X)$ (or $\text{Cl}(f(X))$) $\subseteq N_k^n$.*

PROOF: The sufficiency of the theorem follows from the fact that every uniformly 0-dimensional mapping does not decrease the dimension [Ka1, Theorem 3.3].

Assume that $\dim X \leq k$ and $\varepsilon > 0$. Let $\{H_i\}$ be a sequence of $(n - k - 1)$ -dimensional planes in \mathbb{R}^n such that $\mathbb{R}^n - N_k^n = \cup H_i$. We set

$$\mathcal{S}_i = \{f \in T(X) : \text{Cl}(f(X)) \cap H_i = \emptyset\} \text{ for } i \in \mathbb{N}, \text{ and}$$

$$\mathcal{T} = \{f \in T(X) : f \text{ is uniformly 0-dimensional}\}.$$

Then \mathcal{S}_i is dense and open in $T(X)$, and \mathcal{T} is a dense G_δ -set in $T(X)$ [Ka1, Theorem 2.15]. Hence $\cap \mathcal{S}_i \cap \mathcal{T}$ is dense in $T(X)$, and there exists $f \in \cap \mathcal{S}_i \cap \mathcal{T}$ with $d(1_X, f) < \varepsilon$. Then f is an ε -translation of X with $\text{Cl}(f(X)) \subseteq N_k^n$. \square

We don't know whether Theorem 6 is valid for unbounded subspace X .

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