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Abstract. In this paper we consider a number of sequence and function spaces that are known to be homeomorphic to the countable product of the linear space σ. The spaces we are interested in have a canonical imbedding in both a topological Hilbert space and a Hilbert cube. It turns out that when we consider these spaces as subsets of a Hilbert cube then there is only one topological type. For imbeddings in the countable product of lines there are two types depending on whether the space is contained in a σ-compactum or not.

Keywords: Hilbert space, Hilbert cube, $\mathcal{F}_{\sigma\delta}$-absorber, ambient homeomorphism, function space, p-summable sequence

Classification: 57N20

1. Introduction

The focus of our investigation are so-called $\mathcal{F}_{\sigma\delta}$-absorbers in topological Hilbert spaces and Hilbert cubes ($\mathcal{F}_{\sigma\delta}$ stands for the class of all absolute $F_{\sigma\delta}$-sets). $\mathcal{F}_{\sigma\delta}$-absorbers are the "maximal" elements for that Borel class. The standard example of an $\mathcal{F}_{\sigma\delta}$-absorber is the subset $\sigma^{\mathbf{N}}$ in the product space $s^{\mathbf{N}}$, where $s = \mathbf{R}^{\mathbf{N}}$ and

$$\sigma = \{x \in s : x_i = 0 \text{ for all but finitely many } i\}.$$

Let X be an arbitrary countable completely regular space that is not discrete. Let $C_{\mathrm{p}}(X)$ stand for the subset of the product space $\mathbf{R}^X$ consisting of the continuous functions from X into $\mathbf{R}$. It was shown by Dobrowolski, Marciszewski and Mogilski [7], [4] that $C_{\mathrm{p}}(X)$ is a generalized $\mathcal{F}_{\sigma\delta}$-absorber (and hence homeomorphic to $\sigma^{\mathbf{N}}$) whenever $C_{\mathrm{p}}(X) \in \mathcal{F}_{\sigma\delta}$. Jan van Mill proved essentially in [11] that the pair $(\mathbf{R}^X, C_{\mathrm{p}}(X))$ is homeomorphic to $(s^{\mathbf{N}}, \sigma^{\mathbf{N}})$ provided that X is a metrizable space that is not locally compact. This led Dobrowolski and Mogilski [8, 6.11] to ask the following question: is (s, c_0) (or, equivalently, is $(\mathbf{R}^{\widehat{\mathbf{N}}}, C_{\mathrm{p}}(\widehat{\mathbf{N}}))$) homeomorphic to $(s^{\mathbf{N}}, \sigma^{\mathbf{N}})$, where

$$c_0 = \{x \in s : \lim_{i\to\infty} x_i = 0\}$$

and $\widehat{\mathbf{N}}$ is the convergent sequence? The answer to this question is no because $\sigma^{\mathbf{N}}$ contains a copy of Hilbert space that is closed in $s^{\mathbf{N}}$ where as c_0 is contained in the σ-compactum Σ consisting of the bounded sequences in s. We investigate the natural extension of the question to: for which X is $(\mathbf{R}^X, C_{\mathrm{p}}(X))$ homeomorphic to $(s^{\mathbf{N}}, \sigma^{\mathbf{N}})$?

Theorem 1.1. *The pairs* $(\mathbf{R}^X, C_{\mathrm{p}}(X))$ *and* $(s^{\mathbf{N}}, \sigma^{\mathbf{N}})$ *are homeomorphic if and only if* X *is not compact and* $C_{\mathrm{p}}(X) \in \mathcal{F}_{\sigma\delta}$.

If $p \in (0, \infty)$ let l_p be the subset of s consisting of the p-summable sequences. Put $\tilde{l}_p = \bigcap_{q>p} l_q$ for $p \in [0, \infty)$. It was shown by Dobrowolski and Mogilski [9], [4] that every $\tilde{l}_p$ is homeomorphic to $\sigma^{\mathbf{N}}$ but it is easily seen that $\tilde{l}_p$ is not an $\mathcal{F}_{\sigma\delta}$-absorber in s. We have the following result:

Theorem 1.2. *If* X *is compact and* $p \in [0, \infty)$ *then* $(\mathbf{R}^X, C_{\mathrm{p}}(X))$ *and* $(s, \tilde{l}_p)$ *are homeomorphic to* (s, c_0).

Consider the canonical compactifications $\widehat{\mathbf{R}}^{\mathbf{N}}$ and $\widehat{\mathbf{R}}^X$ of s and $\mathbf{R}^X$, where $\widehat{\mathbf{R}} = [-\infty, \infty]$. Throughout this paper the Hilbert cube Q is represented by $\widehat{\mathbf{R}}^{\mathbf{N}}$ and its pseudointerior s by $\mathbf{R}^{\mathbf{N}}$. In the Hilbert cube the distinction between the two types of imbeddings disappears:

Theorem 1.3. *If* $C_{\mathrm{p}}(X) \in \mathcal{F}_{\sigma\delta}$ *and if* $p \in [0, \infty)$ *then* $(\widehat{\mathbf{R}}^X, C_{\mathrm{p}}(X))$ *and* $(Q, \tilde{l}_p)$ *are both homeomorphic to* $(Q^{\mathbf{N}}, \sigma^{\mathbf{N}})$.

2. Absorbing systems

The material in this section has been taken from the papers [5] and [6]. For background information on infinite-dimensional topology see Bessaga and Pełczyński [2] or van Mill [12].

Throughout this section let E denote either a topological Hilbert space or Hilbert cube. Let Γ be a fixed index set. A collection $\mathcal{X} = (X_\gamma)_{\gamma \in \Gamma}$ of subsets of the space E (formally the pair $(E, \mathcal{X})$) is called a *Z-system* if $\bigcup\{X_\gamma : \gamma \in \Gamma\}$ is contained in a σZ-set of E. Let Δ be a subset of Γ. We say that a Z-system $(E, \mathcal{X})$ is *Δ-imbeddable in* (*Δ-homeomorphic to*) a Z-system $(E', \mathcal{Y})$ if there exists a closed imbedding (homeomorphism) $f : E \to E'$ such that $f^{-1}(Y_\gamma) = X_\gamma$ for each $\gamma \in \Delta$. The map f is called a *Δ-imbedding* (*Δ-homeomorphism*). If $\Delta = \Gamma$ then we simply say that $\mathcal{X}$ is imbeddable in (homeomorphic to) $\mathcal{Y}$. (Maps are assumed to be continuous.)

A Z-system $\mathcal{X}$ is called *reflexively universal* if for every map $f : E \to E$ that restricts to a Z-imbedding on some closed set $K \subset E$, there exists a Z-imbedding $g : E \to E$ that can be chosen arbitrarily close to f with the properties: $g|K = f|K$ and $g^{-1}(X_\gamma) \setminus K = X_\gamma \setminus K$ for every $\gamma \in \Gamma$. A Z-system $\mathcal{X}$ is called *reflexively universal rel* P (a subset of E) if for every map $f : E \to E$ that restricts to a Z-imbedding on some closed set $K \subset E$, there exists a Z-imbedding $g : E \to E$ that can be chosen arbitrarily close to f with the properties: $g|K = f|K$, $g(E \setminus K) \subset P$, and $g^{-1}(X_\gamma) \setminus K = X_\gamma \setminus K$ for every $\gamma \in \Gamma$. In our applications P is usually the pseudointerior s of Q.

These notions come together in the following result (see [5, Theorem 2.1] and [6, Theorem 2.1]).

Theorem 2.1.

(a) *Let $\mathcal{X}$ and $\mathcal{Y}$ be reflexively universal Z-systems in E respectively E'. If $\mathcal{X}$ is Δ-imbeddable in $\mathcal{Y}$ and $\mathcal{Y}$ is Δ-imbeddable in $\mathcal{X}$ then $\mathcal{X}$ is Δ-homeomorphic to $\mathcal{Y}$.*

(b) *Let $\mathcal{X}$ and $\mathcal{Y}$ be reflexively universal rel s in Q and assume that $\bigcup_\gamma X_\gamma$ and $\bigcup_\gamma Y_\gamma$ are contained in a σ-compact subset of s. If $\mathcal{X}$ is Δ-imbeddable in $\mathcal{Y}$ and $\mathcal{Y}$ is Δ-imbeddable in $\mathcal{X}$ then $\mathcal{X}$ is Δ-homeomorphic to $\mathcal{Y}$ via a homeomorphism that preserves s.*

PROOF: We prove part (b). The proof for (a) is essentially the same. Let $\bigcup_\gamma X_\gamma \cup \bigcup_\gamma Y_\gamma \subset \bigcup_i A_i$ and let $B = Q \setminus s = \bigcup_i B_i$, where $\emptyset = A_0 \subset A_1 \subset A_2 \subset \cdots \subset s$ and $\emptyset = B_0 \subset B_1 \subset B_2 \subset \cdots$ are sequences of Z-sets in Q. By induction we shall construct sequences of homeomorphisms $f_i : Q \to Q$ and $g_i = f_i \circ \cdots \circ f_0$ with the properties (for each $\gamma \in \Delta$):

$$A_i \cap X_\gamma = A_i \cap g_i^{-1}(Y_\gamma),$$
$$A_i \cap g_i(X_\gamma) = A_i \cap Y_\gamma,$$
$$g_i(B) = B,$$
$$f_i|\,(g_{i-1}(A_{i-1} \cup B_{i-1}) \cup A_{i-1} \cup B_{i-1}) = 1,$$

where 1 denotes the identity map. Put $f_0 = 1$.

Assume that f_i has been constructed. Put $K = g_i(A_i) \cup A_i$ and observe that $g_i(X_\gamma) \cap K = Y_\gamma \cap K$. Let $p : Q \to Q$ be a Δ-imbedding of the system $\mathcal{X}$ into $\mathcal{Y}$. Then the inverse of $p \circ g_i^{-1}$ is defined on a closed subset of Q and can therefore be extended to a map $r : Q \to Q$. Since $\mathcal{Y}$ is reflexively universal rel s and K is a subset of s we can approximate r by a Z-imbedding $\tilde{r} : Q \to s$ with the properties $\tilde{r}^{-1}(Y_\gamma) = Y_\gamma$ for each $\gamma \in \Delta$ and $\tilde{r}$ coincides with r on $p \circ g_i^{-1}(K)$. Let α be the Z-imbedding $\tilde{r} \circ p \circ g_i^{-1}$ and note that α fixes K and that it has the property $\alpha^{-1}(Y_\gamma) = g_i(X_\gamma)$ for each $\gamma \in \Delta$. Observe that $\alpha|g_i(A_{i+1}) \cup A_i$ is a homeomorphism between compacta in s and hence it can be extended to a homeomorphism $\tilde{\alpha}$ of Q. Without loss of generality we may assume that $\tilde{\alpha}(g_i(B)) = B$ and $\alpha|g_i(B_i) \cup B_i = 1$. This homeomorphism satisfies in addition:

$$\tilde{\alpha}^{-1}(Y_\gamma) \cap g_i(A_{i+1}) = g_i(X_\gamma \cap A_{i+1}).$$

By a similar argument we can find a homeomorphism $\tilde{\beta}$ of Q that fixes the set $\tilde{\alpha} \circ g_i(A_{i+1} \cup B_{i+1}) \cup A_i \cup B_i$ and that has the properties $\tilde{\beta}(B) = \tilde{\alpha} \circ g_i(B)$ and

$$\tilde{\beta}^{-1}(\tilde{\alpha} \circ g_i(X_\gamma)) \cap A_{i+1} = Y_\gamma \cap A_{i+1}.$$

If we put $f_{i+1} = \tilde{\beta}^{-1} \circ \tilde{\alpha}$ then one can easily verify the induction hypothesis for $i+1$. Since $\tilde{\alpha}$ and $\tilde{\beta}$ and hence f_{i+1} can be chosen arbitrarily close to the identity we may assume that $h = \lim_{i\to\infty} g_i$ is a homeomorphism of Q. The function h

maps X_γ onto Y_γ for each $\gamma \in \Delta$ and it maps the pseudoboundary B onto itself. $\square$

A subset A is *locally homotopy negligible in* X if for every map $f : M \to X$ from an absolute neighborhood retract M and for every open cover $\mathcal{U}$ of X there exists a homotopy $h : M \times [0,1] \to X$ such that $\{h(\{x\} \times [0,1])\}_{x \in M}$ refines $\mathcal{U}$, $h(x,0) = f(x)$ and $h(M \times (0,1]) \subset X \setminus A$. A σZ-set and the complement of a capset or fd-capset is always locally homotopy negligible.

For a space X and $* \in X$ we define the weak cartesian product

$$W(X,*) = \{x \in X^{\mathbf{N}} : x_i = * \text{ for all but finitely many } i\}.$$

The following lemma is essentially [5, Lemma 6.2] and [6, Proposition 3.6].

Lemma 2.2.

(a) *Let $\mathcal{X} = (X_\gamma)_{\gamma \in \Gamma}$ be a system in E such that $E \setminus \bigcap_{\gamma \in \Gamma} X_\gamma$ is locally homotopy negligible in E and let $* \in \bigcap_{\gamma \in \Gamma} X_\gamma$. Assume that there exists a homeomorphism $\Phi : E \to E^{\mathbf{N}}$ satisfying*

$$W(X_\gamma, *) \subset \Phi(X_\gamma) \subset X_\gamma^{\mathbf{N}}$$

for all $\gamma \in \Gamma$. Then $\mathcal{X}$ is reflexively universal.

(b) *Let $\mathcal{X} = (X_\gamma)_{\gamma \in \Gamma}$ be a system in Q such that $\bigcap_{\gamma \in \Gamma} X_\gamma$ is a subset of s whose complement is locally homotopy negligible in Q and let $* \in \bigcap_{\gamma \in \Gamma} X_\gamma$. Assume that $(Q, \mathcal{X})$ has a Γ-imbedding into itself whose image is contained in s. If there exists a homeomorphism $\Phi : E \to E^{\mathbf{N}}$ satisfying $s^{\mathbf{N}} \subset \Phi(s)$ and*

$$W(X_\gamma, *) \subset \Phi(X_\gamma) \subset X_\gamma^{\mathbf{N}}$$

for all $\gamma \in \Gamma$ then $\mathcal{X}$ is reflexively universal rel s.

Let Γ be an ordered set and let $\mathcal{M}_\gamma$ be a collection of spaces for each $\gamma \in \Gamma$. Each $\mathcal{M}_\gamma$ is assumed to be *topological* and *closed hereditary*. Let $\mathcal{M}$ stand for the whole system $(\mathcal{M}_\gamma)_{\gamma \in \Gamma}$. Let $\mathcal{X} = (X_\gamma)_{\gamma \in \Gamma}$ be an order preserving indexed collection of subsets of a topological Hilbert cube (Hilbert space) E, i.e. $X_\gamma \subset X_{\gamma'}$ if and only if $\gamma \leq \gamma'$.

The system $\mathcal{X}$ is called $\mathcal{M}$*-universal* if for every order preserving system $(A_\gamma)_\gamma$ in E such that $A_\gamma \in \mathcal{M}_\gamma$ for every $\gamma \in \Gamma$, there is a closed imbedding $f : E \to E$ with $f^{-1}(X_\gamma) = A_\gamma$. The system $\mathcal{X}$ is called *strongly* $\mathcal{M}$*-universal rel* $P \subset E$ if for every order preserving system $(A_\gamma)_\gamma$ in E such that $A_\gamma \in \mathcal{M}_\gamma$ for every $\gamma \in \Gamma$, and for every map $f : E \to E$ that restricts to a Z-imbedding on some compact set K, there exists a Z-imbedding $g : E \to E$ that can be chosen arbitrarily close to f with the properties: $g|K = f|K$, $g(E \setminus K) \subset P$, and $g^{-1}(X_\gamma) \setminus K = A_\gamma \setminus K$ for every γ. If $\mathcal{X}$ is strongly $\mathcal{M}$-universal rel E in E then it is simply called strongly $\mathcal{M}$-universal. Observe that X is strongly $\mathcal{M}$-universal (rel P) whenever X is $\mathcal{M}$-universal and reflexively universal (rel P). If $X_\gamma \in \mathcal{M}_\gamma$ then the converse is also true.

The system X is called $\mathcal{M}$-*absorbing* (rel P) if

(1) $X_\gamma \in \mathcal{M}_\gamma$ for every $\gamma \in \Gamma$,
(2) $\bigcup\{X_\gamma : \gamma \in \Gamma\}$ is a Z-system of E, and
(3) X is strongly $\mathcal{M}$-universal (rel P).

The following uniqueness result follows immediately from Theorem 2.1.

Theorem 2.3.

(a) *If $\mathcal{X}$ and $\mathcal{Y}$ are both $\mathcal{M}$-absorbing systems in E respectively E' then $(E,\mathcal{X})$ and $(E',\mathcal{Y})$ are homeomorphic, i.e. there is a homeomorphism $h : E \to E'$ such that $h(X_\gamma) = Y_\gamma$ for all $\gamma \in \Gamma$. If $E = E'$ then the map h can be found arbitrarily close to the identity.*
(b) *If $\mathcal{X}$ and $\mathcal{Y}$ are both $\mathcal{M}$-absorbing systems rel s in Q and $\bigcup_\gamma(X_\gamma \cup Y_\gamma)$ is contained in a σ-compactum of s, then $(Q,s,\mathcal{X})$ and $(Q,s,\mathcal{Y})$ are homeomorphic, i.e. the homeomorphism h maps the pseudointerior onto itself.*

If the absorbing system consists of just one element X then we say that X is an $\mathcal{M}$-absorber. A *capset* is an absorber for the class of compacta. The standard examples of capsets are Σ in s and Q and the pseudoboundary $B = Q\setminus s$ in Q. An *fd-capset* is an absorber for the class of finite-dimensional compacta. Standard examples are σ in s and Q and

$$l^p_{\mathrm{f}} = \{x \in l^p : x_i = 0 \text{ for all but finitely many } i\}$$

in the Banach space l^p. The examples of $\mathcal{F}_{\sigma\delta}$-absorbers are $\Sigma^{\mathbf{N}}$ and $\sigma^{\mathbf{N}}$ in $s^{\mathbf{N}}$ and $Q^{\mathbf{N}}$.

We finish this section with a few useful lemmas. The first concerns Z-imbeddings (see [6, Lemma 3.2]). Let I denote the interval $[0,1]$.

Lemma 2.4. *Let f and g be functions from a space X into the space E. Let $\varepsilon : X \to I$ be a map and let d be a metric on E such that f and g are ε-close (i.e. $d(f(x),g(x)) \le \varepsilon(x)$ for $x \in X$) and $\varepsilon(x) \le \frac{1}{2}d\big(f(x), f(\varepsilon^{-1}(0))\big)$ for $x \in X$. If f is a Z-imbedding and $g|\varepsilon^{-1}([\delta,1])$ is a Z-imbedding for each $\delta > 0$ then g is a Z-imbedding.*

Recall that since maps into E can be approximated by Z-imbeddings we have that if $f : X \to E$ and $\varepsilon : X \to I$ are continuous maps then there is a $g : X \to I$ that is ε-close to f and with the property $g|\varepsilon^{-1}([\delta,1])$ is a Z-imbedding for each $\delta > 0$.

Lemma 2.5. *If $\mathcal{X}$ is an $\mathcal{M}$-absorbing system in the pseudointerior of the Hilbert cube Q then it is also an $\mathcal{M}$-absorbing system in Q.*

Proof: We only need to look at strong $\mathcal{M}$-universality. Let f be a map from Q to Q, A_γ an order preserving system from $\mathcal{M}$ in Q, and let K be a closed subset in Q. We may assume that f is a Z-imbedding with the property $f(Q \setminus K) \subset s$. Let d be some metric on Q, let d' be a complete metric on s with $d' \geq d$, and let $\varepsilon : Q \to I$ be an arbitrary map that satisfies the conditions $\varepsilon^{-1}(0) = K$ and $\varepsilon(x) \leq \frac{1}{2} d(f(x), f(K))$ for each $x \in Q$. Define the compacta $K_i = \varepsilon([0, 2^{-i+1}])$ for $i = 0, 1, 2, \ldots$. We shall construct inductively a sequence $g_i : Q \setminus K \to s$ of Z-imbeddings with induction hypothesis:

$$g_i^{-1}(X_\gamma) \setminus K_{i+1} = A_\gamma \setminus K_{i+1}.$$

Put $g_0 = f|Q \setminus K$ and assume that g_i has been found. Since we can imbed $Q \setminus K$ as a closed subset of s and since $A_\gamma \setminus \operatorname{int}(K_{i+2}) \in \mathcal{M}$ the strong universality of the system in s implies that we can find a Z-imbedding $g_{i+1} : Q \setminus K \to s$ that is $(\varepsilon 2^{-i-1})$-close to g_i with respect to d' and with the additional properties:

$$g_{i+1}^{-1}(X_\gamma) \setminus K_{i+2} = A_\gamma \setminus K_{i+2},$$
$$g_{i+1}|\overline{Q \setminus K_i} = g_i|\overline{Q \setminus K_i}.$$

Since g_i is a Cauchy sequence with respect to d' we have that $g = \lim_{i\to\infty} g_i$ exists. Obviously, $\tilde{g} = g \cup (f|K)$ is ε-close to f. Since $\tilde{g}|\overline{Q \setminus K_i} = g_i|\overline{Q \setminus K_i}$ we have that $\tilde{g}|\varepsilon^{-1}([\delta, 1])$ is a Z-imbedding for every $\delta > 0$. This means that according to Lemma 2.4 $\tilde{g}$ is a Z-imbedding. One easily verifies that $\tilde{g}^{-1}(X_\gamma) \setminus K = A_\gamma \setminus K$ for every γ. □

The following lemma is a reformulation of [5, Lemma 6.4] with an identical proof.

Lemma 2.6. *If $\mathcal{X}$ is strongly $\mathcal{M}$-universal rel P in Q and Y is a subset of a compact absolute retract M with a locally homotopy negligible complement, then $(X_\gamma \times Y)_\gamma$ is strongly $\mathcal{M}$-universal rel $P \times Y$ in $Q \times M$.*

3. Function spaces in the topology of pointwise convergence

In this section we prove the $C_{\mathrm{p}}(X)$ parts of the theorems in the introduction. We first consider spaces with only one accumulation point, which leads us to free filters on the set $\mathbf{N}$.

Let $\mathfrak{F}_{\mathrm{cof}}$ stand for the Fréchet filter on $\mathbf{N}$, i.e. $\mathfrak{F}_{\mathrm{cof}} = \{A \subset \mathbf{N} : \mathbf{N} \setminus A \text{ is finite}\}$. Throughout this section let $\mathfrak{F}$ stand for an arbitrary filter on $\mathbf{N}$ that is free, i.e. it contains $\mathfrak{F}_{\mathrm{cof}}$. Define the following subspaces of $s = \mathbf{R}^{\mathbf{N}}$:

$$\begin{aligned} c_{\mathfrak{F}} &= \{x \in \mathbf{R}^{\mathbf{N}} : \lim_{\mathfrak{F}} x = 0\} \\ &= \{x \in \mathbf{R}^{\mathbf{N}} : \forall \varepsilon > 0 \, \exists F \in \mathfrak{F} \text{ with } |x_a| \leq \varepsilon \text{ for all } a \in F\} \end{aligned}$$

and for $n \in \mathbf{N}$,

$$X_n(\mathfrak{F}) = \{x \in \mathbf{R}^{\mathbf{N}} : \exists F \in \mathfrak{F} \text{ such that } |x_a| \leq 2^{-n} \text{ for all } a \in F\}.$$

Observe that $\mathcal{X} = (X_n)_n$ is a decreasing sequence of subsets of $\mathbf{R}^{\mathbf{N}}$ with the property that its intersection is $c_{\mathfrak{F}}$.

Proposition 3.1. *If $\mathfrak{F} \neq \mathfrak{F}_{\mathrm{cof}}$ and $c_{\mathfrak{F}}$ is absolute Borel then the system $\mathcal{X}(\mathfrak{F})$ is $\mathcal{F}_\sigma$-universal* (*and hence $c_{\mathfrak{F}}$ is $\mathcal{F}_{\sigma\delta}$-universal*) *in $\mathbf{R}^{\mathbf{N}}$.*

Proof: We shall use the following fact: if A is an $\mathcal{F}_\sigma$-absorber in Q and A' is a σZ-set then for every σ-compactum C in Q there is an imbedding $f : Q \to Q$ such that $f^{-1}(A) = C$ and $f(Q \setminus C) \cap A' = \emptyset$ (cf. [5, Proposition 6.1]).

Since $\mathfrak{F}$ is not the Fréchet filter we may choose an infinite set $N_0 \subset \mathbf{N}$ whose complement is in $\mathfrak{F}$. According to Lutzer and McCoy [10] there exists a partition $\{P_{ijk} : i, j, k \in \mathbf{N}\}$ of $\mathbf{N} \setminus N_0$ consisting of finite sets such that for every $F \in \mathfrak{F}$ there is a $j \in \mathbf{N}$ with

$$F \cap P_{ijk} \neq \emptyset \quad \text{for all } i \text{ and } k.$$

Put $N_i = \bigcup_{j,k=1}^{\infty} P_{ijk}$ and for every $i \in \mathbf{N}$ define the Hilbert cube $Q_i = [-2^{-i+1}, 2^{-i+1}]^{N_i}$. For $i, j, k \in \mathbf{N}$ let π_{ijk} be the projection from Q_i onto the finite-dimensional cell $Z_{ijk} = [-2^{-i+1}, 2^{-i+1}]^{P_{ijk}}$. It is easily verified with the capset characterization theorem in Curtis [3] that

$$C_i = \{x \in Q_i : \exists\, k \in \mathbf{N} \text{ such that } |x_a| \leq 2^{k-a} \text{ for all } a \in N_i\}$$

is an $\mathcal{F}_\sigma$-absorber in Q_i. Observe that for every $x \in C_i$ we have $\lim_{a\to\infty} x_a = 0$. Since P_{ijk} is finite the set

$$B_{ijk} = \{x \in Z_{ijk} : |x_a| \leq 2^{-i} \text{ for some } a \in P_{ijk}\}$$

is compact for every $i, j, k \in \mathbf{N}$. By infinite deficiency the compactum $\bigcap_{k=1}^{\infty} \pi_{ijk}^{-1}(B_{ijk})$ is a Z-set in Q_i and hence

$$D_i = \bigcup_{j=1}^{\infty} \bigcap_{k=1}^{\infty} \pi_{ijk}^{-1}(B_{ijk})$$

is a σZ-set.

Let $A_1 \supset A_2 \supset \cdots$ be a sequence of σ-compacta in Q. Let $f_0 : \mathbf{R}^{\mathbf{N}} \to \mathbf{R}^{N_0}$ be a homeomorphism and let $f_i : \mathbf{R}^{\mathbf{N}} \to Q_i$ $(i \in \mathbf{N})$ be an imbedding such that $f_i^{-1}(C_i) = A_i$ and $f_i(Q_i \setminus A_i)$ does not meet D_i. Consider the closed imbedding

$$f = (f_i)_{i=0}^{\infty} : \mathbf{R}^{\mathbf{N}} \to \mathbf{R}^{N_0} \times \prod_{i=1}^{\infty} Q_i \subset \mathbf{R}^{\mathbf{N}}.$$

Let $x \in A_n$. If $i > n$ then we have $f_i(x) \in Q_i$ and hence all components of $f_i(x)$ are in $[-2^{-n}, 2^{-n}]$. If $i \leq n$ then we have $x \in A_i$ and hence $f_i(x) \in C_i$. Note that only finitely many components of $f_i(x)$ are outside $[-2^{-n}, 2^{-n}]$ and hence $|f(x)_a| > 2^{-n}$ for only finitely many a in $\mathbf{N} \setminus N_0$. This means that $f(x)$ is an element of $X_n(\mathfrak{F})$. If $x \notin A_n$ then we have $f_n(x) \notin D_n$. If F is an arbitrary element of $\mathfrak{F}$ then there is a $j \in \mathbf{N}$ such that F meets P_{njk} for every $k \in \mathbf{N}$. Observe that if $f_n(x) \notin D_n$ then $f_n(x) \notin \pi_{njk}^{-1}(B_{njk})$ for some k. Consequently, we have $|f_n(x)_a| > 2^{-n}$ for all $a \in P_{njk}$. Since F and P_{njk} have at least one a in common we find that $f(x) \notin X_n(\mathfrak{F})$. So we may conclude that $f^{-1}(X_n(\mathfrak{F})) = A_n$. □

The following observation is essentially due to R. Cauty:

Lemma 3.2. *If $(L_\gamma)_\gamma$ is a system of linear subspaces of a Fréchet space E such that $\bigcap_\gamma L_\gamma$ is dense then we have:*

(a) *The system $(L_\gamma \times E)_\gamma$ is reflexively universal in $E \times E$.*
(b) *If E is the pseudointerior s then the system $(L_\gamma \times Q)_\gamma$ is reflexively universal in $Q \times Q$.*

Proof: We prove part (a); the proof for (b) is similar. Let $f = (f_1, f_2) : E \times E \to E \times E$ be a Z-imbedding and let K be a closed subset of $E \times E$. Select an F-norm $\|\cdot\|$ on E and let d be the metric on $E \times E$ that corresponds with the max norm. Let $\varepsilon : E \times E \to I$ be a map such that $\varepsilon^{-1}(0) = K$ and $\varepsilon(x) \leq d(f(x), f(K))/4$. Since $\bigcap_\gamma L_\gamma$ is a dense linear subspace its complement is locally homotopy negligible (see [2, Proposition VIII.3.2]) and we can find a map $\tilde{f}_1 : E \times E \to \bigcap_\gamma L_\gamma$ that is ε-close to f_1. Select now a continuous $\xi : E \times E \to I$ such that $\xi^{-1}(0) = K$ and $\|\xi(x.y)x\| \leq \varepsilon(x, y)$ for each $(x, y) \in E \times E$. Observe that the map $g_1 : E \times E \to E$ given by

$$g_1(x, y) = \tilde{f}_1(x, y) + \xi(x, y)x$$

is 2ε-close to f_1 and has the property $g_1^{-1}(L_\gamma) \setminus K = (L_\gamma \times E) \setminus K$. Select a map $g_2 : E \times E \to E$ such that g_2 and f_2 are ε-close, $g_2|K = f_2|K$, and $g_2|\varepsilon^{-1}([\delta, 1])$ is a Z-imbedding for each $\delta > 0$. Put $g = (g_1, g_2)$ and note that this map is a Z-imbedding according to Lemma 2.4. The map g is 2ε-close to f and it has the property $g^{-1}(L_\gamma \times E) \setminus K = (L_\gamma \times E) \setminus K$. □

Throughout the remainder of this section let X stand for an arbitrary nondiscrete, completely regular, countably infinite space.

Proposition 3.3. *If X is not compact then $C_{\mathrm{p}}(X)$ is reflexively universal in $\mathbf{R}^X$.*

Proof: This follows immediately from Lemma 3.2. Choose an infinite closed discrete subspace A of X. Then $C_{\mathrm{p}}(X)$ is canonically isomorphic in $\mathbf{R}^X$ to the product of $C_{\mathrm{p}}(A) = \mathbf{R}^A$ and $C_{\mathrm{p}}(X; A) = \{f \in C_{\mathrm{p}}(X) : f|A = 0\}$: if $r : X \to A$ is a retraction then

$$\alpha(f) = (f|A, f - (f|A) \circ r) \quad \text{for } f \in \mathbf{R}^X$$

defines a linear homeomorphism with the required property. □

A similar argument shows that if X is not compact then $C_{\mathrm{p}}(X)$ is also reflexively universal in $\widehat{\mathbf{R}}^X$. Since we already showed in [5] that $C_{\mathrm{p}}(X)$ is reflexively universal in $\widehat{\mathbf{R}}^X$ for every metric X we have that $C_{\mathrm{p}}(X)$ is reflexively universal in the Hilbert cube for every X.

Proposition 3.4. *If X is not compact and $C_{\mathrm{p}}(X) \in \mathcal{F}_{\sigma\delta}$ then $C_{\mathrm{p}}(X)$ is an $\mathcal{F}_{\sigma\delta}$-absorber in $\mathbf{R}^X$.*

Proof: We use the method of Dobrowolski, Marciszewski and Mogilski [7]. It is shown in that paper that $C_{\mathrm{p}}(X)$ if it is Borel is contained in a σZ-set. We have the following two cases:

I. The space X does not contain a clopen subset with precisely one accumulation point. Then X can be written as a topological sum $\bigoplus_{i=1}^{\infty} X_i$ of nondiscrete spaces and hence $C_{\mathrm{p}}(X) = \prod_{i=1}^{\infty} C_{\mathrm{p}}(X_i)$ ([7, Proposition 6.1]). According to the proof of [7, Lemma 5.4] the pair (s, σ) is imbeddable in each $(\mathbf{R}^{X_i}, C_{\mathrm{p}}(X_i))$. This means that $(s^{\mathbf{N}}, \sigma^{\mathbf{N}})$ is imbeddable in $(\mathbf{R}^X, C_{\mathrm{p}}(X))$ and hence $C_{\mathrm{p}}(X)$ is $\mathcal{F}_{\sigma\delta}$-universal in $\mathbf{R}^X$.

II. The space X has a clopen subset A with a unique accumulation point a. Since X is not compact we may select an infinite closed discrete subset C. Put $D = A \cup C$ and note that since A is clopen and C is closed and discrete, there is a retraction $r : X \to D$. The neighborhoods of a form a free filter $\mathfrak{F}$ on $\tilde{D} = D \setminus \{a\}$ that is not the Fréchet filter. If $f \in \mathbf{R}^{\tilde{D}}$ then let $\bar{f} : D \to \mathbf{R}$ be the extension of f with $f(a) = 0$. Then $\alpha(f) = \bar{f} \circ r$ defines a closed imbedding of $(\mathbf{R}^{\tilde{D}}, c_{\mathfrak{F}})$ into $(\mathbf{R}^X, C_{\mathrm{p}}(X))$. Since the first pair is $\mathcal{F}_{\sigma\delta}$-universal (Proposition 3.1), so is the second.

It follows from Proposition 3.3 that $C_{\mathrm{p}}(X)$ is strongly $\mathcal{F}_{\sigma\delta}$-universal (and hence $\mathcal{F}_{\sigma\delta}$-absorbing) in $\mathbf{R}^X$ for every non-compact X. Observe that we did not need the condition $C_{\mathrm{p}}(X) \in \mathcal{F}_{\sigma\delta}$ to show strong $\mathcal{F}_{\sigma\delta}$-universality, just that $C_{\mathrm{p}}(X)$ is Borel. □

This proposition implies that the pairs $(\mathbf{R}^X, C_{\mathrm{p}}(X))$ and $(s^{\mathbf{N}}, \sigma^{\mathbf{N}})$ are homeomorphic whenever X is not compact and $C_{\mathrm{p}}(X) \in \mathcal{F}_{\sigma\delta}$. This is one direction of Theorem 1.1.

The other direction is easily seen: if X is compact then $C_{\mathrm{p}}(X)$ is contained in the σ-compactum consisting of the bounded elements of $\mathbf{R}^X$. Therefore $C_{\mathrm{p}}(X)$ cannot contain a copy of Hilbert space that is closed in $\mathbf{R}^X$. On the other hand, $\sigma^{\mathbf{N}}$ contains a copy of s that is closed in $s^{\mathbf{N}}$.

If we combine Proposition 3.4 with Lemma 2.5 and the fact that $(\widehat{\mathbf{R}}^X, C_{\mathrm{p}}(X))$ was shown to be $\mathcal{F}_{\sigma\delta}$-absorbing for metric X in [5] we find:

Proposition 3.5. *If $C_{\mathrm{p}}(X) \in \mathcal{F}_{\sigma\delta}$ then it is an $\mathcal{F}_{\sigma\delta}$-absorber in $\widehat{\mathbf{R}}^X$.*

This result was found independently by Baars, Gladdines and van Mill [1]. Combining Proposition 3.5 and Theorem 2.3 we find half of Theorem 1.3.

We now turn to the case of compact X.

Proposition 3.6. *The space c_0 is an $\mathcal{F}_{\sigma\delta}$-absorber rel s in Q.*

Proof: According to [5, Theorem 6.3] c_0 is an $\mathcal{F}_{\sigma\delta}$-absorber in Q so it suffices to show that c_0 is reflexively universal rel s in Q. We use Lemma 2.2 (b): if

$\Phi : Q \to Q^{\mathbf{N}} = \widehat{\mathbf{R}}^{\mathbf{N}\times\mathbf{N}}$ is a map that simply rearranges coordinates then it obviously satisfies the conditions of part (b) of the lemma with $* = 0$. Also c_0 contains σ so it has a locally homotopy negligible complement.

We now define the imbedding α of (Q, c_0) into (s, c_0). Let $\pi : \widehat{\mathbf{R}} \to [-1, 1]$ be a homeomorphism with $\pi(0) = 0$. If we define for every $x \in Q$ and $n \in \mathbf{N}$,

$$\alpha(x)_{2n-1} = \pi(x_n),$$
$$\alpha(x)_{2n} = 2^{-n} \min\left\{2^n, \max_{i=1,\dots,n} |x_i|\right\},$$

then α is obviously an imbedding of Q into $[-1, 1]^{\mathbf{N}}$.

First, let $x \notin c_0$. If $x \in s$ then $\lim_{n\to\infty} \alpha(x)_{2n-1} = \lim_{n\to\infty} \pi(x_n) \neq 0$ and hence $\alpha(x) \notin c_0$. If, on the other hand, $x_i = \pm\infty$ for some i then $\alpha(x)_{2n} = 1$ for every $n \geq i$ and also $\alpha(x) \notin c_0$.

Now, let $x \in c_0$ and note $\lim_{n\to\infty} \alpha(x)_{2n-1} = \pi(\lim_{n\to\infty} x_n) = 0$. Define the finite number $M = \max_{i\in\mathbf{N}} x_i$ and observe that $0 \leq \alpha(x)_{2n} \leq M2^{-n}$ for every n. Consequently, $\lim_{n\to\infty} \alpha(x)_n = 0$ and $\alpha(x) \in c_0$. So we may conclude that $\alpha^{-1}(c_0) = c_0$. All the conditions of Lemma 2.2 (b) are now satisfied and the proposition is proved. □

The following result follows from Lemma 2.6 and Proposition 3.6. Its proof is identical to the proof of [5, Theorem 6.5]. (Note that a compact X is metrizable and hence $C_{\mathrm{p}}(X) \in \mathcal{F}_{\sigma\delta}$.)

Proposition 3.7. *If X is compact then $C_{\mathrm{p}}(X)$ is an $\mathcal{F}_{\sigma\delta}$-absorber rel $\mathbf{R}^X$ in $\widehat{\mathbf{R}}^X$.*

Applying Theorem 2.3 (b) we find:

Theorem 3.8. *If X is compact then $(\widehat{\mathbf{R}}^X, \mathbf{R}^X, C_{\mathrm{p}}(X))$ is homeomorphic to (Q, s, c_0).*

This proves the $C_{\mathrm{p}}(X)$ part of Theorem 1.2.

4. Sequence spaces

We prove the l_p part of Theorem 1.2 and Theorem 1.3.

Let p be an arbitrary positive real number and define the following function from Q into $[0, \infty]$:

$$|x|_p = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p}.$$

The subspace l_p consists of all x in Q (or s) with $|x|_p < \infty$. Since the expression $|x|_p$ is nonincreasing as a function of p we have $l_p \subset l_q$ whenever $p < q$. So we have an ordered system with index set $(0,\infty)$. Our objective is to show that the spaces

$$\tilde{l}_p = \bigcap_{q>p} l_q \qquad p \in [0,\infty)$$

are $\mathcal{F}_{\sigma\delta}$-absorbers in Q. Since these spaces are contained in the σ-compactum $\Sigma \subset s$ they cannot be $\mathcal{F}_{\sigma\delta}$-absorbers in s. For this reason we shall use the Hilbert cube as ambient space rather than s (cf. the case with compact X in Section 3).

We need some definitions. If A is a countable infinite set then we define the following subspaces of the Hilbert cube $\widehat{\mathbf{R}}^A$: the capset

$$\Sigma'(A) = \{x \in \widehat{\mathbf{R}}^A : \exists\, M \in \mathbf{N} \text{ such that } |x_a| < M \text{ for all but finitely many } a \in A\}$$

and the fd-capset

$$\sigma'(A) = \{x \in \widehat{\mathbf{R}}^A :\ x_a = 0 \text{ for all but finitely many } a \in A\}.$$

In the standard model Q we put $\Sigma' = \Sigma(\mathbf{N})$ and $\sigma' = \sigma(\mathbf{N})$. The sets Σ' and σ' are of course topologically equivalent in Q to Σ respectively σ. Unlike Σ and σ the have the following property: if $x, y \in Q$ differ at only finitely many coordinates then we have $x \in \Sigma'$ (or σ') if and only if $y \in \Sigma'$ (or σ'). This makes Σ' and σ' a superior choice *when the ambient space is a Hilbert cube.*

It is well known that l_p is a capset, i.e. the pair (Q, l_p) is homeomorphic to the pairs (Q, Σ'), $(Q \times Q, Q \times \Sigma')$, and $(Q \times Q, Q \times \sigma')$. The idea is to establish a connection between the system l_p and systems that find their origin in the topological product structure of the Hilbert cube. This leads to the following definitions. If A is a countable dense subset of the interval $(0,\infty)$ and p is a positive real number then

$$Z_p = Z_p(A) = \widehat{\mathbf{R}}^{(0,p]\cap A} \times \Sigma'((p,\infty) \cap A) \subset \widehat{\mathbf{R}}^A$$

and

$$\zeta_p = \zeta_p(A) = \widehat{\mathbf{R}}^{(0,p]\cap A} \times \sigma'((p,\infty) \cap A) \subset \widehat{\mathbf{R}}^A.$$

Both $(Z_p)_p$ and $(\zeta_p)_p$ are ordered systems of capsets. Our objective is to show that the systems l_p, ζ_p, and Z_p are in essence topologically indistinguishable, a result that has the claims made in the introduction as immediate corollaries.

Throughout this section let A be a countable dense subset of $(0,\infty)$. Let $a_1, a_2, \ldots$ enumerate A and let the product topology on $\mathbf{R}^A$ be generated by the metric

$$d(x,y) = \max_{n \in \mathbf{N}} \frac{1}{2n} |\xi(x_{a_n}) - \xi(y_{a_n})|,$$

where $\xi : \widehat{\mathbf{R}} \to [-1,1]$ is a fixed homeomorphism with the property $\xi(0) = 0$. Note that if $x, y \in \widehat{\mathbf{R}}^A$ have their first n coordinates in common then their distance is at most $1/(n+1)$.

The following statement is obvious.

Lemma 4.1. *The collections $(Z_p)_p$, $(\zeta_p)_p$ and $(l_p)_p$ are Z-systems in $\widehat{\mathbf{R}}^A$ respectively Q.*

Lemma 4.2. *The systems Z_p and ζ_p are reflexively universal.*

PROOF: This proof is similar to the proof of Lemma 3.2. Let $f : \widehat{\mathbf{R}}^A \to \widehat{\mathbf{R}}^A$ be a map that restricts to a Z-imbedding on a closed set K. We may assume that f itself is a Z-imbedding. Let $\varepsilon : \widehat{\mathbf{R}}^A \to I$ be a map such that $\varepsilon(x) \le d(f(x), f(K))/2$ for $x \in \widehat{\mathbf{R}}^A$ and $\varepsilon^{-1}(0) = K$. Let A_2 be a sequence in A that converges to 0 and put $A_1 = A \setminus A_2$. Let $\pi_i : \widehat{\mathbf{R}}^A \to \widehat{\mathbf{R}}^{A_i}$ stand for the projection an put $f_i = \pi_i \circ f$. Select a map $g_2 : \widehat{\mathbf{R}}^A \to \widehat{\mathbf{R}}^{A_2}$ that is ε-close to f_2 and with the property that $g_2|\varepsilon^{-1}([\delta, 1])$ is a Z-imbedding for each $\delta > 0$. Select also a map $\tilde{f}_1 : \widehat{\mathbf{R}}^A \to \widehat{\mathbf{R}}^{a_1}$ that is $(\varepsilon/2)$-close to f_1 and that maps the complement of K into the fd-capset $\sigma'(A_1)$. Define for every $n \in \mathbf{N}$ the continuous map $\chi_n : I \to I$ by

$$\chi_n(r) = \min\{1, \max\{0, rn - 1\}\}.$$

Observe that $\chi_n(0) = 0$ and that

$$\chi_n(r) = \begin{cases} 0, & \text{if } rn \le 1 \\ 1, & \text{if } rn \ge 2. \end{cases}$$

We now define the map $g_1 : \widehat{\mathbf{R}}^A \to \widehat{\mathbf{R}}^{A_1}$ by

$$g_1(x)_{a_n} = \tilde{f}_1(x)_{a_n} + \xi^{-1}\big(\chi_n(\varepsilon(x)/2)\xi(x_{a_n})\big)$$

for $x \in \widehat{\mathbf{R}}^A$ and $a_n \in A_1$, where we used the fact that addition is well defined and continuous from $\widehat{\mathbf{R}} \times \mathbf{R}$ to $\widehat{\mathbf{R}}$. Put $g = (g_1, g_2) : \widehat{\mathbf{R}}^A \to \widehat{\mathbf{R}}^A$.

Let $x \in \widehat{\mathbf{R}}^A$. If $x \in K$ then we have $\varepsilon(x) = 0$ and hence $\chi_n(\varepsilon(x)/2) = 0$. This means that $g_1(x) = \tilde{f}_1(x) = f_1(x)$ and $g(x) = f(x)$. If $x \notin K$ then $\varepsilon(x) > 0$ and we can select an $n \in \mathbf{N}$ such that $n\varepsilon(x)/2 \le 1 < (n+1)\varepsilon(x)/2$. The properties of χ guarantee that $g_{a_i}(x) = f_{a_i}(x)$ for each $i \le n$ with $a_i \in A_1$. This means that the distance between $g_1(x)$ and $f_1(x)$ is at most $1/(n+1) < \varepsilon(x)/2$. Consequently, g and f are ε-close. Observe that $g|(\varepsilon^{-1}([\delta, 1]))$ is a Z-imbedding for each $\delta > 0$ since g_2 has that property and hence Lemma 2.4 guarantees that g is a Z-imbedding.

Consider now an $x \notin K$. Choose an n with the properties $n\varepsilon(x) \ge 4$ and $\tilde{f}_1(x)_{a_i} = 0$ for all $i \ge n$ with $a_i \in A_1$ (recall that $\tilde{f}_1(x) \in \sigma(A_1)$). Then $\chi_i(\varepsilon(x)/2) = 1$ and $g_1(x)_{a_i} = 0 + \xi^{-1}(\xi(x_{a_i})) = x_{a_i}$ for all $i \ge n$ with $a_i \in a_1$. So $g(x)_a = x_a$ for all coordinates in A except possibly those in $C = \{a_i : i < n\} \cup A_2$. Since C is a sequence that converges to 0 we have for every $p \in (0, \infty)$ that $(g(x)_a)_{a > p}$ differs at only finitely many coordinates from $(x_a)_{a > p}$ and hence that $g(x) \in Z_p$ (or ζ_p) if and only if $x \in Z_p$ (or ζ_p). ☐

Lemma 4.3. *The system l_p is reflexively universal rel s in Q.*

Proof: This is virtually identical to the proof of Proposition 3.6. The only addition is that the homeomorphism $\pi : \widehat{\mathbf{R}} \to [-1,1]$ should satisfy the condition $\pi(x) = x$ for $|x| \leq \frac{1}{2}$. This guarantees that for every $x \in s$, $\sum_{i=1}^{\infty} |x_i|^p < \infty$ if and only if $\sum_{i=1}^{\infty} |\pi(x_i)|^p < \infty$. □

Proposition 4.4. *The system l_p is imbeddable in Z_p.*

Proof: Write A as a disjoint union of A_0 and A_1, where A_0 is a sequence that converges to 0. Let $a_1, a_2, \ldots$ enumerate A_1. Select an imbedding $\alpha_0 : Q \to \mathbf{R}^{A_0}$. We define $\alpha_1 : Q \to \widehat{\mathbf{R}}^{A_1}$ by

$$\alpha_1(x)_{a_n} = \left(\sum_{i=1}^{n} |x_i|^{a_n} \right)^{1/a_n} \qquad \text{for } x \in Q \text{ and } n \in \mathbf{N}.$$

Note that $0 \leq \alpha_1(x)_{a_n} \leq |x|_{a_n}$. Put $\alpha = (\alpha_0, \alpha_1) : Q \to \widehat{\mathbf{R}}^{A_0} \times \widehat{\mathbf{R}}^{A_1} = \widehat{\mathbf{R}}^{A}$ and observe that α is an imbedding. If $x \in l_q$ and $a \in (q, \infty) \cap A_1$ then we have $\alpha_1(x)_a \leq |x|_a \leq |x|_q$ so $\alpha_1(x)_{a>q}$ is bounded by $|x|_q$. Since $(q,p) \cap A_0$ is finite we may conclude that $\alpha(x)_{a>q}$ is bounded and that $\alpha(x) \in Z_q$. On the other hand if $x \notin l_q$ then we have $|x|_q = \infty$. Let $M \in \mathbf{N}$ be arbitrary. There exists an $n \in \mathbf{N}$ such that $\left(\sum_{i=1}^{n} |x_i|^q\right)^{1/q} > M$. By continuity in q we can find an $\varepsilon > 0$ such that $\left(\sum_{i=1}^{n} |x_i|^r\right)^{1/r} > M$ for each $r \in (q, q+\varepsilon)$. Since A_1 is dense there is an $m > n$ with $a_m \in (q, q+\varepsilon)$. So we have $\alpha_1(x)_{a_m} > M$ and we may conclude that $\alpha_1(x)_{a>q}$ is unbounded and that $\alpha(x) \notin Z_q$. □

Proposition 4.5. *If Δ is a countable dense subset of $(0, \infty)$ then the system Z_p is Δ-imbeddable in ζ_p.*

Proof: We shall use the known fact that there exists a map $v : Q \to Q$ such that $v^{-1}(\sigma') = \Sigma'$. This can easily be seen as follows. The product of Q and the fd-capset σ' is a capset in $Q \times Q$. Since capsets are topologically unique there is a homeomorphism $h : Q \to Q \times Q$ with $h(\Sigma') = Q \times \sigma'$. If we combine h with the projection onto the second coordinate then we have v.

Let $b_0 = 0$ and enumerate $\Delta = \{b_n : n \in \mathbf{N}\}$. Select by induction for every $n \geq 0$ a sequence $A_n \subset A \cap (b_n, c_n)$ that converges to b_n, where $c_n \in (b_n, \infty]$ is the minimum of the compact set

$$(b_n, \infty) \cap \left(\{b_i : i < n\} \cup \bigcup_{i=0}^{n-1} A_i \right).$$

Note that the A_n's are pairwise disjoint. Put $A' = A \setminus \bigcup_{i=0}^{\infty} A_i$. Let $\alpha_0 : \widehat{\mathbf{R}}^{A} \to \mathbf{R}^{A_0}$ be an imbedding and for $n \in \mathbf{N}$ let $\alpha_n : \widehat{\mathbf{R}}^{A \cap (b_n, \infty)} \to \widehat{\mathbf{R}}^{A_n}$ be a map like v above, i.e.

$$\alpha_n^{-1}(\sigma'(A_n)) = \Sigma'(A \cap (b_n, \infty)).$$

We obviously may assume that $\alpha_n(0) = 0$. Define the closed imbedding $\alpha : \widehat{\mathbf{R}}^A \to \widehat{\mathbf{R}}^A$ by

$$\begin{aligned}\alpha(x)_{a\in A_0} &= \alpha_0(x),\\ \alpha(x)_{a\in A_n} &= \alpha_n\big((\max\{0,|x_{a'}|-n\})_{a'>b_n}\big) \quad \text{for} \quad n\in\mathbf{N},\\ \alpha(x)_{a\in A'} &= 0.\end{aligned}$$

If $x \notin Z_{b_n}$ then we have that $(\max\{0,|x_a|-n\})_{a>b_n}$ is still outside of $\Sigma'(A\cap(b_n,p))$. Consequently, $\alpha(x)_{a\in A_n} \notin \sigma'(A_n)$ and $\alpha(x) \notin \zeta^p_{b_n}$. Let $x \in Z^p_{b_n}$ and let m be such that $|x_a| \le m$ for all $a > b_n$ that are outside of some finite set C. Let i be such that $b_i < b_n$. If $i > n$ then A_i and (b_n,∞) are disjoint and if $i < n$ then $A_i \cap (b_n,\infty)$ is finite. Consequently, we have that $(b_n,\infty)\cap\bigcup\{A_i : b_i < b_n\}$ is finite and hence these coordinates are irrelevant to the question whether $\alpha(x)$ is an element of ζ_{b_n} or not. Let i be such that $b_i \ge b_n$. If $i \ge m$ then $\max\{0,|x_a - i|\} = 0$ for $a \in (b_i,\infty)\setminus C$ and hence we have $\alpha(x)_a = 0$ for every $a \in A_i \setminus C$. If $i < m$ then

$$(\max\{0,|x_a|-i\})_{a>b_i} \in \Sigma'(A\cap(b_i,\infty))$$

and hence $\alpha(x)_{a\in A_i}$ is an element of $\sigma'(A_i)$. So we may conclude that $\alpha(x) \in \zeta^p_{b_n}$. □

Proposition 4.6. *If Δ is a countable subset of $(0,\infty)$ then the system ζ_p is Δ-imbeddable in l_p.*

Proof: For technical reasons we shall imbed $\widehat{\mathbf{R}}^A$ into $Q^{\mathbf{N}}$ rather than Q. Enumerate $A = \{a_n : n \ge 2\}$ and $\Delta = \{b_n : n \ge 2\}$. Select for every $n \ge 2$ a δ_n between 0 and a_n such that $[a_n - \delta_n, a_n)$ and $\{b_i : i \le n\}$ are disjoint. Define the continuous map $\chi : [1,\infty) \to I^{\mathbf{N}}$ by

$$\chi(t)_k = t^{-1}\min\{1,\max\{0,t+1-k\}\} \quad \text{for} \quad t\in[1,\infty) \quad \text{and} \quad k\in\mathbf{N}.$$

This map has the following properties: $|\chi(t)|_1 = 1$ and

$$\chi(t)_k = \begin{cases} t^{-1} & \text{for } k \le t\\ 0 & \text{for } k \ge t+1.\end{cases}$$

Put $\chi_q(t)_k = (\chi(t)_k)^{1/q}$ and note that $|\chi_q(t)|_q = 1$. We now define a sequence $(\alpha_n)_{n\in\mathbf{N}}$ of maps from $\widehat{\mathbf{R}}^A$ into Q. Let α_1 be an imbedding of $\widehat{\mathbf{R}}^A$ into $\prod_{i=1}^\infty[0,2^{-i}] \subset Q$ and note that the image of α_1 is contained in $\tilde{l}_0$. If $n \ge 2$ and $x \in \widehat{\mathbf{R}}^A$ then put $\varepsilon_n = \min\{2^{-n+1},|x_{a_n}|\}$. The function $\alpha_n : \widehat{\mathbf{R}}^A \to Q$ is defined by

$$\alpha_n(x) = \begin{cases} \varepsilon_n\chi_{a_n}\left(\varepsilon_n^{-na_n/\delta_n}\right) & \text{for } \varepsilon_n > 0\\ 0 & \text{for } \varepsilon_n = 0.\end{cases}$$

Since $|\alpha_n(x)|_p \leq |\alpha_n(x)|_{a_n} = \varepsilon_n$ we have that α_n is continuous. Noting that $|\alpha_n(x)|_p \leq 2^{-n+1}$ for each $n \in \mathbf{N}$ we may conclude that the sequence $\alpha = (\alpha_n)_{n\in\mathbf{N}}$ forms a continuous map of $\widehat{\mathbf{R}}^A$ into $Q^{\mathbf{N}}$. This function is an imbedding because its first component α_1 is an imbedding.

Assume that x is an element of ζ_q. This means that only finitely many components x_{a_n} with $a_n > q$ are nonzero. We have the following estimate for the q-norm of $\alpha(x)$:

$$\begin{aligned}
\|\alpha(x)\|_q{}^q &= \sum_{n=1}^{\infty} |\alpha_n(x)|_q{}^q \\
&= |\alpha_1(x)|_q{}^q + \sum_{\substack{n=2\\ a_n\leq q}}^{\infty} |\alpha_n(x)|_q{}^q + \sum_{\substack{n=2\\ a_n> q}}^{\infty} |\alpha_n(x)|_q{}^q \\
&\leq |\alpha_1(x)|_q{}^q + \sum_{\substack{n=2\\ a_n\leq q}}^{\infty} |\alpha_n(x)|_{a_n}{}^q + \sum_{\substack{n=2\\ a_n> q\\ x_{a_n}\neq 0}}^{\infty} |\alpha_n(x)|_q{}^q.
\end{aligned}$$

This expression is finite because $|\alpha_1(x)|_q$ is finite, because $|\alpha_n(x)|_{a_n} = \varepsilon_n \leq 2^{-n+1}$ and because the last sum consists of only finitely many terms.

If x is not an element of ζ_q then there exist infinitely many $a_n > q$ such that $x_{a_n} \neq 0$. If moreover $q \in \Delta$ then all but finitely many of those a_n's have the property $a_n - \delta_n > q$. Let a_n be such a coordinate of $\widehat{\mathbf{R}}^A$ and put $t = \varepsilon_n{}^{-na_n/\delta_n}$. Since at least $t-1$ terms of $\chi_{a_n}(t)$ are equal to t^{-1/a_n} we have that

$$\begin{aligned}
|\alpha_n(x)|_q{}^q &\geq \varepsilon_n{}^q(t-1)t^{-q/a_n} \\
&\geq \frac{1}{2}\varepsilon_n{}^q t^{(a_n-q)/a_n} \\
&\geq \frac{1}{2}\varepsilon_n{}^q t^{\delta_n/a_n},
\end{aligned}$$

where we used $t \geq 2$ and $q < a_n - \delta_n$. Substituting the value for t we find $|\alpha_n(x)|_q{}^q \geq \frac{1}{2}\varepsilon_n{}^{q-n} \geq 1$ for all but finitely many a_n's. This means that infinitely many of the terms of the series $\|\alpha(x)\|_q{}^q = \sum_{n=1}^{\infty} |\alpha_n(x)|_q{}^q$ are at least 1 and hence that $\|\alpha(x)\|$ is infinite. □

If we apply Theorem 2.1 to Lemma 4.1, Lemma 4.2, Lemma 4.3, Proposition 4.4, Proposition 4.5 and Proposition 4.6 then we obtain:

Theorem 4.7. *If Δ is a countable dense subset of $(0,\infty)$ then the systems l_p, ζ_p and Z_p are Δ-homeomorphic, i.e. there exist homeomorphisms $\alpha, \beta : Q \to \widehat{\mathbf{R}}^A$ such that $\alpha(l_p) = Z_p$ and $\beta(l_p) = \zeta_p$ for every $p \in \Delta$.*

If $p \in [0,\infty)$ then we define the spaces

$$\tilde{Z}_p = \bigcap_{p<q} Z_q \subset \widehat{\mathbf{R}}^A$$
$$\tilde{\zeta}_p = \bigcap_{p<q} \zeta_q \subset \widehat{\mathbf{R}}^A.$$

If Δ is dense then we have $\tilde{l}_p = \bigcap\{l_q : q \in \Delta \text{ with } p < q\}$, which observation produces:

Corollary 4.8. *The systems $\tilde{l}_p$, $\tilde{\zeta}_p$ and $\tilde{Z}_p$ are homeomorphic.*

Observe that if $\infty = a_0, a_1, a_2, \ldots$ is a decreasing sequence in $[0,\infty]$ that converges to p then we have:

$$\tilde{Z}_p = \widehat{\mathbf{R}}^{(0,p]\cap A} \times \prod_{i=0}^{\infty} \Sigma'([a_{i+1}, a_i) \cap A)$$

and

$$\tilde{\zeta}_p = \widehat{\mathbf{R}}^{(0,p]\cap A} \times \prod_{i=0}^{\infty} \sigma'([a_{i+1}, a_i) \cap A).$$

This leads to:

Corollary 4.9. *The pair $(Q, \tilde{l}_p)$ is homeomorphic to $(Q^{\mathbf{N}}, \Sigma'^{\mathbf{N}})$ and to $(Q^{\mathbf{N}}, \sigma'^{\mathbf{N}})$ and hence also to $(Q^{\mathbf{N}}, \Sigma^{\mathbf{N}})$ and $(Q^{\mathbf{N}}, \sigma^{\mathbf{N}})$.*

This corollary proves the second part of Theorem 1.3 and it means that $\tilde{l}_p$ is just like $\sigma^{\mathbf{N}}$ an $\mathcal{F}_{\sigma\delta}$-absorber in the Hilbert cube, which combines with Lemma 4.3 to:

Theorem 4.10. *The space $\tilde{l}_p$ is an $\mathcal{F}_{\sigma\delta}$-absorber rel s in Q and hence the triple $(Q, s, \tilde{l}_p)$ is homeomorphic to (Q, s, c_0).*

The proof of Theorem 1.2 is now complete.

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