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*Commentationes Mathematicae Universitatis Carolinae*, Vol. 37 (1996), No. 2, 415--418

Persistent URL: <http://dml.cz/dmlcz/118846>

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## A counter-example to some recent existence results on implicit variational inequalities

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*Abstract.* In this note we prove that some recent results on an implicit variational inequality problem for multivalued mappings, which seem to extend and improve some well-known and celebrated results, are not correct.

*Keywords:* quasi-variational inequalities, lower semicontinuity, partition of unity, mini-max

*Classification:* 49J40

### 1. Introduction

Very recently, in [1], J. Fu introduced the following implicit variational inequality problem for multivalued mappings; given two topological vector spaces  $X$  and  $Y$ , two nonempty subsets  $C$  and  $D$  of  $X$  and  $Y$ , respectively, two multivalued mappings  $E : C \rightarrow 2^C$  and  $F : C \rightarrow 2^D$ , two real functions  $f : C \times C \times D \rightarrow \mathbf{R}$  and  $g : C \times C \rightarrow \mathbf{R}$ , with  $f(x, x, y) \geq 0$  for any  $x \in C$  and any  $y \in F(x)$ , find  $(v, u) \in C \times D$  such that

$$(1) \quad v \in E(v), \quad u \in F(v) \quad \text{and} \quad g(v, v) \leq f(v, w, u) + g(v, w) \quad \text{for all} \quad w \in E(v).$$

Such problem extends an implicit variational problem studied by Mosco [2].

In [1], Fu stated the following assertion which he employed to obtain existence results for problem (1) and for some special cases as quasi-variational inequalities (for the basic definitions, we refer to [1]).

**Assertion A** (Theorem 1 of [1]). *Let  $X, Y$  be Hausdorff locally convex topological vector spaces,  $C$  be a nonempty compact convex set of  $X$  and  $D$  be a nonempty closed convex set of  $Y$ . Let  $E : C \rightarrow 2^C$  be upper hemicontinuous with nonempty closed convex values and  $F : C \rightarrow 2^D$  be a mapping with nonempty values. Suppose that  $f : C \times C \times D \rightarrow \mathbf{R}$  satisfies the following conditions:*

- (i) *for each  $x \in C$  and  $y \in F(x)$ ,  $f(x, x, y) \geq 0$ ;*
- (ii) *for any fixed  $x \in C$  and  $y \in D$ , the function  $f(x, u, y)$  of  $u$  is convex;*
- (iii) *for any fixed  $u \in C$ , the function  $\sup_{y \in F(x)} f(x, u, y)$  of  $x$  is upper semi-continuous.*

Then there exists  $x^* \in C$  such that

$$x^* \in E(x^*) \text{ and } \sup_{y \in F(x^*)} f(x^*, u, y) \geq 0 \text{ for all } u \in E(x^*).$$

The aim of this note is to point out that Assertion A, in general, is false, together with several of its consequences obtained in [1]. We shall do this by means of a simple counter-example. We shall also illustrate in detail the gap in the original proof of Assertion A.

**2. The counter-example**

The following example shows that Assertion A, in general, is false.

**Example 2.1.** Let  $X = Y = D = \mathbf{R}$ ,  $C = [0, 1]$ ,  $F(x) \equiv \{1\}$ ,

$$E(x) = \begin{cases} \left[ \frac{3}{4}, 1 \right] & \text{if } x \in \left[ 0, \frac{1}{2} \right[ \\ \left[ 0, 1 \right] & \text{if } x = \frac{1}{2} \\ \left[ 0, \frac{1}{4} \right] & \text{if } x \in \left] \frac{1}{2}, 1 \right], \end{cases}$$

$f(x, u, y) = y(u - x)$ . It is immediate to realize that all the assumptions of Assertion A are satisfied. In particular, we note that the graph of  $E$  is closed, hence, since  $C$  is compact,  $E$  is upper semicontinuous. Therefore, by Lemma 1 of [3],  $E$  is upper hemicontinuous. We note that the only fixed point of  $E$  is  $x^* = \frac{1}{2}$ . However, we have  $\sup_{y \in F(x^*)} f(x^*, u, y) = u - x^* < 0$  for all  $u \in [0, \frac{1}{2}[ \subseteq E(x^*)$ . Thus, Assertion A fails.

**Remarks.** (i) Example 2.1 also shows that Theorems 2, 3 and 9 of [1] are false.

(ii) We note that Theorem 2 of [1], if correct, would imply, in particular (taking into account Theorem 1.4.16 of [4]), that the celebrated Chan and Pang’s existence theorem for generalized quasi-variational inequalities (see Theorem A in [5]) would be true without assuming the lower semicontinuity of the multifunction  $E : C \rightarrow 2^C$ . Example 2.1 shows that such improvement of Chan and Pang’s result is not possible.

We note that the original proof of Assertion A (proof of Theorem 1 in [1]) is arranged as follows.

*First step.* By assuming that the conclusion is false, it is shown that there exists a finite set  $\{p_1, \dots, p_n\} \subseteq X^*$  (where  $X^*$  denotes the topological dual space of  $X$ ) such that

$$C = V_0 \cup \bigcup_{i=1}^n V(p_i),$$

where we put

$$V(p_i) = \left\{ x \in C : \operatorname{Re} \langle p_i, x \rangle - \sup_{z \in E(x)} \operatorname{Re} \langle p_i, z \rangle > 0 \right\},$$

and

$$V_0 = \bigcup_{u \in A} \left\{ x \in C : \sup_{y \in F(x)} f(x, u, y) < 0 \right\},$$

with

$$A = \left\{ u \in C : \exists x \in C \text{ such that } u \in E(x) \text{ and } \sup_{y \in F(x)} f(x, u, y) < 0 \right\},$$

and  $\langle \cdot, \cdot \rangle$  is the usual pairing between  $X^*$  and  $X$ .

*Second step.* The author considers a continuous partition of unity  $\{\beta_0, \beta_1, \dots, \beta_n\}$  subordinated to the open covering  $\{V_0, V(p_1), \dots, V(p_n)\}$  of  $C$ .

*Third step.* The author defines a function  $\varphi : C \times C \rightarrow \mathbf{R}$  by setting

$$\varphi(x, u) = -\beta_0(x) \sup_{y \in F(x)} f(x, u, y) + \sum_{i=1}^n \beta_i(x) \operatorname{Re} \langle p_i, x - u \rangle$$

and observes that by the Ky Fan minimax principle (Theorem A of [3]) there exists  $\hat{x} \in C$  such that

$$(2) \quad \varphi(\hat{x}, u) \leq 0 \text{ for all } u \in C.$$

*Fourth step.* The author claims that inequality (2) is a contradiction. In particular, he claims that if  $\beta_0(\hat{x}) > 0$ , then there exists  $u_0 \in A \subseteq C$  such that  $\varphi(\hat{x}, u_0) > 0$ .

But the gap is exactly here, since the inequality (2) does not imply, in general, any contradiction. To see this, take  $X, Y, D, C, F, f$ , and  $E$ , as in Example 2.1. The reader can easily check that in this case we have  $A = [0, \frac{1}{2}[$ ,  $V_0 = ]0, 1]$ . Since  $V(-1) = [0, \frac{1}{2}[$ , we can take  $n = 1$ ,  $p_1 = -1$ ,

$$\beta_0(x) = \begin{cases} 2x & \text{if } x \in [0, \frac{1}{2}[, \\ 1 & \text{if } x \in [\frac{1}{2}, 1], \end{cases}$$

$$\beta_1(x) = \begin{cases} 1 - 2x & \text{if } x \in [0, \frac{1}{2}[, \\ 0 & \text{if } x \in [\frac{1}{2}, 1]. \end{cases}$$

According to Ky Fan minimax principle, there exists  $\hat{x} \in [0, 1]$  such that

$$\varphi(\hat{x}, u) = (\beta_0(\hat{x}) - \beta_1(\hat{x}))(\hat{x} - u) \leq 0 \text{ for all } u \in [0, 1].$$

In fact, one can take  $\hat{x} = \frac{1}{4}$  since  $\beta_0(\frac{1}{4}) = \beta_1(\frac{1}{4}) = \frac{1}{2}$ , hence  $\varphi(\frac{1}{4}, u) = 0$  for all  $u \in [0, 1]$ . Thus, the contradiction claimed in the final part of the original proof of Assertion A does not hold.

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(Received June 8, 1995)