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Concerning weak*-extreme points

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Abstract. Every separable nonreflexive Banach space admits an equivalent norm such that the set of the weak*-extreme points of the unit ball is discrete.

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Let K be a closed, convex and bounded subset of a Banach space X . A point x of K is called weak*-extreme if it is an extreme point of K^{**} , the weak*-closure of K in X^{**} .

James' theorem implies that the sets $\text{ext } K$ and $\text{ext } K^{**}$ of extreme points of, respectively, K and K^{**} , coincide if and only if the set K is weakly compact.

Godun [G] shows that the existence of an equivalent norm for which $\text{ext } B$ and $w^*\text{-ext } B$ do not coincide characterizes nonreflexive Banach spaces, where $w^*\text{-ext } B$ denotes the set of weak*-extreme points of the unit ball B .

A theorem of Stegall [S] says that a Banach space X which fails the Radon-Nykodým property admits an equivalent norm so that the set $w^*\text{-ext } B$ is empty. Moreover, the distance from X to the set $\text{ext } B^{**}$ is positive.

On the other hand, Phelps [P] shows that if X has the Radon-Nykodým property then every closed, convex and bounded subset of X is the closed convex hull of its strongly exposed points (and such points are weak*-extreme).

The question arises of "how small" the set $w^*\text{-ext } B$ can be. A well-known result of Lindenstrauss and Phelps shows that in an infinite dimensional reflexive Banach space X the set $\text{ext } B$ of extreme points of the unit ball must be uncountable. In particular, if X is separable, the set $\text{ext } B$ cannot be isolated in the norm topology.

Godun, Lin, and Troyanski [GLT] show that if X is separable and nonreflexive, then it admits an equivalent norm such that $w^*\text{-ext } B$ is at most countable. We observe in this note that X can even be renormed so that $w^*\text{-ext } B$ is norm-isolated. We do not know whether or not such an equivalent norm exists for any nonseparable and nonreflexive Banach space. Observe that in [LP] there is an example of a nonseparable reflexive Banach space such that $\text{ext } B$ is norm-isolated.

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In the following we denote by $\text{sp } A$ (respectively, $\text{co } A$) the span (respectively, the convex hull) of a set $A \subset X$.

The weak*-extreme points can be characterized as follows (see, for example, [R] and [GLT]):

Lemma 1. *Let X be a Banach space, K a closed convex bounded subset of X and $x \in K$. The following are equivalent:*

- (i) x is a weak*-extreme point of K ;
- (ii) the open slices of K , containing x , form a neighborhood base for x in the weak topology on K ;
- (iii) if $y_n, z_n \in K$ are such that $\lim \|x - (y_n + z_n)/2\| = 0$, then $\text{weak-lim}(y_n - z_n) = 0$.

We shall use the following characterization of nonreflexive Banach spaces:

Theorem 2 ([J]). *A Banach space X is nonreflexive iff for each $0 < \varepsilon < 1$ there exists a sequence $\{z_n\}$ of norm one elements so that*

$$\text{dist}(\text{sp}\{z_i\}_{i=1}^n, \text{co}\{z_i\}_{i=n+1}^\infty) > \varepsilon$$

for any $n \in \mathbb{N}$.

Theorem 2. *Let X be a separable nonreflexive Banach space. Then X admits an equivalent norm such that the set of weak*-extreme points of the new unit ball is isolated in the norm topology.*

PROOF: Choose $\varepsilon > 0$ and a sequence $\{z_n\}$ in the unit ball of X as in Theorem 2. Clearly, $\{\pm z_n\}$ is ε -discrete. Also $\{z_n\}$ does not have a weak-cluster point in X because

$$\bigcap_{n=1}^\infty \overline{\text{co}}\{z_i\}_{i=n}^\infty = \emptyset.$$

Choose some weak*-cluster point z^{**} of $\{z_n\}$ in X^{**} . Denote by Y the kernel of z^{**} in X^* . Then it is well known that for $x \in X$

$$|x| := \sup\{\langle x^*, x \rangle; x^* \in Y, \|x^*\| \leq 1\}$$

defines an equivalent norm on X (cf. e.g. [GLT]). Denote the unit ball under this norm by D . Because X is separable, there is a norming sequence $\{z_k^*\}$ in the unit sphere of Y , i.e.

$$(1) \quad |x| = \sup\{\langle z_k^*, x \rangle; k \in \mathbb{N}\}$$

for every $x \in X$. Since z_k^* are in the kernel of z^{**} , by passing to a subsequence of $\{z_n\}$ if necessary, we may suppose that

$$(2) \quad \lim_{n \rightarrow \infty} \langle z_n, z_k^* \rangle = 0 \quad \text{for } k \in N.$$

Choose $c > 0$ such that $\|x\| \geq c\|x\|$ for $x \in X$ and denote $\gamma := \varepsilon c/4$. Choose a sequence $\{y_n\}$ dense in the sphere of the ball γD . Clearly the set

$$T := \{\pm(z_n \pm y_n)\}$$

is discrete and symmetric. Moreover T is bounded and

$$\gamma D \subset \overline{\text{co}}T,$$

therefore $U := \overline{\text{co}}T$ is a unit ball of an equivalent norm on X . Now, we have only to follow the proof in [GLT] in order to show that w^* -ext U is a subset of T .

Suppose that some weak*-extreme point x of U is not contained in T . Then by Lemma 1 there exists a sequence $\{\alpha_1 z_{n_i} + \alpha_2 y_{n_i}\}$ (where α_1 and α_2 equal 1 or -1) such that

$$(3) \quad \lim_{i \rightarrow \infty} \langle \alpha_1 z_{n_i} + \alpha_2 y_{n_i}, z_k^* \rangle = \langle x, z_k^* \rangle \quad \text{for } k \in N.$$

Due to (2) we get that

$$(4) \quad |\langle x, z_k^* \rangle| = \left| \lim_{i \rightarrow \infty} \langle y_{n_i}, z_k^* \rangle \right| \leq \gamma,$$

and, because $\{z_k^*\}$ is norming, it follows that $x \in \gamma D$. Consequently, there exists some subsequence $\{y_{m_i}\}$ of the sequence $\{y_n\}$ converging in norm to x . Since

$$y_{m_i} = (y_{m_i} + z_{m_i})/2 + (y_{m_i} - z_{m_i})/2,$$

Lemma 1 implies that the sequence $\{z_{m_i}\}$ converges weakly to zero and this a contradiction to the fact that $\{z_{m_i}\}$ does not have a weak-cluster point in X .

Remark 4. For the unit ball U constructed in the proof of the previous theorem it holds also that the distance between the sets w^* -ext U and $\text{ext } U^{**} \setminus w^*$ -ext U is positive.

PROOF: If x is an extreme point of a weak*-compact set K , then the weak*-open slices containing x form a neighborhood basis for x in the weak*-topology on K . Therefore, for any $x \in \text{ext } U^{**}$ there exists a sequence $\{\alpha_1 z_{n_i} + \alpha_2 y_{n_i}\}$ (where α_1 and α_2 equal 1 or -1) such that (3) and (4) are satisfied. Consequently,

$$\sup\{|\langle x, z_k^* \rangle|; k \in N\} \leq \gamma \quad \text{for } x \in \text{ext } U^{**}.$$

From the definition of T follows that

$$\sup\{\langle x, z_k^* \rangle; k \in N\} \geq 3\gamma \quad \text{for } x \in T.$$

Since $\|z_n^*\| = 1$ for $n \in N$, it holds that the distance of the sets w^* -ext U and $\text{ext } U^{**} \setminus w^*$ -ext U is greater than γ .

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