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Abstract. In this paper we consider nonconvex evolution inclusions driven by time dependent convex subdifferentials. First we establish the existence of a continuous selection for the solution multifunction and then we use that selection to show that the solution set is path connected. Two examples are also presented.

Keywords: subdifferential operator, function of compact type, evolution inclusion, continuous selection, path connectedness, differential variational inequalities, nonlinear parabolic system

Classification: Primary 34G20; Secondary 35K55

1. Introduction.

Recently Cellina-Ornelas [5] considered the multivalued Cauchy problem $\dot{x}(t) \in F(t, x(t))$ a.e., $x(0) = \xi$, with $F(t, x)$ measurable in t, Hausdorff Lipschitz in x and with a nonempty solution set $S(\xi) \subseteq AC(T, \mathbb{R}^n)$. Given $w \in S(\xi_0)$ and K a compact subset containing ξ_0, they proved that we can find a continuous map $u : K \to AC(T, \mathbb{R}^n)$ s.t. $u(\xi) \in S(\xi)$ for all $\xi \in K$ and $u(\xi_0) = w$; i.e. a continuous selector $u(\cdot)$ of the solution multifunction $S : K \to 2^{AC(T,\mathbb{R}^n)} \setminus \{\emptyset\}$ passing through the prescribed value w. The result of Cellina-Ornelas [5] was extended to differential inclusions in Banach spaces by Colombo-Fryszkowski-Rzezuchowski-Staicu [7]. However their formulation precludes the applicability of their work to partial differential equations with multivalued terms (evolution inclusions), which arise often in applications, like obstacle problems, free boundary problems and optimal control of distributed parameter systems. For further details on these applications we refer to Ahmed [1], Chang [6] and Papageorgiou [18]. Very recently Staicu [25] and Papageorgiou [22] considered evolution inclusions and established the existence of a globally continuous selector of the solution multifunction $\xi \to S(\xi)$, which however does not pass from a prescribed value. This then limits the usefulness of their result in the study of the topological structure of the solution set $S(\xi)$.

In this paper, we consider a large class of nonlinear evolution inclusions driven by time dependent subdifferential operators and prove for their solution multifunction $S(\xi)$, a continuous selection theorem analogous to that of Cellina-Ornelas [5] mentioned earlier. So our work here can be viewed as a complement to that of Staicu [25] and Papageorgiou [22]. Having established the existence of a continuous selector for the multifunction $\xi \to S(\xi)$, we then show that for every $\xi \in \operatorname{dom} \varphi(0, \cdot)$, the solution set $S(\xi)$ is path connected. This result extends the work of Staicu-Wu [26], who considered differential inclusions in Banach spaces, with no unbounded operators present. It should be mentioned that very recently DeBlasi-Pianigiani [8] showed that under certain continuity hypotheses on the orientor field (multivalued vector field), the solution set of a class of nonconvex differential inclusions in $\mathbb{R}^n$ is in fact contractible, hence a fortiori path connected. Their approach made use of Choquet's theory on the extremal structure of compact convex sets.

In the last section we present two examples illustrating the applicability of our work.

2. Preliminaries.

Let (Ω, Σ) be measurable space and X a separable Banach space. We will be using the following notation:

$$\begin{aligned} P_{f(c)}(X) &= \{A \subseteq X : \text{ nonempty, closed (and convex)}\} \\ \text{and } \; P_{(w)k(c)}(X) &= \{A \subseteq X : \text{ nonempty, (weakly-) compact (and convex)}\}. \end{aligned}$$

A multifunction (set-valued function) $F : \Omega \to P_f(X)$ is said to be measurable if and only if $\omega \to d(x, F(\omega)) = \inf\{\|x - z\| : z \in F(\omega)\}$ is measurable. Let $\mu(\cdot)$ be a finite measure on (Ω, Σ). By S_F^p, $1 \leq p \leq \infty$, we will denote the set of measurable selectors of $F(\cdot)$ that belong in the Lebesgue-Bochner space $L^p(\Omega, X)$; i.e. $S_F^p = \{f \in L^p(\Omega, X) : f(\omega) \in F(\omega)\mu\text{-a.e.}\}$. This set may be empty. An easy application of Aumann's selection theorem (see Wagner [28, Theorem 5.10]) shows that for a measurable function $F : \Omega \to P_f(X)$, S_F^p is nonempty if and only if $\omega \to \inf\{\|x\| : x \in F(\omega)\} \in Ł_+^p$ with L_+^p being the positive cone of the Lebesgue space $L^p(\Omega, \mathbb{R})$.

On $P_f(X)$ we can define a generalized metric, known as the Hausdorff metric, by setting for $A, B \in P_f(X)$,

$$h(A, B) = \max\left[\sup_{a \in A} d(a, B), \; \sup_{b \in B} d(b, A)\right]$$

where $d(a, B) = \inf\{\|a - b\| : b \in B\}$ and $d(b, A) = \inf\{\|b - a\| : a \in A\}$. It is well known that $(P_f(X), h)$ is a complete metric space.

Let $\varphi : X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$. We will say that $\varphi(\cdot)$ is proper if it is not identically $+\infty$. Assume that $\varphi(\cdot)$ is proper, convex and l.s.c. (usually this family of $\overline{\mathbb{R}}$-valued functions is denoted by $\Gamma_0(X)$). By dom φ, we will denote the effective domain of $\varphi(\cdot)$; i.e. dom $\varphi = \{x \in X : \varphi(x) < +\infty\}$. The subdifferential of $\varphi(\cdot)$ at x is the set $\partial\varphi(x) = \{x^* \in X^* : (x^*, y - x) \leq \varphi(y) - \varphi(x) \text{ for all } y \in \text{dom } \varphi\}$, where $(\cdot, \cdot)$ denotes the duality brackets for the pair (X, X^*). If $\varphi(\cdot)$ is Gâteaux differentiable at x, then $\partial\varphi(x) = \{\varphi'(x)\}$. We say that $\varphi(\cdot)$ is of compact type, if for every $\lambda \in \mathbb{R}_+$, the level set $\{x \in X : \|x\|^2 + \varphi(x) \leq \lambda\}$ is compact.

Finally, recall that if Y is a Hausdorff topological space and $\{U_\alpha\}_{\alpha \in I}$ is a family of open sets covering Y, a "subcover" is any subfamily $\{U_\beta\}_{\beta \in J}$, $J \subseteq I$, also covering Y. If J is finite, then we say that the subcover is finite.

3. Selection theorem.

Let $T=[0,b]$ and H a separable Hilbert space. Using the Riesz-Fréchet theorem, we identify H with its dual (pivot space). The multivalued Cauchy problem under consideration is the following:

$$(\underline{\underline{1}})\qquad \begin{Bmatrix} -\dot{x}(t)\in\partial\varphi(t,x(t))+F(t,x(t)) \text{ a.e.} \\ x(0)=\xi. \end{Bmatrix}$$

We will need the following hypotheses on the data of $(\underline{\underline{1}})$.

$\underline{H(\varphi)}$: $\quad\varphi: T\times H\to\overline{\mathbb{R}}=\mathbb{R}\cup\{+\infty\}$ is a function s.t.

($\underline{1}$) for every $t\in T$, $\varphi(t,\cdot)$ is proper, convex, l.s.c. (i.e. $\varphi(t,\cdot)\in\Gamma_0(H)$) and of compact type,

($\underline{2}$) there exists $0\le\alpha\le 1$ such that for any positive integer r, there exist a constant $K_r>0$, an absolutely continuous function $g_r: T\to\mathbb{R}$ with $\dot{g}_r\in L^2(T)$ if $\alpha\in[0,\frac{1}{2}]$ and $\dot{g}_r\in L^{\frac{1}{1-\alpha}}(T)$ if $\alpha\in[\frac{1}{2},1]$, and a function $h_r: T\to\mathbb{R}$ of bounded variation such that if $t\in T$, $x\in\operatorname{dom}\varphi(t,\cdot)$ with $\|x\|\le r$ and $s\in[t,b]$, there exists $\widehat{x}\in\operatorname{dom}\varphi(s,\cdot)$ satisfying

$$\begin{aligned} &\|\widehat{x}-x\|\le|g_r(s)-g_r(t)|\ (\varphi(t,x)+K_r)^{\alpha} \\ \text{and}\ \ &\varphi(s,\widehat{x})\le\varphi(t,x)+|h_r(s)-h_r(t)|\ (\varphi(t,x)+K_r). \end{aligned}$$

Remark. This hypothesis, which clearly puts very mild restrictions on the t-dependence of $\varphi(t,x)$ is due to Yotsutani [31] and is more general than the ones used in the earlier important works of Watanabe [29], Kenmochi [15] and Yamada [30].

$\underline{H(F)}$: $\quad F: T\times H\to P_f(H)$ is a multifunction s.t.

($\underline{1}$) $t\to F(t,x)$ is measurable,

($\underline{2}$) $h(F(t,x),F(t,y))\le k(t)\|x-y\|$ a.e. with $k(\cdot)\in L^1_+$,

($\underline{3}$) $|F(t,x)|=\sup\{\|v\|: v\in F(t,x)\}\le\alpha(t)+\beta(t)\|x\|$ a.e. with $\alpha,\beta\in L^2_+$.

By a strong solution of $(\underline{\underline{1}})$, we mean a function $x\in C(T,H)$ such that $x(\cdot)$ is strongly absolutely continuous on $(0,b)$, $x(t)\in\operatorname{dom}\varphi(t,\cdot)$ a.e. and satisfies $-\dot{x}(t)\in\partial\varphi(t,x(t))+f(t)$ a.e., $x(0)=\xi$ with $f\in S^2_{F(\cdot,x(\cdot))}$. Recall that since H is a Hilbert space, a strongly absolutely continuous function from $(0,b)$ into H is almost everywhere differentiable (see for example Diestel-Uhl [10, p. 217]). We will denote the set of strong solutions of $(\underline{\underline{1}})$ by $S(\xi)\subseteq C(T,H)$. Under the hypotheses $H(\varphi)$ and $H(F)$ above, for every $\xi\in\operatorname{dom}\varphi(0,\cdot)$, $S(\xi)$ is a nonempty subset of $C(T,H)$ (see Papageorgiou [17, Theorem 3.1] or for an even more general result concerning extremal solutions, see Papageorgiou [20]).

In the proof of our selection theorem, we will need the following simple lemma (see also the proposition in Cellina-Ornelas [5]).

Lemma 3.1. *$\underline{\underline{\text{If}}}$ $\{v_k\}_{k=0}^N\subseteq L^1(T,H)$ and $\{T_k(\xi)\}_{k=0}^N$ is a partition of T into a finite number of subintervals whose endpoints depend continuously on $\xi\in H$ and*

$K \subseteq H$ compact, then there exists $\eta(\cdot) \in L^1_+$ such that for every $\varepsilon > 0$, there exists a $\delta > 0$ and a set $C \subseteq T$ with $\lambda(C) < \varepsilon$ s.t. $\|\xi' - \xi\| < \delta$, $\xi', \xi \in K$ implies $\| \sum_{k=0}^N \chi_{T_k(\xi')}(t)v_k(t) - \sum_{k=0}^N \chi_{T_k(\xi)}(t)v_k(t)\| \le \chi_C(t)\eta(t)$, (here $\lambda(\cdot)$ is the Lebesgue measure on T).

PROOF: We have (with $\chi_{T_k(\xi)}(\cdot)$ being the indicator function of the subinterval $T_k(\xi)$ in the partition):

$$\Big\| \sum_{k=0}^N \chi_{T_k(\xi')}(t)v_k(t) - \sum_{k=0}^N \chi_{T_k(\xi)}(t)v_k(t)\Big\| \le \sum_{k=0}^N |\chi_{T_k(\xi')}(t) - \chi_{T_k(\xi)}(t)| \cdot \|v_k(t)\|$$
$$= \sum_{k=0}^N \chi_{T_k(\xi')\Delta T_k(\xi)}(t) \cdot \|v_k(t)\|.$$

Let $\eta(t) = \sum_{k=0}^N \|v_k(t)\| \in L^1_+$. Since by hypothesis the endpoints of the subintervals in the partition depend continuously on $\xi \in K$ and K is compact, given $\varepsilon > 0$, we can find $\delta > 0$ such that for $\|\xi' - \xi\| < \delta$ we have $\chi_{T_k(\xi')\Delta T_k(\xi)}(t) \le \chi_C(t)$ for all $k \in \{0, 1, 2, \dots, N\}$, with $\lambda(C) < \varepsilon$. Hence for $\|\xi' - \xi\| < \delta$, we have:

$$\Big\| \sum_{k=0}^N \chi_{T_k(\xi')}(t)v_k(t) - \sum_{k=0}^N \chi_{T_k(\xi)}(t)v_k(t)\Big\| \le \chi_C(t) \sum_{k=0}^N \|v_k(t)\| = \chi_C(t)\eta(t).$$

□

Now we can state and prove our selection theorem.

Theorem 3.1. *If hypotheses $H(\varphi)$, $H(F)$ hold, $K \subseteq \operatorname{dom} \varphi(0, \cdot)$ is a nonempty compact set, $\xi_0 \in K$ and $w \in S(\xi_0)$, then there exists a continuous map $u : K \to C(T, H)$ such that $u(\xi_0) = w$ and for all $\xi \in K$, $u(\xi) \in S(\xi)$.*

PROOF: Let $\xi \in K$ and let $p_\xi : L^2(T, H) \to C(T, H)$ be the map that to each $g \in L^2(H)$ assigns the unique solution of the Cauchy problem $-\dot y(t) \in \partial\varphi(t, y(t)) + g(t)$ a.e., $y(0) = \xi$ (see Yotsutani [31]). Since $w \in S(\xi_0)$, by definition there exists $f \in S^2_{F(\cdot, w(\cdot))}$ s.t. $w = p_{\xi_0}(f)$. Set $z_0(\xi) = p_\xi(f)$. A straightforward application of Aumann's selection theorem gives us $r_0(\xi) \in L^2(T, H)$ such that

$$\begin{aligned} & r_0(\xi)(t) \in F(t, z_0(\xi)(t)) \text{ a.e.} \\ \text{and } & \|f(t) - r_0(\xi)(t)\| = d(f(t), F(t, z_0(\xi)(t))) \\ & \le k(t)\|w(t) - z_0(\xi)(t)\| \\ & \le k(t)\|p_{\xi_0}(f)(t) - p_\xi(f)(t)\| \le k(t)\|\xi_0 - \xi\| \text{ a.e.} \end{aligned}$$

Fix $\theta > 0$ and define

$$\delta(\xi) = \begin{cases} \min\left(2^{-3}\theta, \ \frac{\|\xi - \xi_0\|}{2}\right) & \text{if } \xi \ne \xi_0 \\ 2^{-3}\theta & \text{if } \xi = \xi_0. \end{cases}$$

Let $\overset{\circ}{B}(\xi, \delta(\xi)) = \{h \in H : \|\xi - h\| < \delta(\xi)\}$. Clearly $\{\overset{\circ}{B}(\xi, \delta(\xi))\}_{\xi \in K}$ is an open cover of K and from the definition of $\delta(\xi)$, we have that ξ_0 belongs in $\overset{\circ}{B}(\xi_0, \delta(\xi_0))$ only. Since by hypothesis K is compact, we can find $\{\xi_k\}_{k=0}^N$ s.t. $\{\overset{\circ}{B}(\xi_k, \delta(\xi_k))\}_{k=0}^N$ is a finite subcover of K. Let $\{\gamma_k\}_{k=0}^N$ be a locally Lipschitz partition of unity subordinate to this subcover. Set

$$T_0(\xi) = [0, \gamma_0(\xi)b]$$

and $T_k(\xi) = [(\sum_{i=0}^{k-1} \gamma_i(\xi))b,\ (\sum_{i=0}^{k} \gamma_i(\xi))b]$ for $k \in \{1, 2, \dots, N\}$.

Consider the following evolution equation:

$$\left\{ \begin{array}{c} -\dot{y}(t) \in \partial\varphi(t, y(t)) + \displaystyle\sum_{k=0}^{N} \chi_{T_k(\xi)}(t) r_0(\xi_k)(t) \text{ a.e.} \\ y(0) = \xi \in K. \end{array} \right\}$$

From Yotsutani [31], we know that this problem has a unique solution $z_1(\xi)(\cdot) \in C(T, H)$. Let $\mu_0(\xi)(t) = \sum_{k=0}^N \chi_{T_k(\xi)}(t) r_0(\xi_k)(t)$. Then since $z_1(\xi) = p_\xi(\mu_0(\xi))$ and $z_1(\xi') = p_{\xi'}(\mu_0(\xi'))$ and by exploiting the monotonicity of the subdifferential operator, we have:

$$\begin{aligned}
&(-\dot{z}_1(\xi)(t) + \dot{z}_1(\xi')(t), z_1(\xi')(t) - z_1(\xi)(t)) \\
&\le (\mu_0(\xi)(t) - \mu_0(\xi')(t), z_1(\xi')(t) - z_1(\xi)(t)) \text{ a.e.} \\
&\Rightarrow \frac{1}{2}\frac{d}{dt}\|z_1(\xi')(t) - z_1(\xi)(t)\|^2 \le (\mu_0(\xi)(t) - \mu_0(\xi')(t), z_1(\xi')(t) - z_1(\xi)(t)) \text{ a.e.} \\
&\Rightarrow \frac{1}{2}\|z_1(\xi')(t) - z_1(\xi)(t)\|^2 \le \frac{1}{2}\|\xi' - \xi\|^2 \\
&+ \int_0^t \|\mu_0(\xi)(s) - \mu_0(\xi')(s)\| \cdot \|z_1(\xi')(s) - z_1(\xi)(s)\|\, ds.
\end{aligned}$$

Invoking Lemma A. 5, p. 157 of Brezis [4], we get

$$\|z_1(\xi')(t) - z_1(\xi)(t)\| \le \|\xi' - \xi\| + \int_0^t \|\mu_0(\xi)(s) - \mu_0(\xi')(s)\|\, ds.$$

Let $\eta(t) = \sum_{k=0}^N \|r_0(\xi_k)(t)\|$, as in the lemma. Given $\varepsilon > 0$, we can find $\delta_1 > 0$ such that $\int_C \eta(t)\, dt < \frac{\varepsilon}{2}$ for all $C \subseteq T$ measurable with $\lambda(C) < \delta_1$. Also from the lemma we know that corresponding to this $\delta_1 > 0$ we can find $0 < \delta < \frac{\varepsilon}{2}$ such that $\|\xi' - \xi\| < \delta$ implies that $\|\mu_0(\xi)(t) - \mu_0(\xi')(t)\| \le \chi_{\widehat{C}}(t)\eta(t)$ for some $\widehat{C} \subseteq T$ measurable, with $\lambda(\widehat{C}) < \delta_1$. Hence finally, if $\|\xi' - \xi\| < \delta$, $\xi', \xi \in K$, we have

$$\begin{gathered}
\|z_1(\xi')(t) - z_1(\xi)(t)\| \le \frac{\varepsilon}{2} + \int_{\widehat{C}} \eta(s)\, ds < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \ \text{ for all } \ t \in T \\
\Rightarrow \xi \to z_1(\xi) \ \text{ is continuous from } K \text{ into } C(T, H).
\end{gathered}$$

Also note that

$$\begin{aligned}&d(\mu_0(\xi)(t), F(t, z_1(\xi)(t)))\\&\qquad\le d(\mu_0(\xi)(t), F(t, z_0(\xi)(t))) + h(F(t, z_0(\xi)(t)), F(t, z_1(\xi)(t))).\end{aligned}$$

Observe that if B is the Lebesgue-null set postulated by hypothesis $H(F)$ (2) (i.e. on B^c, the h-Lipschitz condition is valid), then for $t \in T_k(\xi) \setminus B$ we have

$$\begin{aligned}&d(\mu_0(\xi)(t), F(t, z_0(\xi)(t)))\\&= d(r_0(\xi_k)(t), F(t, z_0(\xi)(t))) \le h(F(t, z_0(\xi_k)(t))F(t, z_0(\xi)(t)))\\&\le k(t)\|\xi_k - \xi\| \le k(t)\frac{\theta}{2^3}\end{aligned}$$

and this estimate is independent of k, hence valid for all $t \in T \setminus B$.

In addition we have

$$\begin{aligned}&h(F(t, z_0(\xi)(t)), F(t, z_1(\xi)(t)))\\&\le k(t)\|z_0(\xi)(t) - z_1(\xi)(t)\| \le k(t)\int_0^t \|f(s) - \mu_0(\xi)(s)\|\, ds\end{aligned}$$

(the second inequality following as before from the monotonicity of the subdifferential)

$$\begin{aligned}&\le k(t)\int_0^t \sum_{k=0}^N \chi_{T_k}(s)\|f(s) - r_0(\xi_k)(s)\|\, ds\\&\le k(t)\int_0^t \sum_{k=0}^N \chi_{T_k(\xi)}(s)k(s)\|\xi_0 - \xi_k\|\, ds \le \sigma k(t)v(t)\end{aligned}$$

with $\sigma = \operatorname{diam} K$ (the diameter of K; i.e. $\sigma = \sup\{\|\xi - \xi'\| : \xi, \xi' \in K\}$) and $v(t) = \int_0^t k(s)\, ds$.

So finally we have

$$d(\mu_0(\xi)(t), F(t, z_1(\xi)(t))) \le \sigma k(t)v(t) + \frac{\theta}{2^3}k(t) \text{ a.e.}$$

Now we will show by induction that we can have two sequences $\{z_n(\xi)(\cdot)\}_{n\ge 0} \subseteq C(T, H)$ and $\{\mu_n(\xi)(\cdot)\}_{n\ge 0} \subseteq L^2(T, H)$ satisfying the following four properties:

(i) $z_n(\xi)(\cdot) = p_\xi(\mu_{n-1}(\xi))(\cdot)$ for $n \ge 1$, $\xi \in K$,

(ii) $d(\mu_{n-1}(\xi)(t), F(t, z_n(\xi)(t))) \le \sigma k(t)\frac{v^n(t)}{n!} + \frac{\theta}{2^{n+1}}k(t)\sum_{k=0}^n \frac{(2v(t))^k}{k!}$ for $\xi \in K$,

(iii) $\|z_n(\xi)(t) - z_{n-1}(\xi)(t)\| \le \sigma\frac{v^n(t)}{n!} + \frac{\theta}{2^{n+1}}\sum_{k=1}^n \frac{(2v(t))^k}{k!} + \frac{\theta}{2^{n+2}}$,

(iv) there exists $\eta_n \in L^1(T)$ such that for every $\varepsilon > 0$, there exists $\delta > 0$ and $C \subseteq T$ with $\lambda(C) < \varepsilon$ such that $\|\xi' - \xi\| < \delta$, $\xi', \xi \in K$, implies $\|\mu_{n-1}(\xi')(t) - \mu_{n-1}(\xi)(t)\| \le \chi_C(t)\eta_n(t)$.

From what we did in the first part of the proof, the above four properties are satisfied for $n = 1$ (i.e. for $z_0(\xi)(\cdot)$, $z_1(\xi)(\cdot)$, $\mu_0(\xi)(\cdot)$). Assume (for the induction hypothesis) that we have obtained $\{z_m(\xi)(\cdot)\}_{m=0}^{n}$ and $\{\mu_m(\xi)(\cdot)\}_{m=0}^{n-1}$ satisfying (i) $\to$ (iv) above. Via Aumann's selection theorem, we can find $r_n(\xi) : T \to H$ measurable s.t. $r_n(\xi)(t) \in F(t, z_n(\xi)(t))$ and

$$\|r_n(\xi)(t) - \mu_{n-1}(\xi)(t)\| = d(\mu_{n-1}(\xi)(t), F(t, z_n(\xi)(t)))$$
$$\leq \sigma k(t)\frac{v^n(t)}{n!} + \frac{\theta}{2^{n+1}}k(t)\sum_{k=0}^{n}\frac{(2v(t))^k}{k!} \quad \text{(by (ii) and the induction hypothesis).}$$

Because of (iv) and the induction hypothesis, we know that there exists $\delta_n > 0$ s.t. $\|\xi' - \xi\| < \delta_n$, $\xi', \xi \in K$, implies that $\|\mu_{n-1}(\xi')(t) - \mu_{n-1}(\xi)(t)\| \leq \chi_C(t)\eta_n(t)$ for some $C \subseteq T$ measurable such that $\int_C \eta_n(t)\,dt \leq \frac{\theta}{2^{n+4}}$.

Let $\delta_n(\xi) = \min(\delta_n, \frac{\theta}{2^{n+4}}, \frac{\|\xi-\xi_0\|}{2})$ if $\xi \neq \xi_0$ and $\delta_n(\xi_0) = \min(\delta_n, \frac{\theta}{2^{n+4}})$. As before, note that $\{\overset{\circ}{B}(\xi, \delta_n(\xi))\}_{\xi\in K}$ is an open cover for K and K is by hypothesis, compact. So we can find $\xi_k^n \in K$, $k = 0, 1, \ldots, N_n$, $\xi_0^n = \xi_0$ such that $\{\overset{\circ}{B}(\xi_k^n, \delta_n(\xi_k^n))\}_{k=0}^{N_n}$ is a finite subcover of K. Let $\{\gamma_k^n(\cdot)\}_{k=0}^{N_m}$ be a continuous partition of unity subordinate to this subcover. Define $T_0^n(\xi) = [0, \gamma_0^n(\xi)b]$ and $Y_k^n(\xi) = [(\sum_{i=0}^{k-1}\gamma_i^n(\xi))b\ (\sum_{i=0}^{k}\gamma_i^n(\xi))b]$ for $k = 1, 2, \ldots, N_n$. Let $\mu_n(\xi)(t) = \sum_{k=0}^{N_n}\chi_{T_k^n(\xi)}(t)r_n(\xi_k^n)(t) \in L^2(T, H)$ (see the hypothesis $H(F)$ (2)). From Lemma 3.1 we know that $\xi \to \mu_n(\xi)(\cdot)$ is continuous from K into $L^1(T, H)$. Set $z_{n+1}(\xi)(\cdot) = p_\xi(\mu_n(\xi))(\cdot) \in C(T, H)$. Again because of the monotonicity of the subdifferential, for $\xi', \xi \in K$ we have

$$\|z_{n+1}(\xi')(t) - z_{n+1}(\xi)(t)\| \leq \|\xi' - \xi\| + \int_0^t \|\mu_n(\xi')(s) - \mu_n(\xi)(s)\|\,ds,$$

which shows that $\xi \to z_{n+1}(\xi)(\cdot)$ is continuous from K into $C(T, H)$.

Once more exploiting the monotonicity of the subdifferential, we get

$$\|z_{n+1}(\xi)(t) - z_n(\xi)(t)\| \leq \int_0^t \|\mu_n(\xi)(s) - \mu_{n-1}(\xi)(s)\|\,ds$$
$$= \int_0^t \sum_{k=0}^{N_n}\chi_{T_k^n(\xi)}(s)\|r_n(\xi_k^n)(s) - \mu_{n-1}(\xi_k^n)(s)\|\,ds$$
$$+ \int_0^t \sum_{k=0}^{N_n}\chi_{T_k^n(\xi)}(s)\|\mu_{n-1}(\xi_k^n)(s) - \mu_{n-1}(\xi)(s)\|\,ds$$
$$\leq \int_0^t \sum_{k=0}^{N_n}\chi_{T_k^n(\xi)}(s)d(\mu_{n-1}(\xi_k^n)(s), F(s, z_n(\xi_k^n)(s)))\,ds$$
$$+ \int_0^t \sum_{k=0}^{N_n}\chi_{T_k^n(\xi)}(s)\chi_C(s)\eta_n(s)\,ds$$

$$\leq \int_0^t \Big(\sigma k(s)\frac{v^n(s)}{n!} + \frac{\theta}{2^{n+1}}k(s)\sum_{k=0}^{n}\frac{(2v(s))^k}{k!}\Big)\,ds + \frac{\theta}{2^{n+4}}$$
$$= \sigma\frac{v^{n+1}(t)}{(n+1)!} + \frac{\theta}{2^{n+2}}\sum_{k=1}^{n+1}\frac{(2v(t))^k}{k!} + \frac{\theta}{2^{n+4}}\,.$$

Also note that for $t \in T_k^n(\xi) \setminus B$, we have

$$\begin{aligned}
&d(\mu_n(\xi)(t), F(t, z_n(\xi)(t))) = d(r_n(\xi_k^n)(t), F(t, z_n(\xi)(t)))\\
&\leq h(F(t, z_n(\xi_k^n)(t)), F(t, z_n(\xi)(t))) \leq k(t)\|z_n(\xi_k^n)(t) - z_n(\xi)(t)\|\\
&\leq k(t)\|\xi_k^n - \xi\| + \int_0^t \|\mu_{n-1}(\xi_k^n)(s) - \mu_{n-1}(\xi)(s)\|\,ds\\
&\leq k(t)\Big(\frac{\theta}{2^{n+4}} + \frac{\theta}{2^{n+4}}\Big) = \frac{k(t)\theta}{2^{n+3}}
\end{aligned}$$

and this estimate is independent of $k \in \{0, 1, \ldots, N_n\}$, thus valid for all $t \in T \setminus B$. Hence we get

$$\begin{aligned}
&d(\mu_n(\xi)(t), F(t, z_{n+1}(\xi)(t)))\\
&\leq d(\mu_n(\xi)(t), F(t, z_n(\xi)(t))) + h(F(t, z_n(\xi)(t)), F(t, z_{n+1}(\xi)(t)))\\
&\leq \frac{k(t)\theta}{2^{n+3}} + k(t)\|z_n(\xi)(t) - z_{n+1}(\xi)(t)\|\\
&\leq \frac{k(t)\theta}{2^{n+3}} + k(t)\sigma\frac{v^{n+1}(t)}{(n+1)!} + \frac{k(t)\theta}{2^{n+2}}\sum_{k=1}^{n+1}\frac{(2v(t))^k}{k!} + \frac{k(t)\theta}{2^{n+4}}\\
&\leq k(t)\sigma\frac{v^{n+1}(t)}{(n+1)!} + \frac{k(t)\theta}{2^{n+2}}\sum_{k=0}^{n+1}\frac{(2v(t))^k}{k!}\,.
\end{aligned}$$

So our induction is complete and we established the two sequences $\{z_n(\xi)(\cdot)\}_{n\geq 0}$ and $\{\mu_n(\xi)(\cdot)\}_{n\geq 0}$ satisfying (i) $\to$ (iv). Note that from these estimates we have that these sequences are Cauchy in $C(T,H)$ and $L^1(T,H)$ respectively, uniformly on $\xi \in K$. Thus we may assume that $z_n(\xi) \to u(\xi)$ in $C(T,H)$ and $\mu_n(\xi) \to \mu(\xi)$ in $L^1(T,H)$ and both limits are continuous in $\xi \in K$. Also note that because of hypothesis $H(F)\,(\underline{2})$, $\mu(\xi)(\cdot) \in L^2(T,H)$. Furthermore, since $T_0(\xi_0) = [0,b]$, we have $z_n(\xi_0) = 2$ and so $u(\xi_0) = w$. Set $w(\xi) = p_\xi(\mu(\xi))$. Then as before due to the monotonicity of the subdifferential, we have

$$\|z_n(\xi)(t) - w(\xi)(t)\| \leq \int_0^t \|\mu_n(\xi)(s) - \mu(\xi)(s)\|\,ds \to 0 \ \text{ as } \ n \to \infty \Rightarrow u(\xi) = w(\xi).$$

Finally we need to show that $\mu(\xi)(t) \in F(t, u(\xi)(t))$ a.e. Indeed recall that

$$\begin{aligned}
&d(\mu_n(\xi)(t), F(t, z_n(\xi)(t))) \leq \frac{k(t)\theta}{2^{n+3}} \text{ a.e.} \\
&\Rightarrow d(\mu_n(\xi), S^1_{F(\cdot, z_n(\xi)(\cdot))}) \leq \frac{\|k\|_1\theta}{2^{n+3}} \\
&\Rightarrow d(\mu(\xi), S^1_{F(\cdot, u(\xi)(\cdot))}) = 0, \; \xi \in K \\
&\Rightarrow \mu(\xi) \in S^1_{F(\cdot, u(\xi)(\cdot))}, \; \xi \in K \\
&\Rightarrow \mu(\xi)(t) \in F(t, u(\xi)(t)) \text{ a.e.}
\end{aligned}$$

and the exceptional Lebesgue-null set is independent of $\xi \in K$ since both $\xi \to u(\xi)$ and $\xi \to \mu(\xi)$ are continuous. So $u(\xi)(\cdot)$ is the desired continuous selector of $S(\xi)$. □

4. Path connectedness of the solution set.

In this section we use Theorem 3.1 to establish the path connectedness of the solution set $S(\xi)$ for every $\xi \in H$. In the past the problem of connectedness of the solution set of differential inclusions in $\mathbb{R}^n$ and in Banach spaces was studied by several authors. We refer to the works of Himmelberg-Van Vleck [13], Tolstonogov [27], Papageorgiou [19], Deimling-Rao [9] and Hu-Papageorgiou [14]. However, all these works considered convex valued orientor fields, and when done in an infinite dimensional context, they do not allow the presence of unbounded operators. Very recently Papageorgiou [21] extended the above mentioned works to a class of convex evolution inclusions, for which he proved, using Galerkin approximations, that the solution set is connected in $C(T, H)$.

The question of connectedness of the solution set of nonconvex differential inclusions was addressed very recently by DeBlasi-Pianigiani [8] for differential inclusions in $\mathbb{R}^n$ and by Staicu-Wu [26] for differential inclusions in a separable Banach space. As we already mentioned in the introduction DeBlasi-Pianigiani [8] succeeded in proving more, namely that the solution set is contractible.

Here we establish the path connectedness of the solution set $S(\xi)$ of ($\underline{\underline{1}}$) for the autonomous version of ($\underline{\underline{1}}$).

So our multivalued Cauchy problem is now the following

$$(\underline{\underline{1}})' \qquad \left\{ \begin{array}{c} -\dot{x}(t) \in \partial\varphi(x(t)) + F(x(t)) \text{ a.e.} \\ x(0) = \xi. \end{array} \right\}$$

The hypotheses on the date are now simplified as follows:

$\underline{H(\varphi)_1}$ $\quad \varphi : H \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is a proper, convex, l.s.c. function (i.e. $\varphi \in \Gamma(H)$) which is of compact type.

$\underline{H(F)_1}$: $\quad FH \to P_f(H)$ is a multifunction s.t.

($\underline{1}$) $h(F(x), F(y)) \leq \|x - y\|$ with $k > 0$.
($\underline{2}$) $|F(x)| = \sup\{\|v\| : v \in F(x)\} \leq m_1 + m_2\|x\|$ with $m_1, m_2 > 0$.

Theorem 4.1. *If hypotheses $H(\varphi)_1$, $H(F)_1$ hold, then for every $\xi \in \operatorname{dom} \varphi$, $S(\xi)$ is path connected in $C(T, H)$.*

PROOF: Let $x, y \in S(\xi)$ and set $K = y(T) \subseteq \operatorname{dom} \varphi$, a compact subset of H. From Theorem 3.1 we know that there exists $u : K \to C(T, H)$ a continuous map s.t. $u(\xi) = x$ and $u(\xi') \in S(\xi')$ for all $\xi' \in K$. Note that $\lambda \to u(y(\lambda b))$ is continuous from $[0, 1]$ into $C(T, H)$ and $u(y(\lambda b)) \in S(y(\lambda b))$. As in Himmelberg-Van Vleck [13], we define

$$\eta_\lambda(t) = \begin{cases} y(t) & \text{if } \; t \in [0, \lambda b] \\ u(y(\lambda b))(t - \lambda b) & \text{if } \; t \in [\lambda b, b]. \end{cases}$$

Observe that for $\lambda = 0$, $\eta_0 = u(y(0)) = u(\xi) = x$, and for $\lambda = 1$, $\eta_1 = y$.

So to finish the proof that $S(\xi)$ is path connected, we need to show that $\lambda \to \eta_\lambda$ is continuous from $[0, 1]$ into $C(T, H)$. To this end let $\lambda_n \to \lambda$ in $[0, 1]$. We consider two distinct cases:

Case 1. $\lambda_n \leq \lambda$ for all $n \geq n_0 \geq 1$.

(i) $t < \lambda b \Rightarrow t < \lambda_n b$ for all $n \geq n_1 \geq n_0$. Hence by definition we have

$$\begin{aligned} &\eta_{\lambda_n}(t) = \eta_\lambda(t) = y(t) \;\; \text{for all} \;\; n \geq n_1 \geq n_0 \\ &\Rightarrow \sup_{t < \lambda b} \|\eta_{\lambda_n}(t) - \eta_\lambda(t)\| = 0. \end{aligned}$$

(ii) $t \geq \lambda b \Rightarrow t \geq \lambda_n$ for all $n \geq n_0$. So by definition we have

$$\|\eta_{\lambda_n}(t) - \eta_\lambda(t)\| = \|u(y(\lambda_n b))(t - \lambda_n b) - u(y(\lambda b))(t - \lambda b)\|.$$

Note that

$$\begin{aligned} &\|u(y(\lambda_n b))(t - \lambda_n b) - u(y(\lambda b))(t - \lambda b)\| \\ &\leq \|u(y(\lambda_n b))(t - \lambda_n b) - u(y(\lambda b))(t - \lambda_n b)\| \\ &+ \|u(y(\lambda b))(t - \lambda_n b) - u(y(\lambda b))(t - \lambda b)\| \\ &\leq \|u(y(\lambda_n b)) - u(y(\lambda b))\|_{C(T,H)} + \|u(y(\lambda b))(t - \lambda_n b) - u(y(\lambda b))(t - \lambda b)\|. \end{aligned}$$

Recall that $u(y(\lambda_n b)) \to u(y(\lambda b))$ in $C(T, H)$ and that $u(y(\lambda b))(\cdot)$ is uniformly continuous on T. So we get

$$\sup_{t \geq \lambda b} \|\eta_{\lambda_n}(t) - \eta_\lambda(t)\| \to 0 \;\; \text{as} \;\; n \to \infty.$$

Therefore from 1 (i) and (ii) above we get that

$$\eta_{\lambda_n} \to \eta_\lambda \;\; \text{in} \;\; C(T, H).$$

Case 2. $\lambda_n \geq \lambda$ for all $n \geq n_0 \geq 1$.

(i) $t > \lambda b \Rightarrow t > \lambda_n b$ for all $n \geq n_1 \geq n_0$. So by definition we have

$$\|\eta_{\lambda_n}(t) - \eta_\lambda(t)\| = \|u(y(\lambda_n b))(t - \lambda_n b) - u(y(\lambda b))(t - \lambda b)\|.$$

As in Case 1 (ii) above we get that

$$\sup_{t > \lambda b} \|\eta_{\lambda_n}(t) - \eta_\lambda(t)\| \to 0 \quad \text{as} \quad n \to \infty.$$

(ii) $t \leq \lambda b \Rightarrow t \leq \lambda_n b$ for $n \geq n_0$. So by definition we have

$$\sup_{t \geq \lambda b} \|\eta_{\lambda_n}(t) - \eta_\lambda(t)\| = 0.$$

Thus from 2 (i) and (ii) we deduce that

$$\eta_{\lambda_n} \to \eta_\lambda \quad \text{in} \quad C(T, H).$$

Finally for any sequence $\lambda_n \to \lambda$ in $[0,1]$ oscillating around limit λ, there exist subsequences satisfying Case 1 or Case 2. Thus every subsequence of $\{\eta_{\lambda_n}\}_{n \geq 1}$ has a further subsequence converging to η_λ in $C(T,H)$. Hence $\eta_{\lambda_n} \to \eta_\lambda$ in $C(T,H) \Rightarrow \lambda \to \eta_\lambda$ is continuous from $[0,1]$ into $C(T,H) \Rightarrow S(\xi)$ is path connected in $C(T,H)$. $\square$

Remark. This theorem fails under the weaker hypothesis that $F(\cdot)$ is only Hausdorff continuous (i.e. continuous from H into the metric space $(P_f(H), h)$). In fact Pugh [24] produced an example of a single valued differential equation whose solution set is not path connected.

5. Applications.

Let $T = [0,b]$ and $Z \subseteq \mathbb{R}^N$ a bounded domain, with boundary $\Gamma = \partial Z$. Let $D_i = \frac{\partial}{\partial z_i}$, $i = 1, 2, \ldots, N$. Also in what follows for the simplicity, we will write $L^2(Z)$ for $L^2(Z, \mathbb{R})$. Consider the following parabolic control system:

$$(\underline{\underline{2}}) \quad \left\{ \begin{array}{l} \dfrac{\partial x}{\partial t} - \displaystyle\sum_{k=1}^{N} D_k\big(a(z)|D_k x|^{p-2} D_k x\big) + x|x|^{p-2} \\ \qquad = f(z, x(t,z))u(t,z) \text{ a.e. on } T \times Z \\ x \mid_{T \times \Gamma} = 0, x(0,z) = x_0(z), u(t,\cdot) \in U \text{ a.e.}, u(\cdot,\cdot) \text{ measurable}, p \geq 2. \end{array} \right\}$$

The hypotheses on the data of $(\underline{\underline{2}})$ are the following:

$\underline{H(a)}$: $a \in L^\infty(Z)$, with $a(z) \geq c > 0$ for all $z \in C$.

$\underline{H(f)}$: $f : Z \times \mathbb{R} \to \mathbb{R}$ is a function s.t.

($\underline{1}$) $z \to f(z,x)$ is measurable,
($\underline{2}$) $|f(z,x) - f(z,x')| \leq k(z)|x - x'|$ a.e. with $k \in L^2_+(Z)$,
($\underline{3}$) $|f(z,x)| \leq m_1(z) + m_2(z)|x|$ a.e. with $m_1, m_2 \in L^2_+(Z)$.

$\underline{H(U)}$: U is a nonempty, closed and bounded subset of $L^2(Z)$.

$\underline{H_0}$: $x_0(\cdot) \in W_0^{1,p}(Z)$.

By $R(t)$ we denote the reachable set at time $t \in T$, of system ($\underline{\underline{2}}$). So $R(t) = \{x(t,\cdot) \in L^2(Z) : x \in C(T, L^2(Z)) \text{ solves } (\underline{\underline{2}})\} \subseteq L^2(Z)$.

Theorem 5.1. *If hypotheses $H(a)$, $H(f)$, $H(U)$ and H_0 hold, then for all $t \in T$, $R(t)$ is path connected in $L^2(Z)$.*

Proof: Let $H = L^2(Z)$ and define $\varphi : H \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ by

$$\varphi(x) = \begin{cases} \frac{1}{p}\sum_{k=1}^N \int_Z a(z)|D_k x|^p\,dz + \frac{1}{p}\int_Z |x|^p\,dz & \text{if } x \in W_0^{1,p}(Z) \\ +\infty & \text{otherwise.} \end{cases}$$

It is easy to see from the above definition that $\varphi(\cdot)$ is proper, convex and l.s.c. on $H = L^2(Z)$ (i.e. $\varphi(\cdot) \in \Gamma_0(H)$). Also for every $\lambda > 0$, the set

$$L_\lambda = \{x \in L^2(Z) = H : \|x\|^2 + \varphi(x) \le \lambda\}$$

is clearly bounded in $W_0^{1,p}(Z)$. But from the Sobolev embedding theorem we know that $W_0^{1,p}(Z)$ embeds compactly in $L^2(Z)$. So L_λ is compact in $L^2(Z)$, and so we conclude that $\varphi(\cdot)$ is of compact type.

Hence we see that $\varphi(x)$ satisfies the hypothesis $H(\varphi)_1$. As in Barbu [3] (see Proposition 2.9, p. 63), by using Green's identity, we can show that

$$\partial\varphi(x) = -L_p(x)$$

where $L_p(x) = \sum_{k=1}^N D_k(a(z)|D_k x|^{p-2}D_k x) - x|x|^{p-2}$ and $x \in D(L_p(\cdot)) = \{x \in W_0^{1,p}(Z) : L_p(x) \in L^2(Z)\}$.

Next let $F : H \to P_f(H)$ be defined by

$$F(x) = \widehat{f}(x)U$$

with $\widehat{f}(x)$ being the Nemitsky (superposition) operator corresponding to $f(z,x)$; i.e. $\widehat{f}(x)(\cdot) = f(\cdot, x(\cdot)) \in L^2(Z)$ (see the hypothesis $H(f)$).

Note that from the definition of F and the hypotheses $H(f)$ and $H(U)$ we have:

$$h(F(x), F(y)) \le M\|\widehat{f}(x) - \widehat{f}(y)\|_2 \le \widehat{k}\|x-y\|_2$$

where $\widehat{k} = M\|k(\cdot)\|_2$.

Finally note that

$$|F(x)| = \sup\{\|g\|_2 : g \in F(x)\} \le \widehat{m}_1 + \widehat{m}_2\|x\|_2 \text{ a.e.}$$

with $\widehat{m}_1, \widehat{m}_2 > 0$ (see the hypotheses $H(f)$ (3) and $H(U)$).

Because of Aumann's selection theorem, system ($\underline{\underline{2}}$) is equivalent to the following evolution inclusion (deparametrized (i.e. control free) system):

$$(\underline{\underline{2}})' \qquad \begin{Bmatrix} -\dot{x}(t) \in \partial\varphi(x(t)) + F(x(t)) \text{ a.e.} \\ x(0) = x_0. \end{Bmatrix}$$

We have checked that all hypotheses of Theorem 4.1 are satisfied. So the solution set $S(x_0) \subseteq C(T, L^2(Z))$ is path connected. Let $e_t : C(T, L^2(Z)) \to L^2(Z)$ be the evaluation at $t \in T$ map. It is well-known that this is continuous (see for example Dugundji [11]). Since continuous images of path connected sets are path connected (topological invariance of path connectedness), we get that $e_t(S(x_0))$ is path connected in $L^2(Z)$. But $e_t(S(x_0)) = R(t)$. $\square$

Another important class of systems incorporated in the problem ($\underline{\underline{1}}$) are the so called "differential variational inequalities", for which $\varphi(t, x) = \delta_{K(t)}(x)$ with $\delta_{K(t)}(\cdot)$ being the indicator function of the set $K(t)$ (i.e. $\delta_{K(t)}(x) = 0$ if $x \in K(t)$ and $+\infty$ if $x \notin K(t)$). Recalling that $\delta_{K(t)}(x) = N_{K(t)}(x)$ (the normal cone to $K(t)$ at x), we see that differential variational inequalities have the following form:

$$(\underline{\underline{3}}) \qquad \begin{Bmatrix} -\dot{x}(t) \in N_{K(t)}(x(t)) + F(x(t)) \text{ a.e.} \\ x(0) = \xi. \end{Bmatrix}$$

Problems of this form arise in applications like theoretical mechanics (see Moreau [16]) and mathematical economics (see Aubin-Cellina [2] and Henry [12]). A detailed study of such inclusions can be found in Papageorgiou [23].

We assume the following about ($\underline{\underline{3}}$):

$\underline{H(K)}$: $K : T \to P_{f(c)}(\mathbb{R}^n)$ is a multifunction s.t. for $0 \le t \le t' \le b$ we have

$$h(K(t), K(t')) \le \int_t^{t'} m(s)\, ds$$

with $m(\cdot) \in L^1_+$.

Set $\varphi(t, x) = \delta_{K(t)}(x)$. It is easy to check that hypothesis $H(\varphi)$ is satisfied.

So using Theorem 3.1 (with $H = \mathbb{R}^n$) we get the following result concerning the solution set $S(\xi) \subseteq C(T, \mathbb{R}^n)$ of ($\underline{\underline{3}}$):

Theorem 5.2. *$\underline{\text{If}}$ hypotheses $H(K)$ and $H(F)$ (with $H = \mathbb{R}^n$) hold, V is a nonempty compact subset of $K(0)$, $\xi_0 \in V$ and $w \in S(\xi_0)$, $\underline{\text{then}}$ there exists a continuous map $u : V \to C(T, \mathbb{R}^N)$ such that $u(\xi_0) = w$ and for all $\xi \in Kv(\xi) = S(\xi)$.*

Also for the autonomous system we have a structural result concerning its solution set. So the multivalued Cauchy problem under consideration is the following:

$$(\underline{\underline{3}})' \qquad \begin{Bmatrix} -\dot{x}(x) \in N_{K(t)}(x(t)) + F(x(t)) \text{ a.e.} \\ x(0) = \xi. \end{Bmatrix}$$

It is well-known (see for example Aubin-Cellina [2]), that ($\underline{\underline{3}}$)$'$ is equivalent to the "projected" differential inclusion $\dot{x}(t) = \mathrm{proj}\,(-F(x(t)); T_K(x(t)))$ a.e., $x(0) = \xi$. Here $\mathrm{proj}\,(\cdot, T_K(x))$ denotes the metric projection on the tangent cone $T_K(x)$. In many systems, with systems constraints, in describing the effect of the constraint on the dynamics, it can be assumed that the velocity $\dot{x}(t)$ is projected at each instant on the set of allowed directions toward $T_k(x(t))$. This is true for electrical networks with diode nonlinearities, for unilateral problems in mechanics and in mathematical economics, in the study of resource allocation mechanisms. The resulting system is a projected differential inclusion, which is equivalent to ($\underline{\underline{3}}$)$'$.

Theorem 5.3. *If $K \in P_{f(c)}(\mathbb{R}^n)$ and hypothesis $H(F)_1$ (with $H = \mathbb{R}^n$) holds, then for every $\xi \in K$, $S(\xi)$ is path connected in $C(T, \mathbb{R}^n)$.*

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