# Jin Yuan Zhou
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# On subspaces of pseudo-radial spaces

Jin-Yuan Zhou

*Abstract.* It is proved that, under the Martin's Axiom, every $T_1$-space with countable tightness is a subspace of some pseudo-radial space. We also give several characterizations of subspaces of pseudo-radial spaces and conclude that being a subspace of a pseudo-radial space is a local property.

*Keywords:* pseudo-radial spaces, prime spaces, sub pseudo-radial spaces, tightness, Martin's Axiom

*Classification:* 54A35, 54B05, 54D99

## 1. Introduction.

In [1] the authors proposed the following problem: find necessary or sufficient (or both) conditions for a topological space to be a subspace of a pseudo-radial space. They also asked whether, in particular, $N \cup \{p\}$ is a subspace of a pseudo-radial space for $p \in \beta N \setminus N$. In Section 2 we give some necessary and sufficient conditions for a space to be a subspace of a pseudo-radial space. In Section 3 we prove that, under Martin's Axiom, every $T_1$ space with countable tightness is a subspace of a pseudo-radial space. Thus we partly answer the question 3.4 of [1].

**Definition 1.1.** *A subset $A$ of a topological space $X$ is called closed w.r.t. chain-net if for each $x \in X$, if there exists a transfinite sequence in $A$ converging to $x$, then $x \in A$. For any $B \subseteq X$ we denote by $clseq_X B$ the smallest subset of $X$ containing $B$ and closed w.r.t. chain-net.*

**Definition 1.2** (5). *A space is called pseudo-radial if for each $A \subseteq X$, $\overline{A} = clseq_X A$. A space is called sub pseudo-radial if it is a subspace of some pseudo-radial space.*

There was a lot of equivalent definitions of pseudo-radial spaces (see [1] and [2]).

All spaces are assumed to be $T_1$. If $\{X_\alpha : \alpha \in \Sigma\}$ is a family of spaces, we denote by $\oplus_{\alpha\in\Sigma} X_\alpha$ the topological sum of $\{X_\alpha : \alpha \in \Sigma\}$.

## 2. Some characterizations.

We start with a lemma.

**Lemma 2.1.** *Every quotient of a sub pseudo-radial space is sub pseudo-radial.*

Proof: Since every quotient of a pseudo-radial space is pseudo-radial, it is enough to see that for any class $M$ of spaces, if $M$ is closed under quotient mappings, then the class consisting of subspaces of the spaces in $M$ is also closed under quotient mappings. □

We call a space a prime space if it has only one non-isolated point. Given any space $X$ and a point $p$ in $X$, denote by $X_p$ the prime space constructed by making each point, other than $p$, isolated with $p$ retaining its original neighborhoods. We call $X_p$ the prime factor of $X$ at $p$. Obviously, each topological space is the quotient of the topological sum of all its prime factors.

**Proposition 2.2.** *For a space $X$ the following conditions are equivalent:*

(i) *$X$ is sub pseudo-radial,*
(ii) *for every $p$ in $X$, $X_p$ is sub pseudo-radial,*
(iii) *for each subset $A$ of $X$ and $q \in \overline{A}$, there exists a subset $B$ of $A$ such that $q \in \overline{B}$ and $B \cup \{q\}$ is sub pseudo-radial.*

Proof: The implication (i) → (iii) is obvious. The proof of the implication (i) → (ii) is completely the same as that of Proposition 5.1 of [3].

To prove the left two implications, let $Z = \oplus_{p\in X} X_p$ when (ii) holds and $Z = \oplus\{Y : Y \subseteq X$ and $Y$ is sub pseudo-radial$\}$ when (iii) holds. It is easy to see that, in both cases, $X$ is a quotient of $Z$ and $Z$ is sub pseudo-radial. By virtue of Lemma 2.1, $X$ is a pseudo-radial space when (ii) or (iii) holds. □

**Corollary 2.3.** *A space $X$ is sub pseudo-radial if either*

(i) *each subset of $X$ with cardinality not greater than the tightness of $X$ is sub pseudo-radial, or*
(ii) *each point of $X$ has a sub pseudo-radial neighborhood.*

## 3. Countable case.

In this section, $N$ denotes the set of natural numbers. $\beta N$ is the Čech-Stone compactification of the discrete space $N$. If $A$ and $B$ are subsets of $N$, $A \subseteq {}^*B$ means that there exists an $n$ in $N$ such that $A \setminus \{0, 1, 2, \ldots, n-1\} \subseteq B$. A family $\mathcal{A}$ of subsets of $N$ is called an almost disjoint family, shortened as a.d. family, if for any distinct elements $A_1$ and $A_2$ of $\mathcal{A}$, $A_1 \cap A_2$ is finite. We say that $\mathcal{A}$ has sfip (strong finite intersection property) if every nonempty finite subfamily of $\mathcal{A}$ has infinite intersection. We say that $B$ is a pseudo-intersection of $\mathcal{A}$ if $B \subseteq {}^*A$ for each $A$ in $\mathcal{A}$. For any set $A$, $|A|$ denotes the cardinality of $A$; $c$ denotes the cardinality of the power set $\mathcal{P}N$ of $N$.

The following lemma is well-known in set-theory (for example, see 11C of [14]).

**Lemma 3.1** (MA). *For each family $\mathcal{A}$ of subsets of $N$, if $|A| < c$ and $\mathcal{A}$ has sfip, then $\mathcal{A}$ has an infinite pseudo-intersection.*

**Theorem 3.2** (MA). *Every space with countable tightness is sub pseudo-radial.*

Proof: It is a consequence of the following Theorem 3.3 and (i) of Corollary 2.3, □

**Theorem 3.3** (MA). *Every countable space is sub pseudo-radial.*

Proof: By virtue of Proposition 2.2, we only need to prove that every countable prime space is sub pseudo-radial. Let $X = N \cup \{p\}$ be a prime space with the unique non-isolated point $p$. We prove the $X$ is sub pseudo-radial.

W.l.o.g., we assume that $\chi(p,X)=c$. Let $\mathcal{B}$ be a filter base on $N$ such that the set $\{B\cup\{p\}:B\in\mathcal{B}\}$ constitutes a local base at $p$. Let $\mathcal{A}=\mathcal{B}\cup\{A\subseteq N;\ p\in\overline{A}^X$ and $A$ contains no infinite pseudo-intersection of $\mathcal{B}\}$.

Let $\mathcal{A}=\{A_\alpha:\alpha<c\}$ be an enumeration of $\mathcal{A}$ such that for each $A\in\mathcal{A}$, the set $\{\alpha<c:A_\alpha=A\}$ is unbounded in $c$. We construct by induction an almost disjoint sequence $\mathcal{C}=\{C_\alpha:\alpha<c\}$ and a sequence $\{B_\alpha:\alpha<c\}\subseteq\mathcal{B}$ such that

(i) $\forall\,\alpha<c$, $C_\alpha\subseteq A_\alpha$ and $C_\alpha$ is infinite;
(ii) $\forall\,\beta<\alpha<c$, if $A_\beta\in\mathcal{B}$, then $C_\alpha\subseteq^* A_\beta$;
(iii) $\forall\,\alpha<c$, $C_\alpha\cap B_\alpha=\emptyset$.

Assume $\alpha<c$ and we have constructed $\{C_\beta:\beta<\alpha\}$ and $\{B_\beta:\beta<\alpha\}$ satisfying (i) to (iii). We construct $C_\alpha$, $B_\alpha$ as follows.

**Case I.** $A_\alpha\notin\mathcal{B}$. Since $p\in\overline{A}_\alpha^X$, we apply Lemma 3.1 on the family

$$\mathcal{B}'=\{B_\beta\cap A_\alpha:\beta<\alpha\}\cup\{A_\beta\cap A_\alpha:\beta\le\alpha\ \text{ and }\ A_\beta\in\mathcal{B}\}.$$

We obtain an infinite subset $A$ of $A_\alpha$ which is a pseudo-intersection of $\mathcal{B}'$. Since $A$ cannot be a pseudo-intersection of $\mathcal{B}$, there is a $B\in\mathcal{B}$ such that $A\setminus B$ is infinite. Let $C_\alpha=A\setminus B$ and $B_\alpha=B$.

**Case II.** $A_\alpha\in\mathcal{B}$. Let $\mathcal{B}'$ as in the Case I. Since $X$ is a $T_1$ space and $|\mathcal{B}'|<c=\chi(p,X)$, there exists a $B^*\in\mathcal{B}$ such that for each finite subfamily $\mathcal{B}'$ of $\mathcal{B}$, $\bigcap_{B\in\mathcal{B}'}B\setminus B^*$ is infinite. Therefore the family $\mathcal{F}=\{B\setminus B^*:B\in\mathcal{B}\}$ has the sfip. Again by Lemma 3.1, we obtain an infinite $A\subseteq A_\alpha\setminus B^*$ which is a pseudo-intersection of $\mathcal{B}'$. Let $C_\alpha=A$ and $B_\alpha=B^*$. Thus we have finished the induction.

Now we construct a Hausdorff pseudo-radial space $Y$ containing $X$ as a subspace. Let $Y=X\cup(c\times\{0\})$. We define a topology on $Y$ as follows. The set $N$ is open discrete in $Y$. For each $\alpha<c$, let $\{C_\alpha\setminus n\cup\{(\alpha,0)\}:n\in N\}$ be a local base at the point $(\alpha,0)$. For the point $p$, let $\{U(A_\alpha):A_\alpha\in\mathcal{B},\ \alpha<c\}$ be a local base, where $U(A_\alpha)=\{p\}\cup A_\alpha\cup\{(\beta,0):\alpha<\beta<c\}$. It is easy to see that the above topology is well-defined and that $X$ is a subspace of $Y$. $Y$ is Hausdorff because of the above property (iii) and the fact that, for each $B_\alpha$, the set $\{\beta<c:A_\beta=B_\alpha\}$ is unbounded in $c$. We are left to check that $Y$ is pseudo-radial. Let $E\subseteq Y$ and $y\in\overline{E}^Y$. To avoid the trivialities, we assume $y=p$ and $E\subseteq N$. Then $p\in\overline{E}^X$. If $E\in\mathcal{A}$, then $\{(\alpha,0):\alpha<c$ and $A_\alpha=E\}\subseteq clseq_Y E$. Since the set $\{\alpha<c:A_\alpha=E\}$ is unbounded in $c$, $p\in clseq_Y\{(\alpha,0):A_\alpha=E\}$. Thus $p\in clseq_Y E$. If $E\notin\mathcal{A}$, then there exists an infinite subset $E'$ of $E$ which is a pseudo-intersection of $\mathcal{B}$. But this obviously implies that $p\in clseq_X E$. Therefore $p\in clseq_Y E$. We are done. $\square$

**Remark.** For any $p\in\beta N\setminus N$, it is easy to see that $N\cup\{p\}$ is not pseudo-radial. But by Theorem 3.2, we see that it is sub pseudo-radial under the Martin's Axiom. Thus we partly answer the question 4 of [1].

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Department of Mathematics and Statistics, York University, North York, Ontario M3J 1P3, Canada

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