Mohammad Ismail; Andrzej Szymański
Short proofs of two theorems in topology

Commentationes Mathematicae Universitatis Carolinae, Vol. 34 (1993), No. 3, 539--541

Persistent URL: http://dml.cz/dmlcz/118610

Comment.Math.Univ.Carolin. 34,3 (1993)539–541

Short proofs of two theorems in topology

M. Ismail, A. Szymanski

Abstract. We present short and elementary proofs of the following two known theorems in General Topology:

(i) [H. Wicke and J. Worrell] A T_1 weakly $\delta\theta$-refinable countably compact space is compact.
(ii) [A. Ostaszewski] A compact Hausdorff space which is a countable union of metrizable spaces is sequential.

Keywords: countably compact, initially κ-compact, weakly $\delta\theta$-refinable, κ-refinable, sequential

Classification: 54D30, 54D20, 54D55

Throughout this note, κ denotes an infinite cardinal number and all topological spaces are assumed to be T_1.

A space X is called *κ-refinable* if every open cover γ of X has an open refinement λ such that $\lambda = \bigcup_{\alpha<\kappa} \lambda_\alpha$ and for each $x \in X$, there exists $\alpha < \kappa$ such that $1 \leq |\{V \in \lambda_\alpha : x \in V\}| \leq \kappa$. An example of a (hereditary) κ-refinable space is any space that can be represented as a union of $\leq \kappa$ metrizable subspaces.

The ω_0-refinable spaces are the same as weakly $\delta\theta$-refinable spaces, the spaces introduced by H. Wicke and J. Worrell. In 1976 they proved that countably compact weakly $\delta\theta$-refinable spaces are compact [WW]. A slightly different proof of this theorem appears in [B]. See also [A] for a generalization of weak $\delta\theta$-refinability and yet another proof of this theorem. Below, we present a proof which is shorter and much more elementary than these proofs. Moreover, the theorem is more general than that of Wicke and Worrell's.

Recall that a topological space is called *initially κ-compact* if every open cover of it of cardinality $\leq \kappa$ has a finite subcover. Note that 'initially ω_0-compact' is the same as 'countably compact'. The reader is referred to [S] for a survey of initially κ-compact spaces.

Theorem 1. *An initially κ-compact κ-refinable space is compact.*

Proof: Assume the contrary, and let X be an initially κ-compact κ-refinable space which is not compact. Let γ be a maximal open cover of X without a finite subcover. Let $\lambda = \bigcup_{\alpha<\kappa} \lambda_\alpha$ be an open refinement of γ which witnesses the κ-refinability of X. For each $\alpha < \kappa$, and for each $x \in X$, let $\lambda_\alpha(x) = \{V \in \lambda_\alpha : x \in V\}$ and $X_\alpha = \{x \in X : 1 \leq |\lambda_\alpha(x)| \leq \kappa\}$. Then $X = \bigcup_{\alpha<\kappa} X_\alpha$. Since X is initially κ-compact, there exists β such that X_β cannot be covered by κ or less members of γ. Let $W = \bigcup \lambda_\beta$. Since $X_\beta \subseteq W$, $W \notin \gamma$. By the maximality of γ, there exists $U \in \gamma$ such that $X = W \cup U$. Then $X_\beta \setminus U$ cannot be covered by κ or less members of γ.

By induction, we choose a sequence $x_1, x_2, \dots$ of points in $X_\beta \setminus U$ as follows: let $x_1 \in X_\beta \setminus U$ be arbitrary. If $x_1, \dots, x_n$ have already been chosen, then, since $\left|\bigcup_{i=1}^n \lambda_\beta(x_i)\right| \le \kappa$, $X_\beta \setminus U$ is not contained in $\bigcup\left(\bigcup_{i=1}^n \lambda_\beta(x_i)\right)$. Choose $x_{n+1} \in (X_\beta \setminus U) \setminus \bigcup\left(\bigcup_{i=1}^n \lambda_\beta(x_i)\right)$.

Let $S = \{x_1, x_2, \dots\}$. Then $S \subseteq X \setminus U$ and, since $X \setminus U \subseteq W$, no point of $X \setminus U$ is a limit point of S. This is a contradiction, since $X \setminus U$ is countably compact. □

A topological space X is called *sequential* if every nonclosed subset A of X contains a sequence converging to a point in $X \setminus A$.

In [O], A. Ostaszewski proved that a countably compact regular space which can be represented as a union of countably many metrizable spaces is sequential. The proof consists of about four printed pages. Below, we present a short proof based on the Wicke-Worrell Theorem.

Theorem 2. *A countably compact regular space which can be represented as a countable union of metrizable spaces is sequential* (*and compact*).

Proof: Let X be a countably compact regular space, and let $X = \bigcup_{i=1}^\infty X_i$, where each X_i is metrizable. Let A be a non-closed subset of X. Since X is hereditary ω_0-refinable (i.e. hereditarily weakly $\delta\theta$-refinable), A cannot be countably compact. Therefore, there exists a sequence $S = \{x_1, x_2, \dots\}$ in A which has no cluster point in A. Let $Y = \overline{S} \setminus S$. Since Y is non-empty and compact, $Y \cap X_i$ is not nowhere dense in Y , for some i. Hence, $Y \cap X_i$ contains a point which has countable character in Y and thus in $\overline{S}$ as well. Therefore, S contains a subsequence converging to a point in Y. □

The last part of the above proof shows that any countably compact regular space which can be represented as a union of countably many first countable spaces contains a point of countable character. This is essentially the same as Theorem 3 of [O] attributed to M.E. Rudin and K. Kunen there. We have a much stronger theorem of this type which we prove by a different method.

Theorem 3. *Let X be a regular initially ω_1-compact space which can be represented as a union of $\le \omega_1$ subspaces of countable pseudocharacter. Then every non-empty G_δ subset of X contains a point of countable character in X.*

Proof: Let $X = \bigcup\{X_\alpha : \alpha < \omega_1\}$, where each X_α has countable pseudocharacter. Let U be a non-empty G_δ subset of X and suppose that no point of U has countable character in X. By induction, we choose a decreasing sequence $\{F_\alpha : \alpha < \omega_1\}$ of non-empty closed G_δ subsets of X as follows:

If $U \cap X_0 = \emptyset$, let F_0 be an arbitrary non-empty closed G_δ subset of X such that $F_0 \subseteq U$. If $U \cap X_0 \neq \emptyset$, let $x \in U \cap X_0$. Then there exists a G_δ subset V of X such that $V \cap X_0 = \{x\}$. Since X is countably compact and x does not have countable character in X, $\{x\}$ is not G_δ in X. Therefore, $\emptyset \neq U \cap V \neq \{x\}$. Let

F_0 be a non-empty closed G_δ subset of X such that $F_0 \subseteq (U \cap V) \setminus \{x\}$. Then $F_0 \cap X_0 = \emptyset$.

If $\beta < \omega_1$, and for each $\alpha < \beta$, we have chosen F_α, then, since $\bigcap_{\alpha<\beta} F_\alpha$ is a G_δ subset of X, by repeating the above argument with $\bigcap_{\alpha<\beta} F_\alpha$ in place of U and X_β in place of X_0 we can find a non-empty closed G_δ subset F_β of X such that $F_\beta \subseteq \bigcap_{\alpha<\beta} F_\alpha$ and $F_\beta \cap X_\beta = \emptyset$.

Let $F = \bigcap\{F_\alpha : \alpha < \omega_1\}$. Since X is initially ω_1-compact, $F \neq \emptyset$. On the other hand, since $F \cap X_\alpha = \emptyset$, for each $\alpha < \omega_1$, $F = \emptyset$. This is a contradiction. □

References

[A] Arhangel'skiĭ A.V., *The star method, new classes of spaces and countable compactness*, Soviet Math. Dokl. **21** (1980), 550–554.

[B] Burke D.K., *Covering properties*, Handbook of Set Theoretic Topology, North Holland, 1984, pp. 347–422.

[O] Ostaszewski A.J., *Compact σ-metric Hausdorff spaces are sequential*, Proc. Amer. Math. Soc. **68** (1978), 339–343.

[S] Stephenson R.M., Jr., *Initially κ-compact and related spaces*, Handbook of Set Theoretic Topology, North Holland (1984), pp. 603–632.

[WW] Wicke H.H., Worrell J.M., Jr., *Point countability and compactness*, Proc. Amer. Math. Soc. **55** (1976), 427–431.

Department of Mathematics, Slippery Rock University, Slippery Rock, PA 16057, USA

(*Received* January 11, 1993)