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Strong unicity criterion in some space of operators

GRZEGORZ LEWICKI

Abstract. Let X be a finite dimensional Banach space and let $Y \subset X$ be a hyperplane. Let $\mathcal{L}_Y = \{L \in \mathcal{L}(X, Y) : L|_Y = 0\}$. In this note, we present sufficient and necessary conditions on $L_0 \in \mathcal{L}_Y$ being a strongly unique best approximation for given $L \in \mathcal{L}(X)$. Next we apply this characterization to the case of $X = l_\infty^n$ and to generalization of Theorem I.1.3 from [12] (see also [13]).

Keywords: best approximation, strongly unique best approximation, approximation in spaces of linear operators

Classification: Primary 41A65, 41A52, 41A35

0. Introduction.

Let W be a normed linear space and let $V \subset W$ be its nonempty subset. An element $v \in V$ is called a best approximation to $w \in W$ iff

$$(0.1) \quad \|w - v\| = \text{dist}(w, V) = \inf\{\|w - y\|; y \in V\}.$$

If an element v satisfies additionally

$$(0.2) \quad \|w - y\| \geq \|w - v\| + r \cdot \|y - v\| \quad \text{with a constant } r > 0 \\ \text{independent of } y \in V,$$

then v is said to be a strongly unique best approximation (briefly SUBA) to $w \in V$.

The theory of strong uniqueness has its origin in the following result of Newman and Shapiro [11]. Given a compact Hausdorff space T , let $C(T, \mathbb{K})$ denote the Banach space of either complex ($\mathbb{K} = \mathbb{C}$) or real-valued ($\mathbb{K} = \mathbb{R}$) continuous functions on V . If V is a Haar subspace of $C(T, \mathbb{K})$, then for every $w \in C(T, \mathbb{K})$, one can find a constant $r > 0$ such that the best approximation $v \in V$ satisfies one of the following inequalities:

$$(0.3) \quad \|w - y\| \geq \|w - v\| + r \cdot \|y - v\|, \quad \text{for } y \in V$$

if $\mathbb{K} = \mathbb{R}$, and

$$(0.4) \quad \|w - y\|^2 \geq \|w - v\|^2 + r \cdot \|y - v\|^2, \quad \text{for } y \in V$$

in the complex case.

The significance of this notion can be illustrated by Cheney's observation that strong unicity of an optimal element yields the continuity of metric projection (see [6]). One can see that the proof of the convergence of the Remez algorithm depends, in fact, on strong unicity (for an extended version see [9]). For more precise information about strong unicity the reader is referred to [3], [4], [8], [11], [14], [15].

In this note we will investigate strong unicity in the case $W = \mathcal{L}(X)$, the space of all linear operators going from a finite dimensional real Banach space X into itself, and $V = \mathcal{L}_Y(X, Y) = \{L \in \mathcal{L}(X, Y) : L|_Y = 0\}$ (we will write \mathcal{L}_Y for brevity), where $Y \subset X$ is a hyperplane. We characterize strong unicity in terms of the functionals from X^* or Y^* , which is more convenient for applications. Next we apply this characterization to the case of X being an arbitrary three dimensional Banach space and to $X = l_\infty^n$. In particular, we generalize Theorem I.1.3 from [12] (see also [13]) and Theorem 2.5 b) from [10].

Now we introduce some notations which will be used in this note. By S_X we will denote the unit sphere in a Banach space X . The symbol $\text{ext } S_X$ stands for the set of all extremal points of S_X . Given $L \in \mathcal{L}(X)$, we write $\mathcal{P}_Y(L) = \{L_0 \in \mathcal{L}_Y : \|L - L_0\| = \text{dist}(L, \mathcal{L}_Y)\}$. In this note, if nothing special is assumed, X will denote a finite dimensional real Banach space and f a functional from $X^* \setminus \{0\}$. If $Y \subset X$ is a linear subspace and $A \subset X^*$ then $A|_Y$ stands for a set of all restrictions of functionals from A . In the sequel we will use the following

Theorem 0.1 (see [10, Theorem 2.3]). *Assume X is a reflexive space and Y is a Banach space both over the same field \mathbb{K} ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$). Denote by $\mathcal{K}(X, Y)$ the space of all compact operators going from X into Y and let $\mathcal{V} \subset \mathcal{K}(X, Y)$ be a convex set. For given $K \in \mathcal{K}(X, Y)$ and $V \in \mathcal{V}$ put*

$$(0.5) \quad \text{crit}_Y^*(K - V) = \{h \in \text{ext } S_{Y^*} : \|h \circ (K - V)\| = \|K - V\|\}$$

and for every $h \in \text{crit}_Y^*(K - V)$ define

$$(0.6) \quad A_h = \{x \in \text{ext } S_X : h(K - V)x = \|K - V\|\}.$$

Then we have:

- (a) $V \in \mathcal{P}_{\mathcal{V}}(K)$ (the set of all best approximants to K in \mathcal{V}) if and only if for every $U \in \mathcal{V}$ there exists $h \in \text{crit}_Y^*(K - V)$ with $\inf\{re(h(U - V)x) : x \in A_h\} \leq 0$.
- (b) V is a SUBA to K in \mathcal{V} with a constant $r > 0$ if and only if for every $U \in \mathcal{V}$ there exists $h \in \text{crit}^*(K - V)$ with $\inf\{re(h(U - V)x) : x \in A_h\} \leq r \cdot \|U - V\|$.

I. The main result.

We start with two preliminary remarks.

Remark 1.1. For $L \in \mathcal{L}(X)$ let us set

$$(1.1) \quad \text{crit}(L) = \{x \in S_X : \|Lx\| = \|L\|\}.$$

Assume $L_0 \in \mathcal{P}_Y(L)$, $Y = \ker f$, $\|f\| = 1$ and $\|L - L_0\| > \|L|_Y\|$. Put

$$(1.2) \quad C_{L-L_0} = \{x \in \text{crit}(L - L_0) : f(x) > 0\}.$$

Then C_{L-L_0} is a nonempty closed set, $C_{L-L_0} \cap -C_{L-L_0} = \emptyset$ and $C_{L-L_0} \cup -C_{L-L_0} = \text{crit}(L - L_0)$.

PROOF: It is clear that the set $A = \{x \in \text{crit}(L - L_0) : f(x) \geq 0\}$ is closed. Since $\text{dist}(L, \mathcal{L}_Y) > \|L|_Y\|$ and Y is a hyperplane,

$$A = \{x \in \text{crit}(L - L_0); f(x) > 0\},$$

which proves that C_{L-L_0} is closed. The fact $\text{crit}(L - L_0) \cap Y = \emptyset$ implies immediately that $C_{L-L_0} \cup -C_{L-L_0} = \text{crit}(L)$. By (1.2) $C_{L-L_0} \cap -C_{L-L_0} = \emptyset$. \square

Remark 1.2. Let $L \in \mathcal{L}(X)$, $\text{dist}(L, \mathcal{L}_Y) > \|L|_Y\|$, $L_0 \in \mathcal{L}_Y$. Define

$$(1.3) \quad D_{L-L_0} = \{h \in \text{crit}_X^*(L - L_0) : C_{L-L_0} \cap A_h \neq \emptyset\}$$

(see (0.6)) and if $L \in \mathcal{L}(X, Y)$,

$$(1.4) \quad D_{L-L_0}^Y = \{h \in \text{crit}_Y^*(L - L_0) : C_{L-L_0} \cap A_h \neq \emptyset\}.$$

Then D_{L-L_0} ($D_{L-L_0}^Y$ resp.) is a compact set, $D_{L-L_0} \cap -D_{L-L_0} = \emptyset$ ($D_{L-L_0}^Y \cap -D_{L-L_0}^Y = \emptyset$ resp.).

PROOF: Assume $h \in \text{cl}(D_{L-L_0})$ and let $\{h_n\} \subset D_{L-L_0}$, $h_n \rightarrow h$. By (1.3), for every $n \in \mathbb{N}$ there exists $x_n \in C_{L-L_0} \cap A_{h_n}$, i.e. $h_n(L - L_0)x_n = \|L - L_0\|$. Passing to the subsequence, if necessary, we may assume $x_n \rightarrow x$. By Remark 1.1, $x \in C_{L-L_0}$. Note that $h(L - L_0)x = h_n(L - L_0)x + (h - h_n)(L - L_0)x = h_n(L - L_0)x_n + h_n(L - L_0)(x - x_n) + (h - h_n)(L - L_0)x$. Since the last two terms tend to 0 as $n \rightarrow \infty$, $h(L - L_0)x = \|L - L_0\|$ and consequently $x \in A_h$. Since $x \in C_{L-L_0}$, by (1.3) $h \in D_{L-L_0}$. Note that $\text{dist}(L, \mathcal{L}_Y) > \|L|_Y\|$ implies $D_{L-L_0} \cap -D_{L-L_0} = \emptyset$. The proof for the set $D_{L-L_0}^Y$ goes on in the same manner, so we omit it. \square

Now we state the main result of this note.

Theorem 1.3. Assume $L \in \mathcal{L}(X)$ and let $Y = \ker f$, $\|f\| = 1$. Assume furthermore that $\text{dist}(L, \mathcal{L}_Y) > \|L|_Y\|$ and let $L_0 \in \mathcal{L}_Y$. Then the following conditions are equivalent:

- (a) L_0 is a SUBA to L in \mathcal{L}_Y ($L_0 \in \mathcal{P}_Y$ resp.),
- (b) $0 \in \text{int conv } D_{L-L_0}|_Y$ ($0 \in \text{conv } D_{L-L_0}|_Y$ resp.).

PROOF: Assume L_0 is a SUBA to L in \mathcal{L}_Y and let $0 \notin \text{int conv } D_{L-L_0}|_Y$. It means that there exists $\psi \in Y^{**}$ with $\psi(h) \geq 0$ for every $h \in D_{L-L_0}|_Y$ (we may assume $\|\psi\| = 1$). Since Y is finite dimensional, $\psi = y$ for some $y \in S_Y$. Define $L_1 = f(\cdot) \cdot y$ and note that $L_1 \in \mathcal{L}_Y$. By (1.3) and Remark 1.2, for every $h \in D_{L-L_0}$ we have

$$\begin{aligned} \inf\{h(L_1x) : x \in A_h\} &= \inf\{f(x) \cdot h(y) : x \in A_h\} = \\ &= h(y) \cdot \inf\{f(x) : x \in A_h\} \geq 0 > -r \cdot \|L_1\| \quad \text{for every } r > 0. \end{aligned}$$

By Theorem 0.1 L_0 is not a SUBA to L in $\mathcal{L} \mid_Y$; a contradiction. Since by Remark 1.2 the set D_{L-L_0} is compact and consequently $\text{conv } D_{L-L_0}$ is also compact, the proof of the second case goes on the same line.

To prove the converse, let us define a function $g : S_Y \rightarrow \mathbb{R}$ by

$$(1.5) \quad g(y) = \inf\{g_h(y) : h \in D_{L-L_0}\} \quad \text{for } y \in S_Y;$$

where $g_h(y) = \inf\{f(x) \cdot h(y) : x \in A_h\}$. Note that the function $S_Y \ni y \rightarrow f(x) \cdot h(y)$ is continuous and consequently the functions g_h and g are upper-semicontinuous.

Now assume $0 \in \text{int conv } D_{L-L_0} \mid_Y$. It means that for every $y \in S_Y$ there exists $h \in D_{L-L_0}$ with $h(y) < 0$. (If no, then $D_{L-L_0} \mid_Y \subset \{h \in Y^* : h(y) \geq 0\}$ for some $y \in S_Y$ and consequently $\text{int conv } D_{L-L_0} \mid_Y \subset \{h \in Y^* : h(y) > 0\}$. But $0 \in \text{int conv } D_{L-L_0}$; a contradiction.) Since $\text{dist}(L, \mathcal{L}_Y) > \|L \mid_Y\|$ and Y is a hyperplane, for every $y \in S_Y$, $g(y) < 0$. Since g is upper-semicontinuous, the value $\gamma = \max\{g(y) : y \in S_Y\}$ is attained in some point $y_0 \in S_Y$ and consequently $\gamma < 0$. We show that L_0 is a SUBA to L in \mathcal{L}_Y with $r = -\gamma$. To do this, fix $L_1 \in \mathcal{L}_Y \setminus \{0\}$. It is clear that $L_1 = f(\cdot) \cdot y_1$ for some $y_1 \in Y \setminus \{0\}$. Put $y_2 = y_1/\|y_1\|$, fix $\varepsilon > 0$ and take $h \in D_{L-L_0}$ with $g_h(y_2) < g(y_2) + \varepsilon$. Note that

$$\begin{aligned} g_h(y_2) &= \inf\{f(x) \cdot h(y_2) : x \in A_h\} = \inf\{h(L_1 x)/\|y_1\|; x \in A_h\} \leq \\ &\leq g(y_2) + \varepsilon \leq -r + \varepsilon, \quad \text{which gives } \inf\{h(L_1 x) : x \in A_h\} \leq -(r - \varepsilon) \cdot \|L_1\|. \end{aligned}$$

Following Theorem 0.1 and Remark 1.2, L_0 is SUBA to L in \mathcal{L}_Y with $r - \varepsilon$ for every $\varepsilon > 0$ and consequently with r . The proof is complete. \square

Remark 1.4. If $L \in \mathcal{L}(X, Y)$ then the set D_{L-L_0} in Theorem 1.3 can be replaced by $D_{L-L_0}^Y$ (see (1.4)).

As an immediate consequence of Theorem 1.3 we get

Corollary 1.5. Assume $L \in \mathcal{L}(X)$, $L_0 \in \mathcal{P}_Y(L)$, $\|L - L_0\| > \|L \mid_Y\|$. Then the set $D_{L-L_0} \mid_Y$ is linearly dependent. If $L \in \mathcal{L}(X, Y)$, the same holds for $D_{L-L_0}^Y$.

Reasoning as in [10, Theorem 2.5] we may show

Remark 1.6. The constant r defined in Theorem 1.3 is the best possible.

Now we will point out when the assumption $\text{dist}(L, \mathcal{L}_Y) > \|L \mid_Y\|$ is fulfilled.

Remark 1.7. Assume X is a Banach space and let $Y \subset X$ be its complemented subspace. Let $\mathcal{P}(X, Y) = \{P \in \mathcal{L}(X, Y) : P \mid_Y = \text{id}\}$. Take $P_0 \in \mathcal{P}(X, Y)$ and note that $\text{dist}(P_0, \mathcal{L}_Y) = \inf\{\|P\| : P \in \mathcal{P}(X, Y)\} = \lambda(X, Y)$.

In many cases of hyperplanes, $\text{dist}(P_0, \mathcal{L}_Y) > \|P_0 \mid_Y\| = 1$ (see e.g. [2], [5]).

It is well known (see e.g. [12]) that if X is not a Hilbert space then there exists a hyperplane Y in X satisfying $\lambda(X, Y) > 1$. If $\text{dist}(P_0, \mathcal{L}_Y) > 1$ then it is easy to show that $\text{dist}(L, \mathcal{L}_Y) > \|L \mid_Y\|$ if $\|L - P_0\| < \text{dist}(P_0, \mathcal{L}_Y) - 1$.

Now we show an estimation from above of the number $\text{dist}(L, \mathcal{L}_Y)$.

Proposition 1.8. *Assume X is a Banach space and let Y be its complemented subspace. Then for every $L \in \mathcal{L}(X, Y)$*

$$\|L|_Y\| \leq \text{dist}(L, \mathcal{L}_Y) \leq \lambda(X, Y) \cdot \|L|_Y\|.$$

PROOF: Fix $L \in \mathcal{L}(X, Y)$ and $\varepsilon > 0$. Take $P_\varepsilon \in \mathcal{P}(X, Y)$ with $\|P_\varepsilon\| < \lambda(X, Y) + \varepsilon$ and put $L_\varepsilon = L \circ (I - P_\varepsilon)$. It is clear that $L_\varepsilon \in \mathcal{L}_Y$. Compute,

$$\|L - L_\varepsilon\| = \|L - L \circ (I - P_\varepsilon)\| = \|L \circ P_\varepsilon\| \leq \|L|_Y\| \cdot \|P_\varepsilon\|,$$

which gives the desired result. \square

Corollary 1.9. *Assume that $\lambda(X, Y) = 1$. Then*

$$\text{dist}(L, \mathcal{L}_Y) = \|L|_Y\| \quad \text{for every } L \in \mathcal{L}(X, Y).$$

In particular if there exists $P_0 \in \mathcal{P}(X, Y)$ with $\|P_0\| = 1$, then the operator $L_0 = L \circ (I - P_0) \in P_Y(L)$.

Since in the case when Y is a hyperplane we have $\lambda(X, Y) \leq 2$ (for more precise results see [1], [7], [12, p. 84], we immediately get

Corollary 1.10. *Assume $Y \subset X$ is a hyperplane. Then*

$$\|L|_Y\| \leq \text{dist}(L, \mathcal{L}_Y) \leq 2 \cdot \|L|_Y\| \quad \text{for every } L \in \mathcal{L}(X, Y).$$

II. Applications.

Now we apply Theorem 1.3 to generalize Theorem I.1.3 from [12] (see also [13]).

Theorem 2.1. *Assume X is a three dimensional Banach space and let $Y \subset X$ be a hyperplane. Assume furthermore $L \in \mathcal{L}(X, Y)$, $\text{dist}(L, \mathcal{L}_Y) > \|L|_Y\|$. Then there exists $L_0 \in \mathcal{L}_Y$ which is a SUBA to L in \mathcal{L}_Y .*

PROOF: Since \mathcal{L}_Y is a finitely dimensional linear space, the set $\mathcal{P}_Y(L)$ is nonempty. Take an arbitrary $L_0 \in \mathcal{P}_Y(L)$. By Theorem 1.3 and Remark 1.4 it is sufficient to show that $0 \in \text{int conv } D_{L-L_0}^Y$ (see 1.4). Assume on the contrary that it is not true. Following Theorem 1.3, $0 \in \text{conv } D_{L-L_0}^Y$. Since $\dim Y = 2$, $0 = \alpha \cdot h_1 + (1 - \alpha) \cdot h_2$, where $h_1, h_2 \in D_{L-L_0}^Y$ and $\alpha \in (0, 1)$. Since $\|h_1\| = \|h_2\| = 1$, we easily get $\alpha = 1/2$. Consequently $h_1 = -h_2$ which gives $h_1 \in D_{L-L_0}^Y \cap -D_{L-L_0}^Y$; a contradiction with Remark 1.2. \square

Remark 2.2. *The assumption $\text{dist}(L, \mathcal{L}_Y) > \|L|_Y\|$ in Theorem 2.1 is essential. Take e.g. $X = l_\infty^3$, $Y = \ker f$, $f = (1/2, 1/2, 0)$. It is easy to check that the operators $P_1 = \text{Id} - f(\cdot) \cdot (2, 0, 0)$ and $P_2 = \text{Id} - f(\cdot) \cdot (0, 2, 0) \in \mathcal{P}(X, Y)$, $P_1 \neq P_2$, $\|P_1\| = \|P_2\| = 1$. Consequently the set $P_Y(P_1) \supset \{0, P_1 - P_2\}$ and strong unicity does not hold.*

Remark 2.3. *The assumption $\dim X = 3$ in Theorem 2.1 is essential. Take e.g. $X = l_\infty^4$, $Y = \ker f$, $f = (1/3, 1/3, 1/3, 0)$. It is well known (see [5] or [12]) that $\lambda(X, Y) = 4/3$. Take $P_0 \in \mathcal{P}(X, Y)$ with $\|P_0\| = 4/3$ (the formula for such a projection is given for example in [12, p. 104]). Then $\text{dist}(P_0, \mathcal{L}_Y) = \lambda(X, Y) = 4/3 > 1 = \|P_0|_Y\|$. By Theorem 2.5 b) of [10], 0 is not a SUBA to P_0 in \mathcal{L}_Y .*

Now we use Theorem 1.3 to extend Theorem 2.5 b) from [10].

Theorem 2.4. *Assume $X = l_\infty^n$, $Y = \ker f$, $\|f\|_1 = 1$. Let $L \in \mathcal{L}(X)$ and let $\text{dist}(L, \mathcal{L}_Y) > \|L|_Y\|$ (by Theorem 1 from [5] and Remark 1.7 such operators exist if and only if $|f_i| < 1/2$ for $i = 1, \dots, n$). If $|f_i| > 0$ for $i = 1, \dots, n$ then there exists $L_0 \in \mathcal{L}_Y$ which is a SUBA to L in \mathcal{L}_Y .*

PROOF: Since \mathcal{L}_Y is a finitely dimensional space the set $\mathcal{P}_Y(L)$ is nonempty. Fix $L_0 \in \mathcal{P}_Y(L)$ and let $D_{L-L_0} = \{\phi_1, \dots, \phi_k\}$ where $\phi_i = \mp e_{j(i)}$ for $i = 1, \dots, k$. By Theorem 1.3, $0 \in \text{conv } D_{L-L_0}|_Y$. Hence $0 = \sum_{i=1}^l \lambda_i \cdot \phi_i$, $l \leq k$, $\lambda_i > 0$, $\sum_{i=1}^l \lambda_i = 1$. Since $\dim Y = n - 1$, by Carathéodory's Theorem we can assume $l \leq n$. We will show that $l = n$. To do this, by Corollary 1.5, it is sufficient to show that for each $i \in \{1, \dots, n\}$ the set $E_i = \{e_1, \dots, e_{i-1}, e_{i+1}, \dots, e_n\}$ is total over Y . So assume $\sum_{i=1}^n f_i \cdot y_i = 0$ and $e_j(y) = 0$ for $j \neq i$. It means that $y_j = 0$ for $j \neq i$ and $f_i \cdot y_i = 0$. Since $f_i \neq 0$, $y_i = 0$. Consequently $l = n$ and $0 \in \text{int conv } D_{L-L_0}|_Y$. By Theorem 1.3, L_0 is a SUBA to L in \mathcal{L}_Y . The proof is complete. \square

Note that Remark 2.3 shows that the assumption $f_i \neq 0$ for $i = 1, \dots, n$ in Theorem 2.4 is essential.

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