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Commentationes Mathematicae Universitatis Carolinae, Vol. 33 (1992), No. 1, 173--179

Persistent URL: `http://dml.cz/dmlcz/118483`

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Abstract. Let $(e_\beta : \mathbf{Q} \to Y_\beta)_{\beta\in\mathbf{Ord}}$ be the large source of epimorphisms in the category **Ury** of Urysohn spaces constructed in [2]. A sink $(g_\beta : Y_\beta \to X)_{\beta\in\mathbf{Ord}}$ is called natural, if $g_\beta \circ e_\beta = g_{\beta'} \circ e_{\beta'}$ for all $\beta, \beta' \in \mathbf{Ord}$. In this paper natural sinks are characterized. As a result it is shown that **Ury** permits no $(Epi, \mathcal{M})$-factorization structure for arbitrary (large) sources.

Keywords: epimorphism, Urysohn space, cointersection, factorization, natural sink, periodic, cowellpowered, ordinal

Classification: 18A20, 18A30, 18B30, 54B30, 54C10, 54D10, 54D35, 54G20

Introduction.

In [2] a large source $(e_\beta : \mathbf{Q} \to Y_\beta)_{\beta\in\mathbf{Ord}}$ of epimorphisms in **Ury** was constructed, showing that **Ury** is not cowellpowered. The purpose of this paper is twofolded:

(a) Every natural sink $(g_\beta : Y_\beta \to X)_{\beta\in\mathbf{Ord}}$ is defined uniquely by $g_1 : \mathbf{Q} \to X$. Y_β can be as large as might be required. How does g_β look? There are rare instances in General Topology where a smallness condition has overall consequences, e.g. the arbitrary product of separable spaces fulfills the countable chain condition.

(b) Because of non-cowellpoweredness, some categorical theorems should not be applicable in **Ury**. The investigation of sinks $(g_\beta : Y_\beta \to X)_{\beta\in\mathbf{Ord}}$ have as a result the non-existence for any $\mathcal{M}$ of a $(Epi, \mathcal{M})$-factorization structure for (large) sources.

Notation. $\mathbf{Q}$ ($\mathbf{Q}^+ = \mathbf{Q}^+ \cup \{0\}$) are the (positive) rationals. $\mathbf{R}$, $\mathbf{N} = \mathbf{N} \cup \{0\}$ are the real and the natural numbers, respectively. ω_0 is the first infinite ordinal. w is a fixed positive irrational number. ϵ, δ are real numbers > 0. h, l, m, n are elements of $\mathbf{N}$. **Ord** is the class of ordinal numbers. $\alpha, \beta, \gamma, \kappa, \lambda, \xi, \tau$ are ordinal numbers. $[0, 1)$, $[-1, +1]$ are as usual intervals of real numbers. $d(x, y)$ is the euclidean distance of $x, y \in \mathbf{R}$ or of $x, y \in \mathbf{R} \times \mathbf{R}$. If $B \subseteq \mathbf{R}$ (or $\subseteq \mathbf{R} \times \mathbf{R}$), then $d(x, B) := \inf\{d(x, b) | \, b \in B\}$. $U(x, \epsilon)$ is an ϵ-neighbourhood, taken in $\mathbf{R}$, if $x \in \mathbf{R}$; taken in $\mathbf{R} \times \mathbf{R}$ if $x \in \mathbf{R} \times \mathbf{R}$. Every ordinal τ has a unique representation $\tau = \lambda + n$, where λ is a limit ordinal and n is a finite ordinal (natural number). This representation we will use often. **Top**, **Ury**, $\mathbf{T_3}$ are the categories of topological, Urysohn, and regular T_1 spaces, respectively. Recall that a topological space is

*Grants from the Foundation for Research Development and the University of Cape Town to the Categorical Topology Research Group are acknowledged.

called Urysohn, if distinct points have disjoint closed neighbourhoods. clA is, as usual, the topological closure of $A \subseteq X$, $clA =: \overline{A}$. If $(X, \mathcal{X}) \in \mathbf{Top}$ and $A \subseteq X$, then $cl_\theta A := \{x \in X | x \in U \in \mathcal{X} \Rightarrow clU \cap A \neq \emptyset\}$, $cl_{\theta\theta} A := \bigcap\{cl_\theta clU | A \subseteq U \in \mathcal{X}\}$. $cl_\theta^{\omega_0} A := \bigcup_{\mathbf{N}} cl_\theta^n A$, where $cl_\theta^0 A := A$ and $cl_\theta^{n+1} A := cl_\theta cl_\theta^n A$. $\triangle := \{(x,x)|x \in X\}$.

Definition 1. A sink $(g_\beta : Y_\beta \to X)_{\beta \in \mathbf{Ord}}$ is called natural, if $g_\beta \circ e_\beta = g_1 \circ e_1$ for all $\beta \in \mathbf{Ord}$.
$X_0 = \mathbf{Q} \times \{0\} \times \{1\}$, $X_\alpha = \mathbf{Q} \times (\mathbf{Q} \cap [0,1)) \times \{\alpha\}$, $\alpha > 0$,
$Y_1 = X_0$, $Y_\beta = \bigcup\{X_\alpha | \alpha < \beta\}$, $\beta > 1$,
$e_\beta : \mathbf{Q} \to Y_\beta$ is defined by $e_\beta(q) = (q, 0, 1)$ for all $q \in \mathbf{Q}$.
$K(r,s,\alpha,\epsilon) = \{(u,v,\alpha) \in X_\alpha | v > 0 \wedge d((u,v),(r - s/w, 0)) < \epsilon\}$, $0 < \epsilon < 1$.
Y_β becomes a Urysohn space equipped with the following sets forming neighbourhood bases of (r,s,α) (see [2]):

$\alpha = 1:$

$s \neq 0: \; K(r,s,1,\epsilon) \cup \{(r,s,1)\} =: U(r,s,1,\epsilon).$

$s = 0: \; K(r,0,1,\epsilon) \cup \{(u,0,1)|d(u,r) < \epsilon\}.$

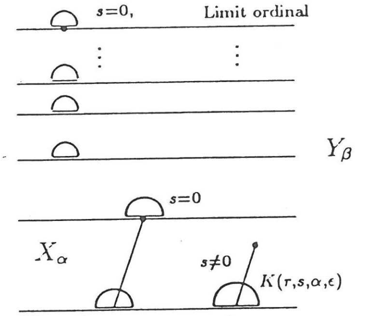

Fig. 1

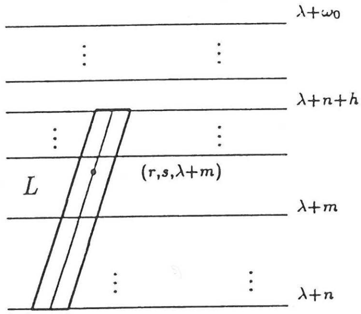

Fig. 2

$\alpha > 1:$

α **limit**:

$$s \neq 0 : K(r,s,\alpha,\epsilon) \cup \{(r,s,\alpha)\} =: U(r,s,\alpha,\epsilon).$$

$$s = 0 : \{(r,0,\alpha)\} \cup K(r,0,\alpha,\epsilon) \cup \bigcup_{\alpha > \tau \geq \gamma} K(r,0,\tau,\epsilon) =: U(r,0,\alpha,\gamma,\epsilon), \; \gamma < \alpha.$$

α **non-limit**:

$$s \neq 0 : K(r,s,\alpha,\epsilon) \cup \{(r,s,\alpha)\} =: U(r,s,\alpha,\epsilon).$$

$$s = 0 : K(r - 1/w, 0, \alpha - 1, \epsilon) \cup K(r,0,\alpha,\epsilon) \cup \{(r,0,\alpha)\} =: U(r,0,\alpha,\epsilon) \text{ (see Fig. 1).}$$

Consider now a limit ordinal λ and $\bigcup_{\mathbf{N}} X_{\lambda+n} =: X_\lambda^\infty$. There is a bijection $\phi : X_\lambda^\infty \to \mathbf{Q} \times \mathbf{Q}^+ \times \{\lambda\}$ given by $\phi(r, s, \lambda + n) := (r, s + n, \lambda)$. We refer to this bijection in the following construction.
Define $\bigcup_{n \le l \le h+n} X_{\lambda+l} \cup \{(r, 0, \lambda + h + n + 1) | r \in \mathbf{Q}\} =: X^h_{\lambda+n}$, $L(r, s, \lambda + m, \epsilon, \lambda + n, h) := \{(u, v+l, \lambda) \in X^h_{\lambda+n} | d((u, v+l), L(r, s, \lambda+m)) \le \epsilon\}$, where $L(r, s, \lambda+m) := \{(x, w(x - r) + s + m) | x \in \mathbf{R}\} \times \{\lambda\}$, i.e. $L(r, s, \lambda + m, \epsilon, \lambda + n, h)$ is the 2ϵ-stripe around $L(r, s, \lambda+m)$ passing through $(r, s, \lambda+m) = \phi^{-1}(r, s+m, \lambda)$ with bottom at $\{(r, 0, \lambda+n) | r \in \mathbf{Q}\}$ and height h. Note that we identify via ϕ the point $(u, v+l, \lambda)$ with $(u, v, \lambda + l)$, where $v < 1$, i.e. $(u, v + l, \lambda) \in X_{\lambda+l}$. The meaning of $h = \omega_0$ is obvious (see Fig. 2).

Lemma 2. *Let $(X, \mathcal{X})$ be a Urysohn space, $A \subseteq X$, $cl_{\theta\theta} A = A$. Define an equivalence relation by $\sim_A := A \times A \cup \triangle$. Then the quotient $X/\sim_A$ is a Urysohn space.*

Proof: Take $x \notin A$. Then there exists $U \in \mathcal{X}$ s.t. $A \subseteq U$ and $x \notin cl_\theta cl U$. Hence there is U_x s.t. $x \in U_x$, $cl U_x \cap cl U = \emptyset$. Since A is θ-closed, distinct points in $X - A$ can be separated by disjoint closed neighbourhoods. □

Lemma 3. *Let $(r, s, \alpha) \in Y_\beta$. We consider basic neighbourhoods of (r, s, α).*
(a) *if $s \ne 0$, then $cl U(r, s, \alpha, \epsilon) = L(r, s, \alpha, \epsilon, \alpha, 1) \cup U(r, s, \alpha, \epsilon)$*
(b) *if $s = 0$, α non-limit, then $cl U(r, 0, \alpha, \epsilon) = L(r, 0, \alpha, \epsilon, \alpha - 1, 2) \cup U(r, 0, \alpha, \epsilon)$*
(c) *if $s = 0$, $\alpha = \lambda$ limit, then $cl U(r, 0, \lambda, \gamma, \epsilon) = \bigcup\{L(r, 0, \tau, \epsilon, \tau, 1) | \alpha \ge \tau \ge \gamma\} \cup U(r, 0, \lambda, \gamma, \epsilon)$*

(see Fig. 3).

Proof: A point (u, v, α) is in the closure of $K(r, s, \alpha)$ if $d(r - s/w, u - v/w) \le \epsilon$ and $0 \le v \le 1$. Where in case $v = 1$ the point $(u, 1, \alpha)$ is identified with $(u, 0, \alpha+1)$. □

Lemma 4. *If λ is a limit ordinal and $n, m \in \mathbf{N}$, then $cl_\theta^{\omega_0} L(r, s, \lambda+m, \epsilon, \lambda+n, h) = L(r, s, \lambda + m, \epsilon, \lambda, \omega_0)$.*

Proof: By induction with the help of Lemma 3 (b). □

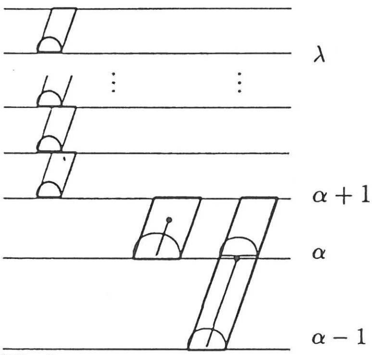

Fig.3

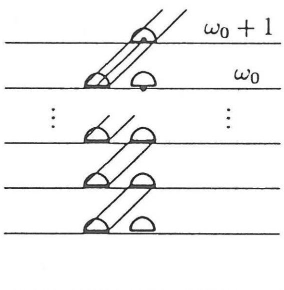

Fig.4

Lemma 5. *Let λ be a limit ordinal, $\lambda < \beta$. Then $\bigcup_{\lambda \le \alpha < \beta} X_\alpha = Y_\beta - Y_\lambda$ and $cl_{\theta\theta}(Y_\beta - Y_\lambda) = Y_\beta - Y_\lambda$.*

PROOF: Take $(r,s,\alpha) \notin Y_\beta - Y_\lambda$, i.e. $(r,s,\alpha) \in Y_\lambda$. We will construct disjoint closed neighbourhoods of (r,s,α) and of $Y_\beta - Y_\lambda$. It is $\alpha < \lambda$ and λ is limit ordinal. The set $\{(u,v,\gamma) \in Y_\beta | (\gamma > \alpha+2) \vee ((\gamma = \alpha+2) \wedge (v > 0))\}$ is an open set containing $Y_\beta - Y_\lambda$. Its closure is $\{(u,v,\gamma) \in Y_\beta | \gamma \ge \alpha+2\}$. By Lemma 3 above, the closure of an open basic neighbourhood of (r,s,α) is contained in $Y_{\alpha+2}$. □

Remark 6. Are there non-constant natural sinks on Y_β? We must find a sink $(g_\beta : Y_\beta \to X)_{\beta \in \mathbf{Ord}}$ coinciding on $\mathbf{Q} \subseteq Y_\beta$. As we know, each morphism $g_\beta : Y_\beta \to X$ is defined by its values on the countable set $\mathbf{Q}$. Since Y_γ is subspace of Y_β, if $\gamma < \beta$, g_β can be regarded as continuous extension of g_γ. Hence there are not so many morphisms into X. Additionally X has fixed weight, character, cardinality, etc.. All these cardinality functions have no bound on the class Y_β, $\beta \in \mathbf{Ord}$. The answer is given by the following

Example 7.

(a) Let λ be a limit ordinal. We apply Lemma 2 and Lemma 5. Take $A = Y_{\lambda+1} - Y_\lambda$. Then $Y_{\lambda+1}/\sim_A =: Y_\lambda \cup \{*\} =: Y_\lambda^*$ is a Urysohn space. Define $g_\beta : Y_\beta \to Y_\lambda^*$ by

$$g_\beta(r,s,\alpha) = \begin{cases} (r,s,\alpha) & \text{if } \alpha < \lambda \\ * & \text{if } \alpha \ge \lambda, \alpha < \beta. \end{cases}$$

(b) Define $\sin_\beta : Y_\beta \to [-1,+1]$ by $\sin_\beta(r,s,\alpha) = \sin(2\pi(wr - s))$ for all $(r,s,\alpha) \in Y_\beta$. Let $U(r,s,\alpha,\delta) \subseteq Y_\beta$ be a basic neighbourhood of (r,s,α). If $(p,q,\tau) \in U(r,s,\alpha,\delta)$, then $d(r - \overline{s}/w, p - q/w) < \delta(1 + 1/w^2)^{1/2} =: \delta c$, where

$$\overline{s} = \begin{cases} 1 & \text{if } \tau = \alpha - 1 \\ s & \text{otherwise.} \end{cases}$$

(c appears for geometrical reasons, as one can see in Fig. 3 or Fig. 4: some points of $U(r,s,\alpha,\epsilon)$ lie outside $L(r,s,\alpha,\epsilon,\alpha,1)$.) Now take an ϵ-neighbourhood $U(\sin(2\pi(wr_0 - s_0)),\epsilon)$. The mapping $x \mapsto \sin(2\pi wx)$ is continuous. There is a δ-neighbourhood $U(r_0 - s_0/w, \delta)$ s.t. $p - q/w \in U(r_0 - s_0/w, \delta) \Rightarrow \sin(2\pi w(p - q/w)) \in U(\sin(2\pi(wr_0 - s_0)), \epsilon)$. Now assume $\tau = \alpha - 1$, then $s_0 = 0$. Take p, q with $d(p - q/w, r_0 - 1/w) < \delta c$, of course $d(p - q/w + 1/w, r_0) < \delta c$,but $\sin(2\pi w(p - q/w)) = \sin(2\pi w(p - q/w + 1/w))$ and hence $\sin(2\pi(wp - q)) \in U(\sin(2\pi wr_0), \epsilon)$.

(c) Define $P_\beta : Y_\beta \to [0,1]$,$\beta \ge 1$, by

$$P_\beta(r,s,\alpha) = \begin{cases} \frac{1}{1 + (r - \frac{s+n}{w})^2} & \text{if } \alpha = n < \omega_0 \\ 0 & \text{if } \alpha \ge \omega_0, \alpha < \beta. \end{cases}$$

The proof of continuity is similar to (b): The mapping $x \mapsto \frac{1}{1+x^2}$ is continuous. Fix $U(P_\beta(r_0, s_0, n), \epsilon) \subseteq [0,1]$. There is $\delta > 0$ s.t. $p - \frac{q+n}{w} \in$

$U((r_0 - \frac{s_0+n}{w}), \delta c)$ implies $\frac{1}{1+(p-\frac{q+n}{w})^2} \in U(P_\beta(r_0, s_0, n), \epsilon)$, showing that even $P_\beta[L(r_0, s_0, n, \delta c, 0, \omega_0)] \subseteq U(P_\beta(r_0, s_0, n), \epsilon)$. Finally P_β is arbitrary small on $U(r_0, 0, \omega_0, n, \delta)$, if n increases.

(d) Take Y^*_λ,$\lambda > \omega_0$, from (a) and a limit ordinal $\xi < \lambda$. If $(r, s, \alpha) \in Y^*_\lambda$, $\alpha = \xi + n$, then $cl_{\theta\theta} L(r, s, \alpha, \epsilon, \xi, \omega_0) = L(r, s, \alpha, \epsilon, \xi, \omega_0) =: F$ and $Y^*_\lambda / \sim_F$ is a Urysohn space. Combination with (a) gives many different sinks $(g_\beta : Y_\beta \to Y^*_\lambda / \sim_F)_{\beta \in \mathbf{Ord}}$.

Definition 8. Let $(g_\beta : Y_\beta \to X)_{\beta \in \mathbf{Ord}}$ be a natural sink.

(a) (g_β) is called periodic, if for all $\beta > \omega_0$ and for all $\alpha, \tilde{\alpha} \geq \omega_0$; $\alpha, \tilde{\alpha} < \beta$: $g_\beta(r, s, \alpha) = g_\beta(r, s, \tilde{\alpha})$.

(b) (g_β) is called eventually periodic, if there exists an ordinal τ, s.t. for all $\beta > \tau$ and for all $\alpha, \tilde{\alpha} \geq \tau$; $\alpha, \tilde{\alpha} < \beta$: $g_\beta(r, s, \alpha) = g_\beta(r, s, \tilde{\alpha})$.

Theorem 9. *Let $(g_\beta : Y_\beta \to X)_{\beta \in \mathbf{Ord}}$ be a natural sink and let X be a T_3-space. Then (g_β) is periodic.*

Proof: Take $x \in X$. For every neighbourhood W_x of x there is a neighbourhood V_x of x with $clV_x \subseteq W_x$, and of course $cl_\theta clV_x = clV_x$, $cl_\theta^{\omega_0} clV_x = clV_x$. If $A \subseteq Y_\beta$, then $g_\beta[cl_\theta^{\omega_0} A] \subseteq cl_\theta^{\omega_0} g_\beta[A] = \overline{g_\beta[A]}$. Take a basic neighbourhood $U(r, s, n, \delta) =: U$ of (r, s, n). Then $cl_\theta^{\omega_0}[U] = L(r, s, n, \delta, 0, \omega_0) \cup U$. For every neighbourhood W of $g_\beta(r, s, n)$ there is $\delta > 0$ s.t. $g_\beta[L(r, s, n, \delta, 0, \omega_0)] \subseteq W$. Consider $(p, 0, \omega_0)$ and $(p, 0, \omega_0+1)$. For every neighbourhood $U(g_\beta(p, 0, \omega_0))$ there is a neighbourhood $U(p, 0, \omega_0, k, \delta)$ s.t. $g_\beta[L(p, 0, \omega_0, \delta, \omega_0, \omega_0) \cup \bigcup_{n \geq k} L(p, 0, n, \delta, 0, \omega_0)] \subseteq U(g_\beta(p, 0, \omega_0))$. Assume $g_\beta(p, 0, \omega_0) \neq g_\beta(p, 0, \omega_0 + 1)$. Then we have open sets U_1, V_1, U_0 s.t. $g_\beta(p, 0, \omega_0 + 1) \in U_1 \subseteq \overline{U_1} \subseteq V_1$, $g_\beta(p, 0, \omega_0) \in U_0 = U(g_\beta(p, 0, \omega_0))$, $V_1 \cap U_0 = \emptyset$. Now take an open basic neighbourhood $U(p, 0, \omega_0+1, \delta) =: U^*$ fulfilling $g_\beta[U^*] \subseteq U_1$. Then $clU^* = L(p, 0, \omega_0+1, \delta, \omega_0, 2) \cup U^* = L(p-1/w, 0, \omega_0, \delta, \omega_0, 2) \cup U^*$. Let $O^* \subseteq g_\beta^{-1}[V_1]$ be an open set containing $\overline{U^*}$. Then for all $(u, 0, \omega_0) \in \overline{U^*}$ there is $\delta_u < \delta, k_u \in \mathbf{N}$ s.t. $K(u, 0, l, \delta_u) \subseteq O^*$ for all $l \geq k_u$ (see Fig. 4). Take a $(x, 0, l) \in K(u, 0, l, \delta_u)$, $l \geq \max\{k_u, k\}$. Then $(x, 0, l) \in L(p, 0, l+1, \delta, 0, \omega_0)$. But $g_\beta(x, 0, l) \in V_1$ and $g_\beta(x, 0, l) \in g_\beta[L(p, 0, l+1, \delta, 0, \omega_0)] \subseteq U_0$, contradicting $V_1 \cap U_0 = \emptyset$. Hence g_β assumes the same values on $\mathbf{Q}_{\omega_0} := \{(r, 0, \omega_0) | r \in \mathbf{Q}\}$ and $\mathbf{Q}_{\omega_0+1} := \{(r, 0, \omega_0 + 1) | r \in \mathbf{Q}\}$. Since $cl_\theta \mathbf{Q}_{\omega_0} = \overline{X_{\omega_0}}$ and $cl_\theta \mathbf{Q}_{\omega_0+1} = \overline{X_{\omega_0+1}}$, this is also true for X_{ω_0} and X_{ω_0+1} (there is another argument: the two mappings $h_1 := g_\beta / X_{\omega_0+1}$ and h_0 defined by $h_0(r, s, \omega_0 + 1) := g_\beta(r, s, \omega_0)$ are continuous and coincide on $\mathbf{Q}_{\omega_0+1}$). The function $\tilde{g_\beta} : Y_\beta \to X$ defined by

$$\tilde{g_\beta}(r, s, \alpha) = \begin{cases} g_\beta(r, s, \alpha) & \text{if } \alpha < \omega_0 \\ g_\beta(r, s, \omega_0) & \text{if } \alpha \geq \omega_0, \alpha < \beta \end{cases}$$

is continuous and coincides with g_β on $\mathbf{Q}$, hence $g_\beta = \tilde{g_\beta}$. To show continuity, take $K(r - 1/w, 0, \alpha - 1, \epsilon) \cup K(r, 0, \alpha, \epsilon) \cup \{(r, 0, \alpha)\} =: U$, an ϵ-neighbourhood of $(r,0,\alpha)$, $\alpha > \omega_0$ non-limit. Then $\tilde{g_\beta}[U] = \tilde{g_\beta}[K(r-1/w,0,\omega_0,\epsilon) \cup K(r,0,\omega_0+1,\epsilon) \cup \{(r,s,\omega_0)\}] = \tilde{g_\beta}[K(r-1/w,0,\omega_0,\epsilon) \cup K(r,0,\omega_0,\epsilon) \cup \{(r,s,\omega_0)\}] = g_\beta\,[K(r-1/w,0,\omega_0,\epsilon) \cup K(r,0,\omega_0+1,\epsilon) \cup \{(r,s,\omega_0)\}]$. α limit or $s \neq 0$ is not a problem. □

Theorem 10. *Let $(g_\beta : Y_\beta \to X)_{\beta \in \mathbf{Ord}}$ be a natural sink in* **Ury**, *then (g_β) is eventually periodic.*

Proof: Let $\chi(X)$ be the character of X. Let λ be a regular limit ordinal with $(cof(\lambda) =)\ \lambda > \max\{\chi(X), \omega_0\}$. Take a space Y_β with $\beta > \lambda$. Look at the bottom edge of $X_\lambda \subseteq Y_\beta$: $\{(p,0,\lambda) | p \in \mathbf{Q}\}$. Let $\mathcal{U}(g_\beta(p,0,\lambda))$ be a neighbourhood base of cardinality $\leq \chi(X)$ of the point $g_\beta(p,0,\lambda)$ in X. For every $V \in \mathcal{U}(g_\beta(p,0,\lambda))$ there exists a neighbourhood $U(p,0,\lambda,\kappa^p_V,\epsilon^p_V) =: U$ of $(p,0,\lambda)$ in Y_β fulfilling $g_\beta[U] \subseteq V$. We have $|\{(\kappa^p_V,\epsilon^p_V) | V \in \mathcal{U}(g_\beta(p,0,\lambda))\}| < \chi(X)$, then also $\sup\{\kappa^p_V | V \in \mathcal{U}(g_\beta(p,0,\lambda))\} =: \kappa^p < \lambda$ and $\sup\{\kappa^p | p \in \mathbf{Q}\} =: \kappa < \lambda$. This shows that for every $V \in \mathcal{U}(g_\beta(p,0,\lambda))$, $p \in \mathbf{Q}$, there is $U := U(p,0,\lambda,\kappa,\epsilon^p_V)$ s.t. $g_\beta[U] \subseteq V$. Now take $\tau < \lambda$, $\tau \geq \kappa$. We show $g_\beta(p,0,\lambda) = g_\beta(p,0,\tau)$. If we assume the contrary, then there are neighbourhoods $U(g_\beta(p,0,\lambda))$ and $U(g_\beta(p,0,\tau))$ of $g_\beta(p,0,\lambda)$ and $g_\beta(p,0,\tau)$ respectively with disjoint intersection. Of course $g_\beta[clU(p,0,\lambda,\kappa,\epsilon^p_{U(g_\beta(p,0,\lambda))})] \subseteq clU(g_\beta(p,0,\lambda))$, but $(p,0,\tau) \in clU(p,0,\lambda,\kappa,\epsilon^p_{U(g_\beta(p,0,\lambda))})$ is a contradiction. We are ready to show $g_\beta(p,0,\tau) = g_\beta(p,0,\tau+1)$ for a limit ordinal $\tau < \lambda, \tau \geq \kappa$. (There is such a limit ordinal.) But this is easy, since $g_\beta(p,0,\tau) = g_\beta(p,0,\lambda) = g_\beta(p,0,\tau+1)$. By the same argument such as in the proof of the previous Theorem 9 we get $g_\beta = \tilde{g_\beta}$ for all $\beta \in \mathbf{Ord}$, where

$$\tilde{g_\beta}(r,s,\alpha) = \begin{cases} g_\beta(r,s,\alpha) & \text{if } \alpha < \tau \\ g_\beta(r,s,\lambda) & \text{if } \alpha \geq \tau, \alpha < \beta. \end{cases}$$

□

Remark 11. The proof illustrates that for showing $g_\beta(p,0,\tau) = g_\beta(p,0,\tau+1)$ for all $p \in \mathbf{Q}$ we only need the Hausdorff property of X. To show $g_\beta = \tilde{g_\beta}$ we need the Urysohn separation axiom, of course.

Example 12. Let $\tau > 1$. Define a new Urysohn space $Y^{\tau+1}_\tau$ on the set $Y_{\tau+1} = Y_\tau \cup X_\tau$, where

(a) the neighbourhoods of the points (r,s,α), $\alpha < \tau$, and of (r,s,τ), $s \neq 0$, are not changed.
(b) neighbourhood bases of $(r,0,\tau) \in X_\tau$ are $U(r,0,\tau,\gamma,\epsilon) \cup K(r-1/w,0,\tau,\epsilon)$. Note that $(r,1,\tau) \notin X_\tau$ by definition. Define $g_\beta : Y_\beta \to Y^{\tau+1}_\tau$ by

$$\tilde{g_\beta}(r,s,\alpha) = \begin{cases} (r,s,\alpha) & \text{if } \alpha < \tau \\ (r,s,\tau) & \text{if } \alpha \geq \tau, \alpha < \beta. \end{cases}$$

(g_β) is an eventually periodic sink, if $\tau > \omega_0$. The maps g_β in case $\beta > \tau$ are pressing down all X_α above τ onto the modified X_τ. The topology on $Y^{\tau+1}_\tau$ is coarser then on $Y_{\tau+1}$.

Remark 13.

(a) The proof of the previous Theorem will apply for arbitrary X which has a coarser T_3-topology. This shows that in all eventually periodic, and non periodic examples the codomain has no coarser T_3-topology (Y_{ω_0} has!).

(b) All sinks are uniquely defined by $g_1 : \mathbf{Q} \to X$. In all the examples, except 7 (b) and 7 (c), g_1 is the identity on $\mathbf{Q}$. This illustrates that in determining $(g_\beta : Y_\beta \to X)_{\beta \in \mathbf{Ord}}$, g_1 is as important as the codomain X of the sink.

(c) Theorem 9 is a strong restriction, however the arithmetic sum of the functions in 7 (b) and 7 (c) provides a new sink.

(d) After these preparations have been completed it becomes an easy task to show that **Ury** allows no $(Epi, \mathcal{M})$-factorization structure for (large) sources and any $\mathcal{M}$. Since **Ury** has coequalizers this means that **Ury** is not (Epi, extremal Mono Source)-category (see [1, Proposition 15.8 (3)]).

Lemma 14. *Let a sink $(g_i : X_i \to Y)_I$ in* **Top** *be given s.t. for all $j \in I$ and all $x_j, y_j \in X_j$, $x_j \neq y_j$, there is a sink $(h_i : X_i \to Z_{x_j y_j})_I$ with $h_j(x_j) \neq h_j(y_j)$ and a continuous map $h : Y \to Z_{x_j y_j}$ s.t. $h \circ g_j = h_j$, then for all $i \in I$, $g_i : X_i \to Y$ is injective.*

Proof: Assume we can find $j \in I$, $x_j, y_j \in X_j$,$x_j \neq y_j$, s.t. $g_j(x_j) = g_j(y_j)$. Take h_i, h as in the Lemma. Then $h_j(x_j) = h \circ g_j(x_j) = h \circ g_j(y_j) = h_j(y_j)$, contradicting $h_j(x_j) \neq h_j(y_j)$. □

Theorem 15. *The large source $(e_\beta : \mathbf{Q} \to Y_\beta)_{\beta \in \mathbf{Ord}}$ consisting of epimorphisms has no cointersection in* **Ury**.

Proof: Assume there is a cointersection $(f_\beta : Y_\beta \to Z)_{\beta \in \mathbf{Ord}}$. Take $x_\beta, y_\beta \in Y_\beta$, $x_\beta \neq y_\beta$. There is a limit ordinal $\lambda > \beta$. Take the natural sink $(g_\beta : Y_\beta \to Y_\lambda^{\lambda+1})$ from Example 12 or the natural sink $(g_\beta : Y_\beta \to Y_\lambda^*)$ from Example 7. By Lemma 14, all f_β are injective, which is impossible. □

Corollary 16. **Ury** *is no $(Epi, \mathcal{M})$-category for any $\mathcal{M}$.*

Proof: If we had some $\mathcal{M}$ giving a factorization structure for large sources, then by [1, Corollary 15.16 (1)] cointersections exist, which contradicts Theorem 15. □

Remark 17. If we only allow T_3-spaces as codomains of a natural sink, then by Theorem 9 each such sink factorizes through $Y_{\omega_0}^{\omega_0+1}$, which gives some kind of cointersection.

References

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(*Received* December 6, 1991)