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A CLASS OF WEIGHTED COMPOSITION OPERATORS ON H^2

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Summary. A fractional linear transformation, mapping the unit disc into itself, gives rise to a weighted composition operator on the Hardy space H^2. Such operators have been recently used in [11] in connection with an extremal problem from operator theory. In this paper, we investigate the basic properties of these operators and determine their spectra. The results can be compared to those for unweighted composition operators on various spaces [1], [3], [5].

Keywords: fractional linear transformations, composition operators.

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In the present paper we investigate the relation between a fractional linear transformation

$$\varphi(A)\colon\ z \mapsto \frac{az+b}{cz+d},$$

given by a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and the corresponding composition operator $m(A)$ defined on H^2 by the formula

$$(m(A)f)(z) = \frac{1}{cz+d} f(\varphi(A)(z)).$$

Operators of this form have been recently used by V. Pták [11] to obtain an explicit expression for the operator realizing the maximum of $\|T^n\|$ as T ranges over the set of all contractions on n-dimensional Hilbert spaces such that the spectral radius of T does not exceed a given bound $r < 1$. It turns out that the maximum is attained for the operator

$$S^* \mid \operatorname{Ker}(S^* - \alpha)^n,$$

where S is the shift operator on H^2 and α is a number of modulus r. The mapping $A \mapsto m(A)$ was used to express this extremal operator as a matrix with respect to an orthonormal basis in H^2. Besides, some basic properties of the operators $m(A)$ (inverses, adjoints, etc.) were established.

The present paper is devoted to a deeper study of the operators $m(A)$. For their definition to be meaningful it is, of course, necessary that $\varphi(A)$ map the open unit disc $\mathbb{D}$ into itself. It is not difficult to give a description of the corresponding matrices A; this is done in Section 1. The result is that $\varphi(A)(\mathbb{D}) \subset \mathbb{D}$ if and only if

$$A^*QA \leqq |\det A|\, Q\,,$$

Q being the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. In Section 2, the operators $m(A)$ are defined and shown to be bounded on H^2. In Section 3, a criterion for compactness is established. The operator $m(A)$ is compact if, and only if, $\varphi(A)(\mathbb{D})$ is bounded away from the boundary of $\mathbb{D}$. Sections 4–10 are concerned with the spectra and the spectral radii of the operators $m(A)$. It turns out that these depend on the position of $\varphi(A)(\mathbb{D})$ as a subset of $\mathbb{D}$, as well as on the number and location of the fixed points of the mapping $\varphi(A)$. When $\varphi(A)$ is a Möbius transformation of $\mathbb{D}$ onto $\mathbb{D}$, the spectrum of $m(A)$ is either a finite set (when $\varphi(A)$ is periodic) or a circle; the latter fact follows from a result of Kitover [10]. In other cases, the spectrum may assume various forms: it can be a sequence of numbers tending to zero, or a disc, or even a spiral approaching the origin. These sections are mostly technical in character and the final results are stated in full detail at the end of this article as Theorems 10 and 11.

The operators $m(A)$ are examples of weighted composition operators on H^2, i.e. composition operators followed by a multiplication. Operators of this type, acting on the disc algebra $A(\mathbb{D})$ rather than H^2, were first studied by Kamowitz [6], [7], who gave conditions for compactness and, in some cases, determined their spectra. Kitover [10] studied such operators on general spaces of analytic functions in the case when the composition operator is invertible (weighted automorphisms). In the context of the present paper, his results apply to the case when $\varphi(A)$ maps $\mathbb{D}$ onto $\mathbb{D}$. Unweighted composition operators on H^2 (or, more generally, H^p) as well as $A(\mathbb{D})$ have been investigated by many authors, and conditions for compactness, nuclearity, etc., as well as descriptions of their spectra in many cases, are known; see, for instance, [3], [5] and [1], where also more of the rich bibliography on this subject can be found. The work of Kitover [9] deals with composition operators on spaces of continuous functions. Many papers are devoted to the study of composition operators (both weighted and unweighted) on Banach algebras, of which we mention [6] and [8] as examples.

1. FRACTIONAL LINEAR TRANSFORMATIONS

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a regular 2×2 complex matrix. Then the fractional linear transformation

$$\varphi(A)\colon\ z \mapsto \frac{az + b}{cz + d}$$

is a 1-to-1 mapping of the Gaussian sphere $\mathbb{G} = \mathbb{C} \cup \{\infty\}$ onto itself. A short computation reveals that, for A, B regular 2×2 matrices,

$$\varphi(A)\,\varphi(B) = \varphi(AB)\,,$$
$$\varphi(B^{-1}) = \varphi(B)^{-1}\,,$$
$$\varphi(A) = \varphi(B) \Leftrightarrow A = tB \quad \text{for some} \quad t \in \mathbb{C} \setminus \{0\}\,.$$

Let

$$\mathscr{M} = \left\{ \begin{pmatrix} \varepsilon q & -\varepsilon\alpha q \\ -\bar{\alpha} q & q \end{pmatrix} \;\middle|\; |\varepsilon| = 1,\ |\alpha| < 1,\ q = (1 - |\alpha|^2)^{-1/2} \right\}.$$

We see that for $M \in \mathscr{M}$, $\varphi(M)$ is a Möbius transformation of $\mathbb{D}$, the unit disc, onto itself; clearly every Möbius transformation can be expressed in this way. Also, a short computation gives

$$M^*QM = Q \quad \text{for all} \quad M \in \mathscr{M}\,,$$

where

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Our next task is to determine when $\varphi(A)(\mathbb{D}) \subset \mathbb{D}$.

Theorem 1. *Let A be a regular 2×2 complex matrix. Then the following assertions are equivalent:*

(1) $\varphi(A)(\mathbb{D}) \subset \mathbb{D}$

(2) $A^*QA \leqq |\det A| \,.\, Q$

(3) $\exists t > 0\colon\ A^*QA \leqq tQ\,.$

Proof. (1) $\Rightarrow$ (2). Suppose $\varphi(A)(\mathbb{D}) \subset \mathbb{D}$. Because $\varphi(A)(\mathbb{D})$ is a disc, there are three possibilities: either it lies inside $\mathbb{D}$, or it touches the boundary $\partial\mathbb{D}$ at exactly one point, or it is the whole of $\mathbb{D}$.

If $\varphi(A)(\mathbb{D})$ lies inside $\mathbb{D}$, then there exists a Möbius transformation $\varphi(M)$, $M \in \mathscr{M}$, such that $\varphi(M)\,\varphi(A)(\mathbb{D})$ is a disc centered at the origin. Further, there exists a Möbius transformation $\varphi(N)$, $N \in \mathscr{M}$, such that $\varphi(M)\,\varphi(A)\,\varphi(N)(0) = 0$. Denote $B = MAN$; then $\varphi(B)(\mathbb{D})$ is a disc centered at the origin, and $\varphi(B)(0) = 0$. If r is the radius of $\varphi(B)(\mathbb{D})$, then the Schwarz lemma applies to the function $\varphi(B)/r$ and gives $\varphi(B)(z) = \lambda z$ for some λ, $|\lambda| = r < 1$, and so

$$B = \begin{pmatrix} \lambda t & 0 \\ 0 & t \end{pmatrix} \quad \text{for some} \quad t \in \mathbb{C} \setminus \{0\}\,.$$

Now

$$B^*QB - |\det B| \,.\, Q = \begin{pmatrix} |\lambda t|^2 - |\lambda t^2| & 0 \\ 0 & |\lambda t^2| - |t|^2 \end{pmatrix} =$$
$$= |t|^2 \begin{pmatrix} |\lambda|\,(|\lambda| - 1) & 0 \\ 0 & |\lambda| - 1 \end{pmatrix} \leqq 0\,,$$

i.e. $B^*QB \leqq Q|\det B|$. Using the relations $|\det A| = |\det B|$, $B = MAN$ and $N^*QN = Q$, we see that this is the same as

$$N^*A^*M^*QMAN \leqq |\det A|\, N^*QN\,.$$

Because N is invertible ($N^{-1} = QN^*Q$), this is equivalent to

$$A^*M^*QMA \leqq |\det A| \,.\, Q\,.$$

Finally, $M^*QM = Q$, so

$$A^*QA \leqq |\det A| \,.\, Q\,.$$

If $\varphi(A)(\mathbb{D})$ is all of $\mathbb{D}$, then we can proceed in the same manner as above, taking $M = I$ (the identity matrix). In this case we even get $A^*QA = |\det A| \,.\, Q$.

In the remaining case, one can again choose $M \in \mathcal{M}$ such that $\varphi(M)\,\varphi(A)(\mathbb{D}) = \{z \in \mathbb{C} \mid |z - 1/2| < 1/2\}$, and then $N \in \mathcal{M}$ such that $\varphi(M)\,\varphi(A)\,\varphi(N)(0) = 1/2$. Denoting $B = MAN$, the Schwarz lemma applies to the function $2\,\varphi(B) - 1$ and gives

$$\varphi(B)(z) = \frac{\varepsilon z + 1}{2}$$

for some $\varepsilon \in \partial\mathbb{D}$, so

$$B = \begin{pmatrix} \varepsilon t & t \\ 0 & 2t \end{pmatrix} \quad \text{for some} \quad t \in \mathbb{C} \setminus \{0\}\,,$$

and, again

$$B^*QB - |\det B|\, Q = |t|^2 \begin{pmatrix} -1 & \bar{\varepsilon} \\ \varepsilon & -1 \end{pmatrix} \leqq 0\,,$$

which implies $A^*QA \leqq Q|\det A|$ just in the same way as in the preceding cases.

(2) $\Rightarrow$ (3). Trivial.

(3) $\Rightarrow$ (1). For $z \in \mathbb{C}$, let $x_z = \begin{pmatrix} z \\ 1 \end{pmatrix}$. By assumption, we have

$$\langle A^*QAx_z, x_z\rangle \leqq t\langle Qx_z, x_z\rangle \quad \forall z \in \mathbb{C}\,.$$

If $z \in \mathbb{D}$, then

$$t\langle Qx_z, x_z\rangle = t(|z|^2 - 1) < 0\,,$$

and so

$$0 > \langle A^*QAx_z, x_z\rangle = \langle QAx_z, Ax_z\rangle = |az + b|^2 - |cz + d|^2\,.$$

This implies $cz + d \neq 0$ and

$$0 > \left|\frac{az + b}{cz + d}\right|^2 - 1 = |\varphi(A)(z)|^2 - 1\,, \quad \text{i.e.} \quad \varphi(A)(z) \in \mathbb{D}\,.$$

This completes the proof of Theorem 1.

By slight variations in the proof above, it is possible to prove the first four assertions of the following theorem. The fifth is a consequence of the first four.

Theorem 2. *Let A be a regular 2×2 complex matrix. Then*

$$\varphi(A)(\mathbb{D}) \subset \mathbb{D} \Leftrightarrow A^*QA \leqq Q|\det A|\,;$$
$$\varphi(A)(\mathbb{D}) \supset \mathbb{D} \Leftrightarrow A^*QA \geqq Q|\det A|\,;$$
$$\varphi(A)(\mathbb{D}) \subset \mathbb{G} \setminus \overline{\mathbb{D}} \Leftrightarrow A^*QA \geqq -Q|\det A|\,;$$
$$\varphi(A)(\mathbb{D}) \supset \mathbb{G} \setminus \overline{\mathbb{D}} \Leftrightarrow A^*QA \leqq -Q|\det A|\,;$$

[*the set* $\mathbb{G} \setminus (\varphi(\partial\mathbb{D}) \cup \partial\mathbb{D})$ *has four connected components*] $\Leftrightarrow$ *both* $A^*QA \pm \pm Q|\det A|$ *are indefinite.*

Remark. For A singular, $A \neq 0$, the mapping $\varphi(A)$ can be defined, too. It will be a constant function (the constant infinity is also allowed). One can then prove the following modification of Theorem 2:

Theorem 2′. *Let $A \neq 0$ be a singular 2×2 complex matrix. Then*

$$\varphi(A)(\mathbb{D}) \subset \overline{\mathbb{D}} \Leftrightarrow A^*QA \leqq 0\,;$$
$$\varphi(A)(\mathbb{D}) \subset \partial\mathbb{D} \Leftrightarrow A^*QA = 0\,;$$
$$\varphi(A)(\mathbb{D}) \subset \mathbb{G} \setminus \mathbb{D} \Leftrightarrow A^*QA \geqq 0\,.$$

We omit the easy proof of this theorem, as it will not be used in the sequel.

2. COMPOSITION OPERATORS

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ again be a regular 2×2 matrix and suppose that $\varphi(A)(\mathbb{D}) \subset \mathbb{D}$. For a function f on $\mathbb{D}$, define

$$(m(A)f)(z) = \frac{1}{cz + d} f\left(\frac{az + b}{cz + d}\right).$$

This is also a function on $\mathbb{D}$, which is analytic on $\mathbb{D}$ if f is. In fact, a little more is true:

Theorem 3. *Let A be a regular 2×2 matrix such that $\varphi(A)(\mathbb{D}) \subset \mathbb{D}$. Then $m(A)$ is a bounded linear operator on H^2.*

Proof. For any function f on $\mathbb{D}$,

$$\frac{1}{2\pi}\int_0^{2\pi} |(m(A)f)(re^{it})|^2 \, dt =$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{|cre^{it} + d|^2} \cdot |f(\varphi(A)(re^{it}))|^2 \, dt =$$

$$= \frac{1}{2\pi r|\det A|} \cdot \oint_{\Gamma_r} |f(y)|^2 \, ds(y),$$

where Γ_r is the closed curve $\{\varphi(re^{it}) | \, t \in \langle 0, 2\pi\rangle\}$ and $ds(y)$ is the line element on Γ_r,

$$ds(y) = \left|\frac{d}{dt}\varphi(re^{it})\right| dt = |\varphi'(re^{it}) \, rie^{it}| \, dt = \frac{|\det A| \, r}{|cre^{it} + d|^2} \, dt \, .$$

Now if $f \in H^2$, then $|f|^2$ has a harmonic majorant u on $\mathbb{D}$, and so

$$\oint_{\Gamma_r} |f(y)|^2 \, ds(y) \leqq \oint_{\Gamma_r} u(y) \, ds(y) \, .$$

By the mean value theorem for harmonic functions, the last integral equals

$$(\text{length of } \Gamma_r) \, u(\gamma_r) \, ,$$

where γ_r is the center of $\Gamma_r (\Gamma_r$ is a circle). Combining the above results and letting r tend to 1, we get

$$\|m(A) f\|_{H^2}^2 \leqq \frac{(\text{length of } \Gamma_1)}{2\pi|\det A|} \, u(\gamma_1) \, .$$

Because u is harmonic on $\mathbb{D}$, Harnack's inequality gives

$$u(\gamma_1) \leqq \frac{1 + |\gamma_1|}{1 - |\gamma_1|} \, . \, u(0) \, ;$$

finally, u can be chosen so that $u(0) = \|f\|_{H^2}^2$. Summing up, we see that

$$\|m(A) f\|_{H^2}^2 \leqq \frac{(\text{length of } \varphi(A)(\partial\mathbb{D}))}{2\pi|\det A|} \cdot \frac{1 + |\text{center of } \varphi(A)(\partial\mathbb{D})|}{1 - |\text{center of } \varphi(A)(\partial\mathbb{D})|} \cdot \|f\|_{H^2}^2 \, , \tag{1}$$

which completes the proof of Theorem 3.

Before going on, we list some properties of the operators $m(A)$. In everything what follows, these operators are considered as operators on H^2.

Proposition 4. *Let A, B be regular 2×2 complex matrices.*

(i) *If $\varphi(A)(\mathbb{D}) \subset \mathbb{D}$ and $\varphi(B)(\mathbb{D}) \subset \mathbb{D}$, then also $\varphi(AB)(\mathbb{D}) \subset \mathbb{D}$ and $m(AB) = m(B)\, m(A)$.*

(ii) *If $\varphi(A)(\mathbb{D}) \subset \mathbb{D}$ and $\varphi(A^{-1})(\mathbb{D}) \subset \mathbb{D}$, then $m(A^{-1}) = m(A)^{-1}$.*

(iii) *If $M \in \mathcal{M}$, then $m(M)$ is unitary.*

Proof. (i) $\varphi(AB)(\mathbb{D}) \subset \mathbb{D}$ because $\varphi(AB) = \varphi(A)\,\varphi(B)$; the formula $m(AB) = m(B)\, m(A)$ follows after a short computation.

(ii) Take $B = A^{-1}$ in (i).

(iii) If $M \in \mathcal{M}$, then $|\det M| = 1$, $\varphi(M)(\mathbb{D}) = \mathbb{D}$ and $\varphi(M)(\partial\mathbb{D}) = \partial\mathbb{D}$, so the formula (1) gives $\|m(M)\| \leqq 1$. Because $M \in \mathcal{M}$ implies $M^{-1} \in \mathcal{M}$, we have also $\|m(M^{-1})\| \leqq 1$. Finally, $m(M^{-1}) = m(M)^{-1}$, so $m(M)$ is an invertible isometry and hence it is unitary.

3. COMPACTNESS

Now we are ready to establish a criterion for compactness of the operators $m(A)$.

Suppose $\varphi(A)(\mathbb{D}) \subset \mathbb{D}$. Looking at the beginning of the proof of Theorem 1, we see that there always exist $M, N \in \mathcal{M}$ such that $MAN \overset{\text{df.}}{=} B \overset{\text{df.}}{=} tB_0$, where $t \in \mathbb{C} \setminus \{0\}$ and B_0 is

$$\text{either } \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}, \quad |\lambda| \leqq 1, \quad \text{or } \begin{pmatrix} \varepsilon/2 & 1/2 \\ 0 & 1 \end{pmatrix}, \quad |\varepsilon| = 1,$$

depending on whether $\varphi(A)(\mathbb{D})$ lies inside $\mathbb{D}$ ($|\lambda| < 1$) or is all of $\mathbb{D}$ ($|\lambda| = 1$), or touches $\partial\mathbb{D}$ at one point. According to Proposition 4, $m(B) = (1/t)\, m(B_0) = m(N)\, m(A)\, m(M)$ with $m(N)$ and $m(M)$ unitary. It follows that $m(A)$ is compact if and only if $m(B_0)$ is.

Let us first dispose of the second case. We have $B_0 = B_0' B_\varepsilon$, where

$$B_\varepsilon = \begin{pmatrix} \varepsilon & 0 \\ 0 & 1 \end{pmatrix}, \quad B_0' = \begin{pmatrix} 1/2 & 1/2 \\ 0 & 1 \end{pmatrix}.$$

It follows that $m(B_0) = m(B_\varepsilon)\, m(B_0')$, with $m(B_\varepsilon)$ unitary (because $B_\varepsilon \in \mathcal{M}$); hence, it suffices to consider the case $\varepsilon = 1$. Then

$$(m(B_0) f)(z) = f\left(\frac{z+1}{2}\right).$$

Ror $\operatorname{Re} z < 0$, let $\arg z$ be the branch of the argument which takes its values in $(\pi/2, 3\pi/2)$. Denote

$$g(z) = \ln |z - 1| + \mathrm{i} \arg (z - 1),$$

and

$$f_\alpha(z) = \exp(\alpha\, g(z)), \quad \alpha \in \mathbb{C}.$$

If $\alpha \in \langle 0, +\infty)$, we have

$$|f_\alpha(z)| = \exp \operatorname{Re}(\alpha\, g(z)) = |z - 1|^\alpha \leqq 2^\alpha \quad \text{for all} \quad z \in \mathbb{D},$$

so $f_\alpha \in H^\infty \subset H^2$; furthermore,

$$(m(B_0) f_\alpha)(z) = f_\alpha\left(\frac{z+1}{2}\right) = \exp\left[\alpha g\left(\frac{z+1}{2}\right)\right] =$$
$$= \exp \alpha[g(z) - \ln 2] = \mathrm{e}^{-\alpha \ln 2} f_\alpha(z).$$

This means $\exp(-\alpha \ln 2) \in \sigma_p(m(B_0))$. As α runs through $\langle 0, +\infty)$, $\exp(-\alpha \ln 2)$ runs through $(0, 1\rangle$. So $\langle 0, 1\rangle \subset \sigma(m(B_0))$ and $m(B_0)$ cannot be compact. (We remark that what we have just done was exhibiting some quite elementary solutions to Schroeder's equation

$$f\left(\frac{z+1}{2}\right) = \lambda f(z), \quad z \in \mathbb{D};$$

for an exhaustive treatise on this matter, see [2], especially Proposition 4.4.)

Now let $B_0 = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}$. If $|\lambda| = 1$, then $B_0 \in \mathcal{M}$ and, in view of Proposition 4, $m(B_0)$ is unitary, i.e. not compact. We are going to show that for $|\lambda| < 1$, $m(B_0)$ is compact.

Assume $f_n \in H^2$, $f_n \to 0$ weakly. Any weakly convergent sequence is bounded, so $\|f_n\| \leqq c$ for all n, for some $c > 0$. For $x \in \mathbb{D}$, denote by $g_x(z) = (1 - \bar{x}z)^{-1}$ the reproducing kernel at x. Then $f_n \to^w 0$ implies $f_n(x) = \langle f_n, g_x\rangle \to 0$ for every $x \in \mathbb{D}$. If $|x| = |\lambda|$, then

$$|f_n(x)| = |\langle f_n, g_x\rangle| \leqq \|f_n\| \, \|g_x\| \leqq c\|g_x\| = \frac{c}{(1 - |\lambda|^2)^{1/2}}.$$

Thus one can use the Lebesgue dominated convergence theorem to conclude that

$$\|m(B_0) f_n\|^2 = \frac{1}{2\pi} \int_0^{2\pi} |f_n(\lambda e^{it})|^2 \, dt \to 0.$$

Thus the operator $m(B_0)$ maps weakly convergent sequences into norm convergent ones, and so must be compact.

Summing up, we have proved

Theorem 5. *Let A be a regular 2×2 matrix, $\varphi(A)(\mathbb{D}) \subset \mathbb{D}$. Then $m(A)$ is a compact operator if and only if* $\mathrm{cl}(\varphi(A)(\mathbb{D})) \subset \mathbb{D}$.

Remark. This theorem may be compared with Theorem 1 of Kamowitz [6] which yields a similar criterion for compactness of operators $m(A)$ acting on $A(\mathbb{D})$, the disc algebra.

4. SPECTRAL PROPERTIES

The rest of this paper is devoted to spectral properties of the operators $m(A)$. The results somewhat resemble those for common (unweighted) composition operators on H^2, cf. [3] or [5], for example.

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a regular 2×2 complex matrix such that $\varphi(A)(\mathbb{D}) \subset \mathbb{D}$. There are four cases to distinguish:

First case: $\mathrm{cl}(\varphi(A)(\mathbb{D})) \subset \mathbb{D}$;

Second case: $\varphi(A)(\mathbb{D})$ touches $\partial\mathbb{D}$ at exactly one point, which is not a fixed point of $\varphi(A)$;

Third case: $\varphi(A)(\mathbb{D})$ touches $\partial\mathbb{D}$ at exactly one point, which is a fixed point of $\varphi(A)$;

Fourth case: $\varphi(A)(\mathbb{D}) = \mathbb{D}$.

We are going to determine the spectra in all four cases; that is the contents of Sections 5–8, respectively. The fourth case is made easier by a theorem of Kitover [10], combined with the known results about composition operators on H^∞.

The following fact will be frequently used in the sequel: if φ is a nonconstant fractional linear transformation, different from the identity, then it has exactly one or two fixed points in $\mathbb{G}$.

5. FIRST CASE

$\mathrm{cl}(\varphi(A)(\mathbb{D})) \subset \mathbb{D}$. In this case, $\varphi(A)$ has a fixed point z_0 in $\mathbb{D}$; namely, $\{z_0\} = \bigcap_1^\infty \mathrm{cl}\,\varphi(A)^n(\mathbb{D})$. We assert that it has one more fixed point in $\mathbb{G}$; it is a consequence of the following lemma.

Lemma 6. *If φ is a nonconstant fractional linear transformation having only one fixed point and $\varphi(K) \subset K$ for some open disc K in $\mathbb{G}$, then the fixed point lies in ∂K. (By an open disc in $\mathbb{G}$, we mean an open disc in the complex plane, or its exterior in $\mathbb{G}$, or an open half-plane.)*

Proof. We can suppose the fixed point to be ∞. Because it is to be the only fixed point, $\varphi(z) = z + b\ \forall z \in \mathbb{C}$, for some $b \in \mathbb{C}$, $b \neq 0$. Now $\varphi(K) \subset K$ clearly implies that K must be a half-plane, and so $\infty \in \partial K$.

Because $z_0 \in \mathbb{D}$ and $\varphi(A)(\mathbb{D}) \subset \mathbb{D}$, we see that $\varphi(A)$ (which is not the identity) has exactly one more fixed point, z_1, in $\mathbb{G}$; clearly $z_1 \notin \mathbb{D}$. Denote $\Omega = \bigcup_{n=1}^\infty \varphi(A)^{-n}(\mathbb{D})$; this is an open subset of $\mathbb{G}$.

Lemma 7. $\Omega = \mathbb{G} \setminus \{z_1\}$.

Proof. Choose a fractional linear transformation ψ sending 0 into z_0 and ∞ into z_1, and let $\varphi = \psi^{-1}\,\varphi(A)\,\psi$. Then φ has 0 and ∞ as fixed points, so it has the form $\varphi(z) = az$, $a \in \mathbb{C}$, $a \neq 0$. Also $\mathrm{cl}\,\varphi(K) \subset K$, where $K = \psi^{-1}(\mathbb{D})$ is some open disc containing 0; this forces $|a| < 1$. Now it is already clear that $\bigcup_{n=1}^\infty \varphi^{-n}(K) = \mathbb{G} \setminus \{\infty\}$, which establishes the lemma.

Now we are ready to determine $\sigma(m(A))$. By Theorem 5, $m(A)$ is compact, so $\sigma(m(A))$ = zero plus all nonzero eigenvalues. So, let us pick $\lambda \neq 0$ and try to find an $f \in H^2$ such that

$$(2) \qquad f(z) = \frac{1}{\lambda(cz + d)} f(\varphi(A)(z)) \quad \forall z \in \mathbb{D}\,.$$

Suppose first $c \neq 0$, so $z_1 \neq \infty$. We assert, via the relation (2), that f admits a holomorphic continuation to Ω, which is $\mathbb{G} \setminus \{z_1\}$ by the preceding lemma. The only problems could arise when $cz + d = \infty$ or $cz + d = 0$. Because $\varphi(A)(-d/c) = \infty$, the former occurs first. So suppose f is already defined in a neighbourhood of $\varphi(A)(\infty) = a/c$; then

$$\lim_{z \to \infty} \frac{1}{\lambda(cz + d)} f(\varphi(A)(z)) = \frac{f(a/c)}{\infty} = 0\,,$$

and by the Riemann removable singularities theorem we can define $f(\infty) = 0$ and f will be analytic in a neighbourhood of ∞. Furthermore, as $f(\infty) = 0$, a finite limit

$$\lim_{z \to -d/c} \frac{1}{\lambda(cz + d)} f(\varphi(A)(z))$$

exists and $f(-d/c)$ can be defined to be this limit.

Thus we arrive at a function f, holomorphic on $\Omega = \mathbb{G} \setminus \{z_1\}$, satisfying (2) there, and with $f(\infty)$ equal to zero. Define

$$F(y) = f\left(\frac{z_0 - yz_1}{1 - y}\right).$$

Then F is holomorphic in $\mathbb{G} \setminus \{\infty\} = \mathbb{C}$ (i.e. F is an entire function), $F(1) = f(\infty) = 0$ and (2) can be transcribed into the form

$$\lambda\, F(y)\,(cz_0 + d)\,\frac{1 - \varrho y}{1 - y} = F(\varrho y) \quad \text{for all} \quad y \in \mathbb{C}\,,$$

where $\varrho = (cz_1 + d)/(cz_0 + d)$. Because $F(1) = 0$, $G(y) = F(y)/(1 - y)$ is also an entire function and satisfies

$$\lambda(cz_0 + d)\, G(y) = G(\varrho y)\,.$$

Comparing the Taylor coefficients on both sides, we see that $G(y) = y^n$ and $\lambda = \varrho^n/(cz_0 + d) = (cz_1 + d)^n/(cz_0 + d)^{n+1}$. Going back, we get

$$f(z) = \frac{(z_0 - z_1)(z - z_0)^n}{(z - z_1)^{n+1}}\,,$$

which belongs to H^2 (even to H^∞), because $z_1 \notin \mathbb{D}$. So the nonzero eigenvalues of $m(A)$ are precisely the numbers $(cz_1 + d)^n/(cz_0 + d)^{n+1}$, $n = 0, 1, 2, \ldots$.

It remains to treat the case $c = 0$. This time $z_1 = \infty$; proceeding in the same way as before, one can show that f possesses an analytic continuation to all of $\mathbb{C}$ and the function $F(y) = f(z_0 - y)$ satisfies

$$\lambda\, F(y) = \frac{1}{d} F\left(\frac{a}{d} y\right), \quad \text{for all} \quad y \in \mathbb{C}\,.$$

Comparing the Taylor coefficients gives $f(z) = (z_0 - z)^n$, which belongs to H^2 and the corresponding eigenvalue is $\lambda = a^n/d^{n+1}$.

This completes the discussion of the first case.

6. SECOND CASE

$\varphi(A)(\mathbb{D})$ touches $\partial\mathbb{D}$ at one point – say, τ – and $\varphi(A)(\tau) \neq \tau$. Then $\mathrm{cl}\, \varphi(A)^2(\mathbb{D}) \subset \subset \mathbb{D}$, and what we know about the „first case" applies to $\varphi(A)^2$: the operator $m(A)^2$ is compact, and so its spectrum, and, consequently, the spectrum of $m(A)$, consists of zero plus the eigenvalues. If $\lambda f = m(A) f$, $\lambda \neq 0$, then $m(A)^2 f = \lambda^2 f$, and so f must be

$$\frac{(z_0 - z_1)(z - z_0)^n}{(z - z_1)^{n+1}} \quad \text{or} \quad (z - z_0)^n,$$

where z_0, z_1 or z_0, ∞, respectively, are the fixed points of $\varphi(A)^2$, and $z_0 \in \mathbb{D}$. Putting these expressions into $m(A) f = \lambda f$ yields

$$\lambda = \frac{(cz_1 + d)^n}{(cz_0 + d)^{n+1}} \quad \text{or} \quad \lambda = \frac{a^n}{d^{n+1}}.$$

Finally, $\{z_0\} = \bigcap\limits_1^\infty \mathrm{cl}\, \varphi(A)^{2n}(\mathbb{D}) = \bigcap\limits_1^\infty \mathrm{cl}\, \varphi(A)^n(\mathbb{D})$, so z_0 is a fixed point of $\varphi(A)$ as well; by Lemma 7, $\varphi(A)$ has exactly one more fixed point. Because this will be also a fixed point for $\varphi(A)^2$, it must be z_1 (or ∞). Thus, we conclude that z_0 and z_1 (or ∞) are the fixed points of $\varphi(A)$ and we see that the result is the same as for the „first case".

7. THIRD CASE

$\varphi(A)(\mathbb{D})$ touches $\partial\mathbb{D}$ at exactly one point τ, and $\varphi(A)(\tau) = \tau$.

Without loss of generality, we may assume $\tau = 1$. Indeed, if $M \in \mathcal{M}$ is such that $\varphi(M)(\tau) = 1$, then $\varphi(MAM^{-1})$ falls into the „third case" with $\tau = 1$, and the operators $m(A)$ and $m(MAM^{-1})$ are unitarily equivalent by Proposition 4.

Further treatment differs, depending on whether $\varphi(A)$ has one or two fixed points.

I. Suppose $\varphi(A)$ has **two fixed points**, 1 and $z_1 \neq 1$; clearly $z_1 \notin \mathbb{D}$. Let $\alpha \in \mathbb{D}$ and

$$M = \begin{pmatrix} \varepsilon q & \varepsilon\alpha q \\ \bar{\alpha} q & q \end{pmatrix}, \quad q = (1 - |\alpha|^2)^{-1/2}, \quad \varepsilon = \frac{1 + \bar{\alpha}}{1 + \alpha}.$$

Then $M \in \mathcal{M}$, and so $m(A)$ and $m(MAM^{-1})$ are unitarily equivalent. Also $\varphi(M)(1) = 1$. If we show that $\varphi(M)(z_1) = \infty$ for some $\alpha \in \mathbb{D}$, then we can assume, without loss of generality, that $z_1 = \infty$. That is the contents of the following lemma.

Lemma 8. *If $z_1 \notin \overline{\mathbb{D}}$, then there exists $\alpha \in \mathbb{D}$ such that $\varphi(M)$ maps z_1 into ∞.*

Proof. It suffices to prove that, while α runs through $\mathbb{D}$, $\varphi(M)(\infty)$ runs through all of $\mathbb{G} \setminus \mathbb{D}$; a moment's thought reveals that this is equivalent to showing that $\varphi(M)(0)$ runs through all of $\mathbb{D}$ as α does. Denote

$$\varphi(M)(0) = \frac{1 + \bar{\alpha}}{1 + \alpha} \alpha = \psi(\alpha) .$$

Let C_r be the circle $|z| = r$; we shall show that $\psi(C_r)$ covers all of C_r, for every $r \in \langle 0, 1)$. $r = 0$ is trivial; so let $r > 0$. Because $\psi(C_r) \subset C_r$, it suffices to show that $\mathrm{Ind}_{\psi(C_r)}\, 0 \neq 0$. But this index equals

$$\frac{1}{2\pi \mathrm{i}} \int_{\psi(C_r)} \frac{\mathrm{d}x}{x} = \frac{1}{2\pi \mathrm{i}} \int_{C_1} \frac{1 - r^2}{(r + z)(1 + rz)} \mathrm{d}z =$$

$$= \mathrm{Res}_{z=-r} \frac{1 - r^2}{(r + z)(1 + rz)} = 1 .$$

We have made use of the fact that $\psi(y) = (1 + (r^2/y))/(1 + y) \,.\, y$ if $|y| = r$, and performed the substitution $x = \psi(rz) = (r^2 + rz)/(1 + rz)$. This proves the lemma.

So we may suppose $z_1 = \infty$. This implies that $A = \begin{pmatrix} a & b \\ 0 & a + b \end{pmatrix}$, $\varphi(A)(z) = = (az + b)/(a + b) = 1 + (a/(a + b))(z - 1)$, and $\varphi(A)(\mathbb{D}) \subset \mathbb{D}$ forces $\varrho \in (0, 1)$, where $\varrho = a/(a + b)$. Define the functions $f_\alpha(z)$, univalent branches of $(z - 1)^\alpha$ on $\mathbb{D}$, in the same way as at the beginning of Section 3. Just as before, one can show that $m(A) f_\alpha = \lambda_\alpha f_\alpha$ with

$$\lambda_\alpha = \frac{1}{a + b} \exp(\alpha \ln \varrho) .$$

A routine argument shows that $f_\alpha \in H^2$ whenever $\mathrm{Re}\, \alpha < -1/2$. If α runs through this half-plane, λ_α runs through the disc with center at the origin and radius

$$\left| \frac{\varrho^{-1/2}}{a + b} \right| = \frac{1}{|a(a + b)|^{1/2}} = |\det A|^{-1/2} .$$

Iterating $\varphi(A)$ gives $\varphi(A)^n(z) = 1 + \varrho^n(z - 1)$, so $\varphi(A)^n(\mathbb{D})$ is the disc centered at $1 - \varrho^n$ with radius ϱ^n. Using the estimate (1) for the norm, we have

$$\|m(A)^n\|^2 \leqq \frac{2\pi\, \varrho^n}{2\pi |\det A|^n} \cdot \frac{1 + (1 - \varrho^n)}{1 - (1 - \varrho^n)} \leqq \frac{2}{|\det A|^n}$$

and the formula for the spectral radius gives

$$\text{spectral radius of } \; m(A) \leqq |\det A|^{-1/2} .$$

Summing up, we see that in this case

$$\sigma(m(A)) = |\det A|^{-1/2}\, \mathbb{D} ,$$

i.e. the spectrum is a disc centered at the origin.

II. It remains to consider the case when $\varphi(A)$ has **only one fixed point** (namely, 1). Let $\psi(z) = 1/(1 - z)$; then $\psi\,\varphi(A)\,\psi^{-1}$ has ∞ as the only fixed point, and so it is the translation by a nonzero vector $b \in \mathbb{C}$. Thus

$$\frac{1}{\varphi(A)(z) - 1} = \frac{1}{z - 1} + b\,,$$

and

$$\varphi(A)(z) = \frac{(1 + b)z - b}{bz + (1 - b)}\,, \quad b \neq 0\,. \tag{3}$$

The requirement $\varphi(A)(\mathbb{D}) \subset \mathbb{D}$ forces $\operatorname{Re} b < 0$.

For the determination of the spectrum, we will employ the argument adopted from Cowen [1, page 102]. First note that (3) implies

$$A = t\begin{pmatrix} 1 + b & -b \\ b & 1 - b \end{pmatrix} \tag{3a}$$

for some $t \in \mathbb{C}$, $t \neq 0$. Because $m(A/t) = t\,m(A)$, it suffices to consider the case $t = 1$. Then the matrices

$$\begin{pmatrix} 1 + b & -b \\ b & 1 - b \end{pmatrix} =^{\text{def}} A_b\,, \quad \operatorname{Re} b < 0\,,$$

satisfy $A_{b_1 + b_2} = A_{b_1} \cdot A_{b_2}$, i.e. they form a multiplicative semigroup, isomorphic, under the correspondence $b \leftrightarrow A_b$, to the additive semigroup of all complex numbers with negative real parts. It follows that the operators $m_b =^{\text{def}} m(A_b)$, $\operatorname{Re} b < 0$, also form a commutative semigroup: $m_{b_1 + b_2} = m_{b_1} \cdot m_{b_2}$.

Lemma 9. *The semigroup m_b is norm-holomorphic (i.e. $b \mapsto m_b$ is a norm-holomorphic operator-valued function).*

Proof. By [4, page 93], Theorem 3.10.1, it suffices to show that for any $f \in H^2$ and $x \in \mathbb{D}$ the function

$$b \mapsto \langle m_b f, g_x \rangle$$

is analytic in the left half-plane. Here, as before, $g_x(z) = (1 - \bar{x}z)^{-1}$ is the reproducing kernel. Now observe that

$$\langle m_b f, g_x \rangle = (m_b f)(x) = \frac{1}{bx + (1 - b)} f\left(\frac{(1 + b)x - b}{bx + (1 - b)}\right),$$

which is certainly analytic in b for any $x \in \mathbb{D}$. Q.E.D.

Consider the norm-closed subalgebra $\mathscr{C}$ of $\mathscr{B}(H^2)$ generated by the m_b's, $\operatorname{Re} b < 0$, and the identity. This is a commutative Banach algebra, and so, according to the Gelfand theory,

$$\sigma(m_b) = \{\Lambda(m_b) \colon \Lambda \text{ is a multiplicative lin. functional on } \mathscr{C}\}\,. \tag{4}$$

Fix such a Λ and consider the function $\lambda(b) =^{\mathrm{def}} \Lambda(m_b)$. By Lemma 9, this function is analytic in the half-plane

$$\mathscr{H} = \{z \in \mathbb{C}\colon \operatorname{Re} z < 0\},$$

and satisfies the semigroup condition

$$\lambda(b_1 + b_2) = \lambda(b_1)\,\lambda(b_2).$$

It follows that either λ vanishes everywhere, or $\lambda(x) = \mathrm{e}^{\beta x}$ for some $\beta \in \mathbb{C}$. Denote by $\mathscr{B}$ the set of all beta's that arise in this way, and add the symbol $+\infty$ to $\mathscr{B}$ in case the possibility $\lambda \equiv 0$ also occurs. Then we have

$$\text{(5)} \qquad \sigma(m_b) = \{\mathrm{e}^{\beta b}\colon \beta \in \mathscr{B}\}, \quad \text{for all} \quad b \in \mathscr{H},$$

where, for a while, we set $\mathrm{e}^{\beta b} = 0$ for $b \in \mathscr{H}$ and $\beta = +\infty$.

For $\operatorname{Re} b < 0$, the disc $\varphi(A_b)(\mathbb{D})$ is contained in $\mathbb{D}$ and touches $\partial\mathbb{D}$ at 1, so it must have center $1 - r_b$ and radius r_b for some $r_b \in (0, 1)$. The norm estimate (1) gives

$$\|m_b\|^2 \leqq \frac{2\pi r_b}{2\pi\,.\,1}\,.\,\frac{1 + (1 - r_b)}{1 - (1 - r_b)} = 2 - r_b \leqq 2,$$

implying $|\lambda(b)| \leqq \|\Lambda\|\,.\,\|m_b\| \leqq \sqrt{2}$ for all $b \in \mathscr{H}$. This forces $\beta = +\infty$ or $\beta > 0$, so

$$\mathscr{B} \subset \langle 0, +\infty\rangle.$$

We are going to show that this inclusion is, in fact, an equality. Let us first prove the following lemma.

Lemma 10. *For any $t > 0$ and $b \in \mathscr{H}$, the operators m_b and m_{tb} are unitarily equivalent.*

Proof. Just as before Lemma 8, consider the matrices

$$M = \begin{pmatrix} \varepsilon q & \varepsilon\alpha q \\ \bar{\alpha} q & q \end{pmatrix}, \quad q = (1 - |\alpha|^2)^{-1/2}, \quad \varepsilon = \frac{1 + \bar{\alpha}}{1 + \alpha},$$

where $\alpha \in \mathbb{D}$. Because $M \in \mathscr{M}$, the operators $m_b = m(A_b)$ and $m(MA_bM^{-1})$ are unitarily equivalent (Proposition 4). A few minutes' calculation reveals that

$$MA_bM^{-1} = A_{b_1},$$

with

$$b_1 = b\,.\,\frac{|1 + \alpha|^2}{1 - |\alpha|^2}.$$

Now observe that as α runs through $\mathbb{D}$ (or, even, only through $(-1, +1)$), the value

$$t = \frac{|1 + \alpha|^2}{1 - |\alpha|^2}$$

runs through all of $(0, +\infty)$, which proves our assertion.

Suppose there exists $\beta_0 \in (0, +\infty)$ such that $\beta_0 \notin \mathscr{B}$. Owing to (5), $\exp(\beta_0 b) \notin \sigma(m_b)$ for all $b \in \mathscr{H}$. According to Lemma 10, $\sigma(m_b) = \sigma(m_{tb})$ whenever $t > 0$; so $\exp(\beta_0 b) \notin \sigma(m_{tb})$, and using (5) once again we conclude that $\beta_0/t \notin \mathscr{B}$. Because this holds for every $t > 0$, $\mathscr{B}$ can contain only 0 and $+\infty$. It follows that every multiplicative linear functional on $\mathscr{C}$ is either 0 on all m_b's, or 1 on all of them. But the m_b's plus the identity generate $\mathscr{C}$; so $\mathscr{C}$ has at most two multiplicative linear functionals. Consequently, it has dimension at most 2, which clearly is not the case.

This contradiction shows that $(0, +\infty) \subset \mathscr{B}$. Looking at (5) and recalling that a spectrum is always a closed set, we see that $\mathscr{B} = \langle 0, +\infty \rangle$.

Summing up, we have shown that, for all $b \in \mathscr{H}$,

$$\sigma(m_b) = \{e^{\beta b}: \beta \in \langle 0, +\infty)\} \cup \{0\} .$$

8. FOURTH CASE

$\varphi(A)(\mathbb{D}) = \mathbb{D}$. Since no new techniques are used in this section, we will proceed a little more briefly. Because $\varphi(A)$ is a Möbius transformation, $A = tA_0$ for some $t \in \mathbb{C} \setminus \{0\}$, $A_0 \in \mathscr{M}$, and $m(A) = (1/t)\, m(A_0)$. It suffices to consider $t = 1$, i.e. $A \in \mathscr{M}$. This implies $m(A)$ is unitary and $\sigma(m(A)) \subset \partial\mathbb{D}$.

The easiest case to handle occurs when $\varphi(A)$ has one fixed point in $\mathbb{D}$ and the other outside $\mathbb{D}$. Conjugating by an appropriate Möbius transformation, one can suppose the former to be 0. Then $\varphi(A)(z) = \varepsilon z$ for some $\varepsilon \in \partial\mathbb{D}$ and $\sigma(m(A))$ is readily seen to be the closure of $\{\varepsilon^n: n = 0, 1, 2, \ldots\}$. This is either the set of all N-th roots of unity, for some N, or the whole of $\partial\mathbb{D}$.

The parabolic case (i.e. $\varphi(A)$ has only one fixed point in $\mathbb{G}$, lying on $\partial\mathbb{D}$) is more difficult; it can be treated in a similar way as in Section 7 - II. We may suppose A to be

$$\begin{pmatrix} 1 + b & -b \\ b & 1 - b \end{pmatrix}, \quad b \neq 0,$$

this time with $\operatorname{Re} b = 0$ (Theorem 2), i.e. $b = si$, $s \in \mathbb{R}$, $s \neq 0$. The operators $T_s \stackrel{\text{def}}{=} m(A_{is})$, $s \in \mathbb{R}$, form a commutative group of unitary operators ($T_{s+t} = T_s T_t$, $T_s^* = T_{-s}$ for $s, t \in \mathbb{R}$). By a classical theorem of M. H. Stone (cf. [12], Theorem 13.37), $T_s = e^{isH}$, where iH is the infinitesimal generator of the group $\{T_s\}_{s\in\mathbb{R}}$, $H^* = H$. Also, $\sigma(T_s)$ is the closure of $e^{is\sigma(H)}$ (cf. [12], Theorem 13.27c). Looking at the proof of Lemma 10, we see that it carries over verbatim to the present situation and, moreover, the operator which establishes the unitary equivalence between T_s and T_{st} depends only on t, not on s:

$$U_t^* T_s U_t = T_{st} \quad \text{for all} \quad t > 0 , \quad s \in \mathbb{R} .$$

It follows that $U_t^* H U_t = tH$, and so $\sigma(H) = t\, \sigma(H)$ for each $t > 0$. Thus, $\sigma(H)$ can be only $\{0\}$, $\langle 0, +\infty)$, $(-\infty, 0)$ or $\mathbb{R}$. Reasoning as at the end of Section 7 rules out the first possibility; consequently,

$$\sigma(T_s) = \partial\mathbb{D} \quad \text{for} \quad s \in \mathbb{R}\,, \quad s \neq 0\,.$$

The case that remains is that $\varphi(A)$ has two fixed points, both of which lie on $\partial\mathbb{D}$. This case seems to be the most difficult. The fixed points may be supposed to be $+1$ and -1 (this can be shown by proving an analogue of Lemma 8 – with $z_1 \in \partial\mathbb{D} \setminus \{1\}$ and -1 instead of ∞) and A may be assumed to be

$$A_r =^{\text{def}} (1 - r^2)^{-1/2} \begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix}, \quad r \in (-1, 1)\,, \quad r \neq 0\,.$$

Now we are going to use a result of A. K. Kitover [10], which is reproduced below for the special case $X = H^2$, $A = H^\infty$, $a = \mathbf{1}$ (consult [10] for this notation).

Theorem. *Suppose that*
1. *the operator $\mathcal{U} = m(A_r)$ is bounded on H^2 and its spectrum lies on $\partial\mathbb{D}$;*
2. *$H^\infty \supset B_1^1$, the space of all functions analytic on $\mathbb{D}$ and such that their second derivative is square-integrable over $\mathbb{D}$;*
3. *when f, g are functions analytic on $\mathbb{D}$, B is a Blaschke product and $f = Bg$, then $f \in H^2 \Leftrightarrow g \in H^2$. (It would even suffice to consider Blaschke products whose zeros are subject to some special condition, but, for our purposes, this formulation will do.)*

Then the spectrum of $\mathcal{U}$ is the same as the spectrum of the operator V, defined on H^∞ by the formula

$$(Vg)(z) =^{\text{def}} g(\varphi(A_r)(z))\,.$$

The only assumption which is not fulfilled at first sight is, perhaps, the second; but note that a function

$$f = \sum_0^\infty a_n z^n$$

belongs to B_1^1 if and only if

$$\sum_0^\infty |a_{n+2}|^2 (n+2)^2 (n+1) < +\infty\,,$$

and that

$$|f(z) - a_1 z - a_0| = |\sum_2^\infty a_n z^n| \leqq \sum_0^\infty |a_{n+2}| \leqq$$
$$\leqq (\sum_0^\infty |a_{n+2}|^2 (n+2)^2 (n+1))^{1/2} (\sum_0^\infty (n+2)^{-2} (n+1)^{-1})^{1/2}$$

whenever $z \in \mathbb{D}$. So indeed, $B_1^1 \subset H^\infty$.

Applying the theorem and recalling that $\sigma(V) = \partial\mathbb{D}$ (see, for example, [8, the second Corollary on page 269]), we conclude that, again, $\sigma(m(A_r)) = \partial\mathbb{D}$, the same result as when $\varphi(A)$ was parabolic.

9. THE SPECTRUM

Summing up the results from all the cases under discussion, we obtain the following theorem.

Theorem 11. *Let* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *be a regular* 2×2 *complex matrix such that* $\varphi(A)(\mathbb{D}) \subset \mathbb{D}$.

(a) *If* $\operatorname{cl} \varphi(A)(\mathbb{D}) \subset \mathbb{D}$, *then* $\varphi(A)$ *has two fixed points* z_0 *and* z_1, $z_0 \in \mathbb{D}$, $z_1 \notin \bar{\mathbb{D}}$, *and*

$$\sigma(m(A)) = \{0\} \cup \left\{\frac{(cz_1 + d)^n}{(cz_0 + d)^{n+1}} : n = 0, 1, \ldots\right\},$$

or, if $c = 0$ (*and* $z_1 = \infty$),

$$\sigma(m(A)) = \{0\} \cup \left\{\frac{a^n}{d^{n+1}} : n = 0, 1, \ldots\right\}.$$

(b) *If* $\varphi(A)(\mathbb{D})$ *touches* $\partial\mathbb{D}$ *at exactly one point* τ *and* $\varphi(A)(\tau) \neq \tau$, *everything is the same as in* (a).

(c) *If* $\varphi(A)(\mathbb{D})$ *touches* $\partial\mathbb{D}$ *at exactly one point* τ, $\varphi(A)(\tau) = \tau$ *and* $\varphi(A)(z) = z$ *for some* $z \neq \tau$, *then*

$$\sigma(m(A)) = |\det A|^{-1/2}\, \bar{\mathbb{D}}.$$

(d) *If* $\varphi(A)(\mathbb{D})$ *touches* $\partial\mathbb{D}$ *at exactly one point* τ, $\varphi(A)(\tau) = \tau$ *and* $\varphi(A)$ *has no other fixed points, then*

$$\frac{1}{\varphi(A)(z) - \tau} = \frac{1}{z - \tau} + q$$

for some $q \in \mathbb{C}$, $\operatorname{Re} q < 0$, *and*

$$\sigma(m(A)) = \{0\} \cup \{te^{\beta q} : \beta \in \langle 0, +\infty)\}$$

for some complex number t *of modulus* $|\det A|^{-1/2}$.

(e) *If* $\varphi(A)(\mathbb{D}) = \mathbb{D}$ *and* N *is the smallest positive integer such that* $\varphi(A)^N = \mathrm{id}$, *then*

$$\sigma(m(A)) = t\{\varepsilon^n : n = 0, 1, \ldots, N - 1\},$$

where $\varepsilon = \exp(2\pi i/N)$ *and* t *is a complex number of modulus* $|\det A|^{-1/2}$.

(f) *If* $\varphi(A)(\mathbb{D}) = \mathbb{D}$ *and* $\varphi(A)^N \neq \mathrm{id}$ *for all* $N = 1, 2, \ldots$, *then*

$$\sigma(m(A)) = |\det A|^{-1/2}\, \partial\mathbb{D}.$$

Proof. Everything is just a restatement of what has been said before, except for the factor $|\det A|^{-1/2}$ in (d)–(f). To clear up this point, note that for any $M \in \mathcal{M}$

$$\det A = \det MAM^{-1},$$

and so the modulus of t in (3a) must be $|\det A|^{1/2}$; it remains to use the fact that $m(tA) = (1/t)\, m(A)$. This settles (d). As for (e) and (f), we have $\varphi(A)^n(\mathbb{D}) = \mathbb{D}$ for all $n = 0, \pm 1, \pm 2, \ldots$. Consequently, the norm estimate (1) gives

$$\|m(A)^n\|^2 \leqq \frac{2\pi}{2\pi|\det A^n|} \cdot \frac{1+0}{1-0} = |\det A|^{-n}$$

for all integers n. Hence, the spectral radius of $m(A)$ equals $|\det A|^{-1/2}$. This concludes the proof.

10. THE SPECTRAL RADIUS

Since we know what the spectrum of $m(A)$ is, we can determine its spectral radius.

Theorem 12. *Let* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *be a regular* 2×2 *complex matrix such that* $\varphi(A)(\mathbb{D}) \subset \mathbb{D}$. *Then*

(a) *If* cl $\varphi(A)(\mathbb{D}) \subset \mathbb{D}$, *then* $\varphi(A)$ *has a unique fixed point* z_0 *in* $\mathbb{D}$ *and*

$$\|m(A)\|_{sp} = \frac{1}{|cz_0 + d|}.$$

(b) *If* $\varphi(A)(\mathbb{D})$ *touches* $\partial\mathbb{D}$ *at exactly one point* τ *and* $\varphi(A)(\tau) \neq \tau$, *the same conclusion as in* (a) *holds.*

(c) *If* $\varphi(A)(\mathbb{D})$ *touches* $\partial\mathbb{D}$ *at exactly one point* τ *and* $\varphi(A)(\tau) = \tau$, *then*

$$\|m(A)\|_{sp} = |\det A|^{-1/2}.$$

(d) *If* $\varphi(A)(\mathbb{D}) = \mathbb{D}$, *the the same conclusion as in* (c) *holds.*

Proof. (a) and (b): by Theorem 11, (a) and (b),

$$\sigma(m(A)) \setminus \{0\} = \left\{\frac{1}{cz_0 + d} \cdot \varrho^n \colon\ n = 0, 1, \ldots\right\},$$

where

$$\varrho = \frac{cz_1 + d}{cz_0 + d} \quad \text{or} \quad \varrho = \frac{a}{d} \quad (\text{if } c = 0).$$

Because $m(A)$ or $m(A)^2$ is compact by Theorem 5, we have $|\varrho| < 1$. This implies $\sup\{|\varrho^n| \colon n = 0, 1, \ldots\} = 1$ and the result follows.

(c): Follows from Theorem 11, part (c) and (d).

(d): Has been proved in the course of the proof of Theorem 11, parts (e)–(f).
The proof is complete.

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Souhrn

O JEDNÉ TŘÍDĚ VÁŽENÝCH KOMPOZIČNÍCH OPERÁTORŮ NA H^2

MIROSLAV ENGLIŠ

Každé lineární lomené transformaci, která zobrazuje jednotkový kruh do sebe, lze přiřadit vážený kompoziční operátor na Hardyho prostoru H^2. Tyto operátory byly nedávno zavedeny v práci [11] v souvislosti s jistým extremálním problémem z teorie operátorů. V této práci se zabýváme jejich základními vlastnostmi až po určení jejich spekter. Analogické výsledky pro jiné třídy kompozičních operátorů lze nalézt v pracích [1], [3], [5].

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