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THE PERRON PRODUCT INTEGRAL AND GENERALIZED LINEAR DIFFERENTIAL EQUATIONS

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Summary. The concept of the Perron product integral due to J. Jarník and J. Kurzweil is investigated. The class of Perron product integrable „point — interval" functions is extended and it is shown that this extension is suitable for the representation of the fundamental matrix of generalized linear differential equations.

Keywords: Perron product integral, generalized linear differential equations.

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INTRODUCTION

In the recent paper [2] of J. Jarník and J. Kurzweil a definition of the Perron product integral is given, which is the „product form" of an analogous concept of the sum integral. In [2] the basic properties of the product integration are developed and the product integral is connected with a relatively wide class of linear ordinary differential equations of the form

$$\dot{u} = a(t)\, u$$

where a is an $n \times n$-matrix valued function.

Here we use the definition from [2] for a slightly more general class of Perron product integrable functions. In Section 1 we consider the properties of the product integral in an analogous way as this was done in [2] and in Section 2 we give further results which can be applied to generalized linear differential equations of the form

$$x(s) = x(a) + \int_a^s \mathrm{d}[A(r)]\, x(r)\,, \quad s \in [a, b]$$

where A is an $n \times n$-matrix valued function of bounded variation on $[a, b]$. The concept of generalized linear differential equations is given e.g. in [3] and [4]. A product integral representation of the fundamental matrix of a generalized linear differential equation is derived under some additional assumptions on the matrix valued function A.

1. THE PERRON PRODUCT INTEGRAL AND THE CONDITION $\mathscr{C}$

Let $n \in \mathbb{N}$ and let $\mathbb{R}^n$ be the n-dimensional Euclidean space. We denote by $L(\mathbb{R}^n)$ the set of all linear operators from $\mathbb{R}^n$ to $\mathbb{R}^n$ (the $n \times n$-matrices) and assume that $\|\cdot\|$ is the corresponding operator norm in $L(\mathbb{R}^n)$.

Let $[a, b] \subset \mathbb{R}$ be a compact interval and let J be the set of all compact subintervals in $[a, b]$, i.e. intervals of the form $[x, y]$, where $a \leqq x \leqq y \leqq b$. Assume that a function $V: [a, b] \times J \to L(\mathbb{R}^n)$ is given.

A finite set

$$\Delta = \{\alpha_0, t_1, \alpha_1, t_2, \alpha_2, \ldots, \alpha_{k-1}, t_k, \alpha_k\}$$

is called a partition of the interval $[a, b]$ if

$$a = \alpha_0 < \alpha_1 < \ldots < \alpha_k = b$$

and

$$t_j \in [\alpha_{j-1}, \alpha_j], \quad j = 1, 2, \ldots, k.$$

Given a function $\delta: [a, b] \to (0, +\infty)$, called a gauge on $[a, b]$, the partition Δ of $[a, b]$ is said to be δ-fine, if

$$I_i = [\alpha_{i-1}, \alpha_i] \subset (t_i - \delta(t_i), t_i + \delta(t_i)), \quad i = 1, 2, \ldots, k.$$

For the function $V: [a, b] \times J \to L(\mathbb{R}^n)$ and a given partition Δ of $[a, b]$ denote

$$P(V, \Delta) = V(t_k, [\alpha_{k-1}, \alpha_k]) V(t_{k-1}, [\alpha_{k-2}, \alpha_{k-1}]) \ldots V(t_1, [\alpha_0, \alpha_1]) =$$
$$= V(t_k, I_k) V(t_{k-1}, I_{k-1}) \ldots V(t_1, I_1).$$

1.1. Definition. A function $V: [a, b] \times J \to L(\mathbb{R}^n)$ is called *Perron product integrable* if there exists $Q \in L(\mathbb{R}^n)$ which is invertible such that for every $\varepsilon > 0$ there is a gauge $\delta: [a, b] \to (0, +\infty)$ on $[a, b]$ such that

$$(1.1) \qquad \|P(V, \Delta) - Q\| < \varepsilon$$

for every δ-fine partition Δ of $[a, b]$.

$Q \in L(\mathbb{R}^n)$ is called the Perron product integral of V over $[a, b]$ and we use the notation $Q = \prod_a^b V(t, \mathrm{d}t)$.

1.2. Remark. This definition follows exactly the line of definition of the Perron product integral given by J. Jarník and J. Kurzweil in their paper [2]. In [2] the notation $(PP) \int_a^b V(t, \mathrm{d}t)$ is used for Q. It has to be mentioned that the set of δ-fine partitions Δ of $[a, b]$ is nonempty for every given gauge δ on $[a, b]$ (see e.g. [4]). Therefore the notion of Perron product integrability given in Definition 1.1 makes sense.

Because the space $L(\mathbb{R}^n)$ with the operator norm $\|\cdot\|$ is a Banach space (i.e. complete), it is easy to see that the following holds.

1.3. Proposition. *Let* $V: [a, b] \times J \to L(\mathbb{R}^n)$ *be given. The following two conditions are equivalent.*

(i) *There is a* $Q \in L(\mathbb{R}^n)$ *such that for every* $\varepsilon > 0$ *there is a gauge* $\delta: [a, b] \to (0, +\infty)$ *such that* $\|P(V, \Delta) - Q\| < \varepsilon$ *for any* δ*-fine partition* Δ *of* $[a, b]$.

(ii) *For every* $\varepsilon > 0$ *there exists a gauge* $\delta: [a, b] \to (0, +\infty)$ *such that* $\|P(V, \Delta_1) - P(V, \Delta_2)\| < \varepsilon$ *for any* δ*-fine partitions* Δ_1, Δ_2 *of* $[a, b]$.

In the sequel we will assume that the function $V: [a, b] \times J \to L(\mathbb{R}^n)$ satisfies the following condition.

Condition $\mathscr{C}$.

(1.2) $V(t, [t, t]) = I$ *for every* $t \in [a, b]$, *where* $I \in L(\mathbb{R}^n)$ *is the identity operator in* $L(\mathbb{R}^n)$;

(1.3) *for every* $t \in [a, b]$ *and* $\zeta > 0$ *there exists* $\sigma > 0$ *such that*

$$\|V(t, [x, y]) - V(t, [t, y]) V(t, [x, t])\| < \zeta$$

for all $x, y \in [a, b]$, $t - \sigma < x \leqq t \leqq y < t + \sigma$;

(1.4) *for every* $t \in [a, b)$ *there is an invertible* $V_+(t) \in L(\mathbb{R}^n)$ *such that*

$$\lim_{y \to t+} \|V(t, [t, y]) - V_+(t)\| = 0, \text{ i.e.}$$

$$\lim_{y \to t+} V(t, [t, y]) = V_+(t)$$

and for every $t \in (a, b]$ *there is an invertible* $V_-(t) \in L(\mathbb{R}^n)$ *such that*

$$\lim_{x \to t-} \|V(t, [x, t]) - V_-(t)\| = 0, \quad \textit{i.e.}$$

$$\lim_{x \to t-} V(t, [x, t]) = V_-(t).$$

1.4. Remark. In [2] it is assumed that the function $V: [a, b] \times J \to L(\mathbb{R}^n)$ satisfies the following condition

(1.5) *for every* $t \in [a, b]$ *and* $\zeta > 0$ *there is* $\sigma > 0$ *such that*

$$\|V(t, [x, y]) - I\| < \zeta$$

for all $x, y \in [a, b]$, $t - \sigma < x \leqq t \leqq y < t + \sigma$.

Since we have

$$\begin{aligned} V(t, [x, y]) - V(t, [t, y]) V(t, [x, t]) &= V(t, [x, y]) - V(t, [t, y]) + \\ &+ V(t, [x, t]) - I + (V(t, [t, y]) - I)(V(t, [x, t]) - I) = \\ &= V(t, [x, y]) - I + I - V(t, [t, y]) - V(t, [x, t]) + \\ &+ I - (V(t, [t, y]) - I)(V(t, [x, t]) - I) \end{aligned}$$

we have also

$$\|V(t, [x, y]) - V(t, [t, y])\, V(t, [x, t])\| \leqq$$
$$\leqq \|V(t, [x, y]) - I\| + \|V(t, [t, y]) - I\| + \|V(t, [x, t]) - I\| +$$
$$+ \|V(t, [t, y]) - I\| \,.\, \|V(t, [x, t]) - I\| \,.$$

This inequality implies that if $V: [a, b] \times J \to L(\mathbb{R}^n)$ satisfies (1.5) then

$$\|V(t, [x, y]) - V(t, [t, y])\, V(t, [x, t])\| < 3\zeta + \zeta^2$$

for all $x, y \in [a, b]$, $t - \sigma < x \leqq t \leqq y < t + \sigma$ and this implies that (1.3) given in condition $\mathscr{C}$ is fulfilled. Moreover (1.5) evidently yields $\lim_{y \to t+} V(t, [t, y]) = I$, $t \in [a, b)$ and $\lim_{x \to t-} V(t, [x, t]) = I$, $t \in (a, b]$ and therefore (1.4) as well as (1.2) from condition $\mathscr{C}$ hold. This means that the condition (1.5) introduced by J. Jarník and J. Kurzweil in [2] implies the condition $\mathscr{C}$ given above.

1.5. Lemma. *Assume that for the function* $V: [a, b] \times J \to L(\mathbb{R}^n)$ *the condition* $\mathscr{C}$ *is satisfied. Then for every* $t \in [a, b]$ *there exists a* $\sigma_1 = \sigma_1(t) > 0$ *such that* $V(t, [x, y]) \in L(\mathbb{R}^n)$ *is invertible provided* $x, y \in [a, b]$, $t - \sigma_1 < x \leqq t \leqq y < t + \sigma_1$.

Proof. Let $t \in [a, b]$ be given. For a given $\zeta > 0$ let $\sigma_1(t) > 0$ be such that for $x, y \in [a, b]$, $t - \sigma_1 < x \leqq t \leqq y < t + \sigma_1$ we have

$$\|V(t, [x, y]) - V(t, [t, y])\, V(t, [x, t])\| < \zeta \tag{1.6}$$

and

$$\|V(t, [x, t]) - V_-(t)\| < \zeta\,, \quad \|V(t, [t, y]) - V_+(t)\| < \zeta \tag{1.7}$$

provided $x, y \in [a, b]$, $t - \sigma_1 < x \leqq t \leqq y < t + \sigma_1$. (1.3) and (1.4) assure the possibility of such a hoice of $\sigma_1 > 0$.

Since $V_-(t)$ and $V_+(t)$ are invertible operators (we define $V_-(a) = I$, $V_+(b) = I$), the operator $V_+(t)\, V_-(t)$ is also invertible with $(V_+(t)\, V_-(t))^{-1} = (V_-(t))^{-1}\, (V_+(t))^{-1}$.

We have evidently

$$V(t, [x, y]) - V_+(t)\, V_-(t) = V(t, [x, y]) - V(t, [t, y])\, V(t, [x, t]) +$$
$$+ (V(t, [t, y] - V_+(t))\, (V(t, [x, t]) - V_-(t)) +$$
$$+ V_+(t)\,.\,(V(t, [x, t]) - V_-(t)) + (V(t, [t, y] - V_+(t))\, V_-(t)\,.$$

Hence

$$\|V(t, [x, y]) - V_+(t)\, V_-(t)\| \leqq \|V(t, [x, y]) - V(t, [t, y])\, V(t, [x, t])\| +$$
$$+ \|V(t, [t, y]) - V_+(t)\| \,.\, \|V(t, [x, t] - V_-(t)\| +$$
$$+ \|V_+(t)\| \,.\, \|V(t, [x, t]) - V_-(t)\| + \|V(t, [t, y] - V_+(t)\| \,.\, \|V_-(t)\|\,,$$

and if $x, y \in [a, b]$, $t - \sigma_1 < x < t < y < t + \sigma_1$ then by (1.6) and (1.7) we have

$$\|V(t, [x, y]) - V_+(t)\, V_-(t)\| \leqq \zeta + \zeta^2 + \zeta(\|V_+(t)\| + \|V_-(t)\|) =$$
$$= \zeta(1 + \|V_+(t)\| + \|V_-(t)\| + \zeta)\,.$$

Since $\zeta > 0$ can be choosen arbitrarily small, the operator $V(t, [x, y])$ is invertible. (It is e.g. sufficient when $\zeta > 0$ is choosen in such a way that $\zeta(1 + \|V_+(t)\| + \|V_-(t)\| + \zeta) < \|(V_-(t))^{-1}\,(V_+(t))^{-1}\|^{-1}$. If e.g. $x = t < y$ then the result comes immediately from the second inequality in (1.7) for a sufficiently small ζ. The case $x < t = y$ is a consequence of the first relation in (1.7) and finally for $x = t = y$ we have $V(t, [x, y]) = I$ and $V(t, [x, y])$ is evidently invertible.

1.6. Lemma. *Assume that $V: [a, b] \times J \to L(\mathbb{R}^n)$ satisfies the condition $\mathscr{C}$. Then for every $t \in [a, b]$ there is a $\sigma_2 = \sigma_2(t) > 0$ such that*

$$\text{(1.8)} \qquad \begin{aligned} &\|V(t, [x, t])\| \leqq \|V_-(t)\| + \tfrac{1}{2}\|(V_-(t))^{-1}\|\,,\\ &\|(V(t, [x, t]))^{-1}\| \leqq 2\|(V_-(t))^{-1}\| \end{aligned}$$

for all $x \in [a, b]$ such that $t - \sigma_2 < x < t$ and

$$\text{(1.9)} \qquad \begin{aligned} &\|V(t, [t, y])\| \leqq \|V_+(t)\| + \tfrac{1}{2}\|(V_+(t))^{-1}\|\,,\\ &\|(V(t, [t, y]))^{-1}\| \leqq 2\|(V_+(t))^{-1}\| \end{aligned}$$

for all $y \in [a, b]$ such that $t < y < t + \sigma_2$.

Proof. Let us prove (1.8), the proof of (1.9) is analogous. Let $t \in (a, b]$; if $t = a$, there is no $x \in [a, b]$ such that $x < t$. $V_-(t) \in L(\mathbb{R}^n)$ is invertible by (1.4). If $B \in L(\mathbb{R}^n)$ and $\|B - V_-(t)\| < \frac{1}{2}\|(V_-(t))^{-1}\|^{-1}$, then by the general result given in [1, VII.6.1] $B^{-1} \in L(\mathbb{R}^n)$ exists and

$$B^{-1} = (V_-(t))^{-1} \sum_{k=0}^{\infty} [(V_-(t) - B)\,(V_-(t))^{-1}]^k\,.$$

Therefore

$$\|B^{-1}\| \leqq \|(V_-(t))^{-1}\| \sum_{k=0}^{\infty} (\|V_-(t) - B\| \,.\, \|(V_-(t))^{-1}\|)^k =$$
$$= \frac{\|(V_-(t))^{-1}\|}{1 - \|V_-(t) - B\| \,.\, \|(V_-(t))^{-1}\|}\,.$$

Since in this case $\|V_-(t) - B\| \,.\, \|(V_-(t))^{-1}\| < \frac{1}{2}$, we have $1 - \|V_-(t) - B\| \,.\, \|(V_-(t))^{-1}\| > \frac{1}{2}$ and consequently

$$\text{(1.10)} \qquad \|B^{-1}\| < 2\|(V_-(t))^{-1}\|\,.$$

By (1.4) there is a $\sigma_2^-(t) > 0$ such that if $x \in [a, b]$, $t - \sigma_2^- < x < t$, then

$$\text{(1.11)} \qquad \|V(t, [x, t]) - V_-(t)\| < \tfrac{1}{2}\|(V_-(t))^{-1}\|^{-1}\,.$$

Hence by (1.10) we have

$$\|(V(t, [x, t])^{-1}\| < 2\|(V_-(t))^{-1}\|$$

and (1.11) implies also

$$\|V(t, [x, t])\| \leqq \|V(t, [x, t]) - V_-(t)\| + \|V_-(t)\| < \tfrac{1}{2}\|(V_-(t))^{-1}\| + \\ + \|V_-(t)\|$$

provided $t - \sigma_2^- < x < t$, i.e. (1.8) holds for such $x \in [a, b]$.

For the case $t \in [a, b)$ we can find a $\sigma_2^+(t) > 0$ such that (1.9) holds for every $y \in [a, b]$, $t < y < t + \sigma_2^+$. Taking $\sigma_2 = \min(\sigma_2^+, \sigma_2^-)$ we obtain the statement of the lemma.

1.7. Theorem. *Let $V: [a, b] \times J \to L(\mathbb{R}^n)$ be Perron product integrable over $[a, b]$ with $\prod_a^b V(t, \mathrm{d}t) = Q$ and assume that for V the condition $\mathscr{C}$ is satisfied.*

Then there exists a constant $K > 0$ such that for every $s \in [a, b]$ the Perron product integrals $\prod_a^s V(t, \mathrm{d}t)$, $\prod_s^b V(t, \mathrm{d}t)$ exist, the equality

$$\prod_s^b V(t, \mathrm{d}t) \prod_a^s V(t, \mathrm{d}t) = \prod_a^b V(t, \mathrm{d}t)$$

holds and

$$\|\prod_a^s V(t, \mathrm{d}t)\| \leqq K, \quad \|(\prod_a^s V(t, \mathrm{d}t))^{-1}\| \leqq K.$$

Proof. Let $\zeta > 0$ be arbitrary. Let $\delta_0: [a, b] \to (0, +\infty)$ be a gauge on $[a, b]$ such that $\delta_0(t) \leqq \min(\sigma_1(t), \sigma_2(t))$, $t \in [a, b]$ where $\sigma_1(t)$, $\sigma_2(t)$ are given in Lemma 1.5 and 1.6 respectively and such that

(1.12) $$\|P(V, \Delta) - Q\| < \tfrac{1}{2}\|Q^{-1}\|^{-1}$$

holds for every δ_0-fine partition Δ of $[a, b]$ and

(1.13) $$\|V(t, [x, y]) - V(t, [t, y]) V(t, [x, t])\| \leqq \zeta$$

for $t, x, y \in [a, b]$, $t - \delta_0(t) < x \leqq t \leqq y < t + \delta_0(t)$. Then the following holds.

(1.14) *For every $t \in [a, b]$ there is a $K_1(t) > 0$ such that*
(i) *if $s \in (t - \delta_0(t), t] \cap [a, b]$ and Δ_1 is a δ_0-fine partition of $[a, s]$ then*

$$\max\{\|P(V, \Delta_1)\|, \|(P(V, \Delta_1))^{-1}\|\} \leqq K_1(t)$$

and
(ii) *if $s \in [t, t + \delta_0(t)] \cap [a, b]$ and Δ_2 is a δ_0-fine partition of $[s, b]$ then*

$$\max\{\|P(V, \Delta_2)\|, \|(P(V, \Delta_2))^{-1}\|\} \leqq K_1(t).$$

For proving (1.14) let us first mention that because we have $\delta_0(t) \leqq \sigma_1(t)$, Lemma 1.5 implies that $V(t, [x, y]) \in L(\mathbb{R}^n)$ is invertible for every $t, x, y \in [a, b]$ such that $t - \delta_0(t) < x \leqq t \leqq y < t + \delta_0(t)$.

In order to prove (i) from (1.14) let Δ_3 be a δ_0-fine partition of $[t, b]$. Let

$$\Delta_1 = \{\alpha_0, t_1, \alpha_1, \ldots, \alpha_{l-1}, t_l, \alpha_l\}$$

be the δ_0-fine partition of $[a, s]$ and let

$$\Delta_3 = \{\alpha_{l+1}, t_{l+2}, \alpha_{l+2}, \ldots, \alpha_{k-1}, t_k, \alpha_k\}$$

be a δ_0-fine partition of $[t, b]$. Set

$$\Delta = \{\alpha_0, t_1, \alpha_1, \ldots, \alpha_{l-1}, t_l, \alpha_l = s,\ t_{l+1} = t,\ \alpha_{l+1} = t,$$
$$t_{l+2}, \alpha_{l+2}, \ldots, \alpha_{k-1}, t_k, \alpha_k\}\,.$$

(In the sequel we will use the notation $\Delta = \Delta_1 \circ (t, [s, t]) \circ \Delta_3$ for this construction of a partition of the interval $[a, b]$; Δ is in fact the union of ordered finite sets in which the ordering preserves the ordering of the components Δ_1, $\{s, t, t\}$, Δ_3; by $\circ$ the union of ordered sets is denoted as it is denoted in [2] too.)

It is evident that Δ is a δ_0-fine partition of $[a, b]$ and that $V(t_i, [\alpha_{i-1}, \alpha_i]) \in L(\mathbb{R}^n)$, $i = 1, 2, \ldots, k$ are invertible. Hence also $P(V, \Delta_1) = V(t_l, [\alpha_{l-1}, \alpha_l])\,.$ $. \, V(t_{l-1}, [\alpha_{l-2}, \alpha_{l-1}]) \ldots V(t_1, [\alpha_0, \alpha_1]) \in L(\mathbb{R}^n)$ and $P(V, \Delta_3) = V(t_k, [\alpha_{k-1}, \alpha_k])\,.$ $. \, V(t_{k-1}, [\alpha_{k-2}, \alpha_k]) \ldots V(t_{l+2}, [\alpha_{l+1}, \alpha_{l+2}]) \in L(\mathbb{R}^n)$ are invertible and (1.12) holds.

By definition we evidently have

$$P(V, \Delta) = P(V, \Delta_3)\, V(t_{l+1}, [\alpha_l, \alpha_{l+1}])\, P(V, \Delta_1) =$$
$$= P(V, \Delta_3)\, V(t, [s, t])\, P(V, \Delta_1)$$

and

$$\|P(V, \Delta_1) - (V(t, [s, t]))^{-1} (P(V, \Delta_3))^{-1} Q\| =$$
$$= \|(V(t, [s, t]))^{-1} (P(V, \Delta_3))^{-1} [P(V, \Delta_3)\, V(t, [s, t])\, P(V, \Delta_1) - Q]\| \leqq$$
$$\leqq \|(V(t, [s, t]))^{-1}\| \,.\, \|(P(V, \Delta_3))^{-1}\| \,.\, \tfrac{1}{2}\|Q^{-1}\|^{-1}\,.$$

Consequently by Lemma 1.6 we obtain

$$\text{(1.15)} \quad \|P(V, \Delta_1)\| \leqq \|P(V, \Delta_1) - (V(t, [s, t]))^{-1} (P(V, \Delta_3))^{-1} Q\| +$$
$$+ \|(V(t, [s, t]))^{-1}\| \,.\, \|(P(V, \Delta_3))^{-1}\| \,.\, \|Q\| \leqq$$
$$\leqq \|(V(t, [s, t]))^{-1}\| \,.\, \|(P(V, \Delta_3))^{-1}\| \,.\, (\tfrac{1}{2}\|Q^{-1}\|^{-1} + \|Q\|) \leqq$$
$$\leqq 2\|(V_-(t))^{-1}\| \,.\, \|(P(V, \Delta_3))^{-1}\| \,.\, (\tfrac{1}{2}\|Q^{-1} + \|Q\|) = K_0(t) > 0\,.$$

On the other hand we have

$$\|(P(V, \Delta_1))^{-1} - Q^{-1} P(V, \Delta_3)\, V(t, [s, t])\| =$$
$$= \|Q^{-1}(Q - P(V, \Delta_3)\, V(t, [s, t])\, P(V, \Delta_1))\, (P(V, \Delta_1))^{-1}\| \leqq$$
$$\leqq \|Q^{-1}\| \,.\, \|P(V, \Delta) - Q\| \,.\, \|(P(V, \Delta_1))^{-1}\| \leqq$$
$$\leqq \|Q^{-1}\| \,.\, \tfrac{1}{2}\|Q^{-1}\|^{-1}\, \|(P(V, \Delta_1))^{-1}\| = \tfrac{1}{2}\|(P(V, \Delta_1))^{-1}\|$$

and consequently by Lemma 1.6 we get

$$\|(P(V,\Delta_1))^{-1}\| \leqq \|(P(V,\Delta_1))^{-1} - Q^{-1}P(V,\Delta_3)\,V(t,[s,t])\| +$$
$$+ \|Q^{-1}\| \,.\, \|P(V,\Delta_3)\| \,.\, \|V(t,[s,t])\| \leqq \tfrac{1}{2}\|(P(V,\Delta_1))^{-1}\| +$$
$$+ \|Q^{-1}\| \,.\, \|P(V,\Delta_3)\| \,.\, (\|V_-(t)\| + \tfrac{1}{2}\|(V_-(t))^{-1}\|)\,,$$

i.e. we obtain the inequality

$$(1.16)\qquad \|(P(V,\Delta_1))^{-1}\| \leqq 2\|Q^{-1}\| \,.\, \|P(V,\Delta_3)\| \,(\|V_-(t)\| + \tfrac{1}{2}\|(V_-(t))^{-1}\|) =$$
$$= K^0(t) > 0\,.$$

Taking $K_-(t) = \max(K_0(t), K^0(t)) > 0$ we conclude by (1.15) and (1.16) that

$$\max\{\|P(V,\Delta_1)\|, \|(P(V,\Delta_1))^{-1}\|\} \leqq K_-(t)$$

holds. A completely analogous reasoning gives also that if $s \in [t, t + \delta_0(t)) \cap [a, b]$ and Δ_2 is a δ_0-fine partition of $[s, b]$ then

$$\max\{\|P(V,\Delta_2)\|, \|(P(V,\Delta_2))^{-1}\|\} \leqq K_+(t)$$

where $K_+(t) > 0$. Putting $K_1(t) = \max(K_-(t), K_+(t))$ we obtain (1.14).

Now we will show that the following is satisfied.

(1.17) *For every $t \in [a, b]$ there is a $K_2(t) > 0$ such that*

$$\max\{\|P(V,\Delta_1)\|, \|(P(V,\Delta_1))^{-1}\|, \|P(V,\Delta_2)\|, \|(P(V,\Delta_2))^{-1}\|\} \leqq K_2(t)$$

if $s \in (t - \delta_0(t), t + \delta_0(t)) \cap [a, b]$ and Δ_1, Δ_2 are arbitrary δ_0-fine partitions of $[a, s]$, $[s, b]$ respectively.

Let us take e.g. $s \in [t, t + \delta_0(t)]$ and set $\Delta = \Delta_1 \circ \Delta_2$. Then $P(V, \Delta) = P(V, \Delta_2)\,P(V, \Delta_1)$ and $P(V, \Delta_2), P(V, \Delta_1) \in L(\mathbb{R}^n)$ are invertible by Lemma 1.5. Since (1.12) holds we have

$$\|P(V,\Delta_2)\,P(V,\Delta_1) - Q\| < \tfrac{1}{2}\|Q^{-1}\|^{-1}$$

and

$$\|P(V,\Delta_1) - (P(V,\Delta_2))^{-1}\,Q\| = \|(P(V,\Delta_2))^{-1}\,(P(V,\Delta_2)\,P(V,\Delta_1) - Q)\| \leqq$$
$$\leqq \|(P(V,\Delta_2))^{-1}\| \,.\, \tfrac{1}{2}\|Q^{-1}\|^{-1}\,.$$

Hence

$$(1.18)\qquad \|P(V,\Delta_1)\| \leqq \|P(V,\Delta_1) - (P(V,\Delta_2))^{-1}\,Q\| + \|(P(V,\Delta_2))^{-1}\| \,.\, \|Q\| \leqq$$
$$\leqq \|(P(V,\Delta_2))^{-1}\|\,(\tfrac{1}{2}\|Q^{-1}\|^{-1} + \|Q\|)\,.$$

On the other hand we have

$$\|(P(V,\Delta_1))^{-1} - Q^{-1}P(V,\Delta_2)\| =$$
$$= \|Q^{-1}(Q - P(V,\Delta_2)\,P(V,\Delta_1))\,(P(V,\Delta_1))^{-1}\| \leqq$$
$$\leqq \|Q^{-1}\| \,.\, \|Q - P(V,\Delta_2)\,P(V,\Delta_1)\| \,.\, \|(P(V,\Delta_1))^{-1}\| <$$
$$< \tfrac{1}{2}\|(P(V,\Delta_1))^{-1}\|$$

and henceforth

$$\|(P(V, \Delta_1))^{-1}\| \leqq \|(P(V, \Delta_1))^{-1} - Q^{-1}P(V, \Delta_2)\| + \\ + \|Q^{-1}\| \cdot \|P(V, \Delta_2)\| \leqq \tfrac{1}{2}\|(P(V, \Delta_1))^{-1}\| + \|Q^{-1}\| \cdot \|P(V, \Delta_2)\| ,$$

i.e.

$$\|(P(V, \Delta_1))^{-1}\| \leqq 2\|Q^{-1}\| \cdot \|P(V, \Delta_2)\| . \tag{1.19}$$

By (ii) from (1.14) we get by (1.18) and (1.19) the estimate

$$\max\{\|P(V, \Delta_1)\|, \|(P(V, \Delta_1))^{-1}\|\} \leqq \\ \leqq K_1(t)\,[2\|Q^{-1}\| + \tfrac{1}{2}\|Q^{-1}\|^{-1} + \|Q\|] = K_L(t) > 0 .$$

Similarly we can show that

$$\max\{\|P(V, \Delta_2)\|, \|(P(V, \Delta_2))^{-1}\|\} \leqq K_R(t), \quad K_R(t) > 0 ,$$

and putting e.g. $K_2(t) = \max(K_L(t), K_R(t)) > 0$ we obtain (1.17).

The sets of the form $(t - \delta_0(t), t + \delta_0(t))$, $t \in [a, b]$ form an open covering of the compact interval $[a, b]$. Hence there is a finite set $\{t_1, t_2, \dots, t_l\} \subset [a, b]$ such that

$$[a, b] \subset \bigcup_{j=1}^{l} (t_j - \delta_0(t_j), t_j + \delta_0(t_j)) .$$

Define $K = \max\{1, K_2(t_1), K_2(t_2), \dots, K_2(t_l)\}$ where $K_2(t)$ is given by (1.17). Then (1.17) implies that the following holds.

(1.20) *There exists a constant $K \geqq 1$ such that*
(i) *if $s \in (a, b]$ and Δ_1 is a δ_0-fine partition of $[a, s]$, then*

$$\max\{\|P(V, \Delta_1)\|, \|(P(V, \Delta_1))^{-1}\|\} \leqq K$$

and
(ii) *if $s \in [a, b)$ and Δ_2 is a δ_0-fine partition of $[s, b]$, then*

$$\max\{\|P(V, \Delta_2)\|, \|(P(V, \Delta_2))^{-1}\|\} \leqq K .$$

Now we prove the following statement

(1.21) *Let $\varepsilon \in (0, \frac{1}{2}\|Q^{-1}\|^{-1})$ be given and let δ be a gauge on $[a, b]$ such that $\delta(t) \leqq \delta_0(t)$, $t \in [a, b]$ and*

$$\|P(V, \Delta) - Q\| < \varepsilon$$

for every δ-fine partition Δ of $[a, b]$.
(i) *If $s \in [a, b)$ and Δ_2, Δ_4 are arbitrary δ-fine partitions of $[s, b]$, then*

$$\|P(V, \Delta_2) - P(V, \Delta_4)\| \leqq 2K\varepsilon .$$

(ii) *If $s \in (a, b]$ and Δ_1, Δ_3 are arbitrary δ-fine partitions of $[a, s]$, then*

$$\|P(V, \Delta_1) - P(V, \Delta_3)\| \leqq 2K\varepsilon$$

(K *is the constant given in* (1.20)).

We prove only (i), the proof of (ii) is similar. Let $s \in [a, b)$. Denote by Δ_1 an arbitrary δ-fine partition of $[a, s]$. Let us put $\Delta_5 = \Delta_1 \circ \Delta_2$ and $\Delta_6 = \Delta_1 \circ \Delta_4$. Δ_5 and Δ_6 are evidently δ-fine partitions of $[a, b]$. Hence

$$\|P(V, \Delta_2) P(V, \Delta_1) - P(V, \Delta_4) P(V, \Delta_1)\| \leqq$$
$$\leqq \|P(V, \Delta_5) - Q\| + \|P(V, \Delta_6) - Q\| \leqq 2\varepsilon$$

and

$$\|P(V, \Delta_2) - P(V, \Delta_4)\| =$$
$$= \|[P(V, \Delta_2) P(V, \Delta_1) - P(V, \Delta_4) P(V, \Delta_1)] (P(V, \Delta_1))^{-1}\| \leqq$$
$$\leqq \|P(V, \Delta_2) P(V, \Delta_1) - P(V, \Delta_4) P(V, \Delta_1)\| \,.\, \|(P(V, \Delta_1))^{-1}\| \leqq 2K\varepsilon$$

by (1.20). The second statement (ii) in (1.21) can be proved analogously.

By (1.21) and by Proposition 1.3 we have the following result.

(1.22) *If* $s \in (a, b)$ *then there exist* $Q^-, Q^+ \in L(\mathbb{R}^n)$ *such that for every* $\varepsilon \in (0, \frac{1}{2}\|Q^{-1}\|^{-1})$ *there is a gauge* $\delta_1 : [a, b] \to (0, +\infty)$ *on* $[a, b]$ *such that*

$$\|P(V, \Delta_1) - Q^-\| < \varepsilon$$

for every δ_1*-fine partition* Δ_1 *of* $[a, s]$ *and*

$$\|P(V, \Delta_2) - Q^+\| < \varepsilon$$

for every δ_1*-fine partition* Δ_2 *of* $[s, b]$.

Assume that $s \in (a, b)$. Let us choose a gauge δ_2 on $[a, b]$ such that $\delta_2(t) \leqq \min(\delta(t), \delta_0(t), \delta_1(t), |t - s|)$ for $t \neq s$ and $\delta_2(s) \leqq \delta_1(s)$. By this choice every δ_2-fine partition $\Delta = \{\alpha_2, t_1, \alpha_1, \ldots, \alpha_{k-1}, t_k, \alpha_k\}$ has the property that there exists a $j \in \{1, 2, \ldots, k\}$ such that $t_j = s$. For a δ_2-fine partition Δ of $[a, b]$ and δ_2-fine partitions Δ_1, Δ_2 of $[a, s]$, $[s, b]$ respectively we have by (1.20) the following inequality

$$(1.23) \quad \|P(V, \Delta) - Q^+Q^-\| \,.\, \|P(V, \Delta) - P(V, \Delta_2) P(V, \Delta_1)\| +$$
$$+ \|P(V, \Delta_2) P(V, \Delta_1) - Q^+Q^-\| \leqq \|P(V, \Delta) - P(V, \Delta_2) P(V, \Delta_1)\| +$$
$$+ \|P(V, \Delta_2) P(V, \Delta_1) - Q^+P(V, \Delta_1) + Q^+(P(V, \Delta_1) - Q^-)\| \leqq$$
$$\leqq \|P(V, \Delta) - P(V, \Delta_2) P(V, \Delta_1)\| + \|P(V, \Delta_2) - Q^+\| \,.\, \|P(V, \Delta_1)\| +$$
$$+ \|Q^+ - P(V, \Delta_2)\| \,.\, \|P(V, \Delta_1) - Q^-\| +$$
$$+ \|P(V, \Delta_2)\| \,.\, \|P(V, \Delta_1) - Q^-\| \leqq$$
$$\leqq \|P(V, \Delta) - P(V, \Delta_2) P(V, \Delta_1)\| + \varepsilon(2K + \varepsilon)\,.$$

For a given δ_2-fine partition

$$\Delta = \{\alpha_0, t_1, \alpha_1, \ldots, \alpha_{j-1}, t_j = s, \alpha_j, t_{j+1}, \alpha_{j+1}, \ldots, \alpha_{k-1}, t_k, \alpha_k\}$$

we put

$$\Delta_- = \{\alpha_0, t_1, \alpha_1, \ldots, t_{j-1}, \alpha_{j-1}\},$$

$$\Delta_+ = \{\alpha_j, t_{j+1}, \alpha_{j+1}, \ldots, \alpha_{k-1}, t_k, \alpha_k\}$$

and

$$\Delta_1 = \Delta_- \circ \{\alpha_{j-1}, \tilde{t}_j = s, \tilde{\alpha}_j = s\},$$

$$\Delta_2 = \{\tilde{\alpha}_{j-1} = s, \tilde{t}_j = s, \alpha_j\} \circ \Delta_+,$$

then Δ_1, Δ_2 are evidently δ_2-fine partitions of $[a, s]$, $[s, b]$ respectively and

$$P(V, \Delta) = P(V, \Delta_+)\, V(s, [\alpha_{j-1}, \alpha_j])\, P(V, \Delta_-),$$

$$P(V, \Delta_1) = V(s, [\alpha_{j-1}, s])\, P(V, \Delta_-),$$

$$P(V, \Delta_2) = P(V, \Delta_+)\, V(s, [s, \alpha_j]).$$

Moreover

$$\|P(V, \Delta) - P(V, \Delta_2)\, P(V, \Delta_1)\| = \|P(V, \Delta_+)\, V(s, [\alpha_{j-1}, \alpha_j])\, P(V, \Delta_-) -$$
$$- P(V, \Delta_+)\, V(s, [s, \alpha_j])\, V(s, [\alpha_{j-1}, s])\, P(V, \Delta_-)\| =$$
$$= \|P(V, \Delta_+)\, [V(s, [\alpha_{j-1}, \alpha_j]) -$$
$$- V(s, [s, \alpha_j])\, V(s, [\alpha_{j-1}, s])]\, P(V, \Delta_-)\| \leqq K^2 \zeta$$

by (1.20) and (1.13) because we have $\alpha_{j-1}, \alpha_j \in [a, b]$ and $s - \delta_0(s) < s - \delta_2(s) < \alpha_{j-1} \leqq s \leqq \alpha_j < s + \delta_2(s) < s + \delta_0(s)$.

Using (1.23) we therefore obtain

$$\|P(V, \Delta) - Q^+ Q^-\| < K^2 \zeta + \varepsilon(2K + \varepsilon).$$

Taking e.g. $\zeta = \varepsilon / K^2$ and using the fact that

$$\|P(V, \Delta) - Q\| < \varepsilon$$

for every δ_2-fine partition Δ of $[a, b]$ (see (1.21)) we obtain

$$\|Q - Q^+ Q^-\| \leqq \|Q - P(V, \Delta)\| +$$
$$+ \|P(V, \Delta) - Q^+ Q^-\| < \varepsilon + \varepsilon + \varepsilon(2K + \varepsilon) = \varepsilon(2 + 2K + \varepsilon)$$

and consequently because $\varepsilon > 0$ can be choosen arbitrarily we get

$$(1.24) \qquad Q = Q^+ Q^- .$$

Since $Q \in L(\mathbb{R}^n)$ is invertible, we have by (1.24) $Q^{-1} Q^+ Q^- = I$ and consequently $Q^{-1} Q^+ \in L(\mathbb{R}^n)$ is the inverse to Q^- ($Q^{-1} Q^+$ is the left inverse to Q^- but we have also $Q^- Q^{-1} Q^+ Q^- = Q^-$ and consequently $Q^- Q^{-1} Q^+ = I$; i.e. $Q^{-1} Q^+$ is also the right inverse to Q^-). Similarly it can be shown that $Q^+ \in L(\mathbb{R}^n)$ is also invertible with $(Q^+)^{-1} = Q^- Q^{-1}$.

This yields by (1.22) that the Perron product integrals $\prod_a^s V(t, dt) = Q^-$, $\prod_s^b V(t, dt) = Q^+$ exist and (1.24) is in fact the equality

$$(1.25) \qquad \prod_a^b V(t, dt) = \prod_s^b V(t, dt) \prod_a^s V(t, dt)$$

from the statement.

The estimates $\|\prod_a^s V(t, dt)\| \leqq K$, $\|(\prod_a^s V(t, dt))^{-1}\| \leqq K$ are simple consequences of (1.20) and of (1.25).

1.8. Lemma. *Assume that $V: [a, b] \times J \to L(\mathbb{R}^n)$ satisfies the condition $\mathscr{C}$ and the Perron product integral $\prod_a^b V(t, dt) = Q$ exists.*

Let us define $\Phi: [a, b] \to L(\mathbb{R}^n)$ by the relations

$$(1.26) \qquad \Phi(a) = I\,, \quad \Phi(s) = \prod_a^s V(t, dt)\,, \quad s \in (a, b]\,.$$

The function Φ is well defined and its values are invertible elements of $L(\mathbb{R}^n)$, $\Phi(b) = Q$.

For a given $\varepsilon > 0$ let $\delta: [a, b] \to (0, +\infty)$ be a gauge on $[a, b]$ such that

$$(1.27) \qquad \|P(V, \Delta) - \Phi(b)\| = \|P(V, \Delta) - \prod_a^b V(t, dt)\| < \varepsilon$$

holds for every δ-fine partition Δ of $[a, b]$. Assume that we have $a \leqq \beta_1 \leqq \xi_1 \leqq \leqq \gamma_1 \leqq \beta_2 \leqq \xi_2 \leqq \gamma_2 \leqq \ldots \leqq \beta_m \leqq \xi_m \leqq \gamma_m \leqq b$ where

$$\xi_j - \delta(\xi_j) < \beta_j \leqq \xi_j \leqq \gamma_j \leqq \xi_j + \delta(\xi_j)\,, \quad j = 1, 2, \ldots, m\,.$$

Then

$$(1.28) \qquad \begin{aligned} &\|(\Phi(\gamma_m))^{-1} V(\xi_m, [\beta_m, \gamma_{m-1}])\, \Phi(\beta_m)\, (\Phi(\gamma_{m-1}))^{-1}\,. \\ &\cdot V(\xi_m, [\beta_{m-1}, \gamma_{m-1}])\, \Phi(\beta_{m-1}) \ldots (\Phi(\gamma_1))^{-1}\,. \\ &\cdot V(\xi_1, [\beta_1, \gamma_1])\, \Phi(\beta_1) - I\| \leqq \|(\Phi(b))^{-1}\|\, \varepsilon\,. \end{aligned}$$

Proof. The function $\Phi: [a, b] \to L(\mathbb{R}^n)$ is well defined by Theorem 1.7 and the same theorem yields also the invertibility of the values of this function. By Theorem 1.7 also the product integral $\prod_c^d V(t, dt)$ exists over every interval $[c, d] \subset [a, b]$.

Let us denote $\gamma_0 = a$ and $\beta_{m+1} = b$.

Since the integral $\prod_{\gamma_j}^{\beta_{j+1}} V(t, dt)$ exists for every $j = 0, 1, \ldots, m$ we have by definition the following:

For every $\eta > 0$ there is a gauge $\delta_j: [\gamma_j, \beta_{j+1}] \to (0, +\infty)$ such that $\delta_j(t) < \delta(t)$, $t \in [\gamma_j, \beta_{j+1}]$ and

$$(1.29) \qquad \|P(V, \Delta_j) - \prod_{\gamma_j}^{\beta_{j+1}} V(t, dt)\| = \|P(V, \Delta_j) - \Phi(\beta_{j+1})\, (\Phi(\gamma_j))^{-1}\| < \eta$$

for every δ_j-fine partition Δ_j of $[\gamma_j, \beta_{j+1}]$, $j = 0, 1, 2, \ldots m$.

For δ_j-fine partitions Δ_j of $[\gamma_j, \beta_{j+1}]$, $j = 0, 1, \ldots, m$ let us set

$$\Delta = \Delta_0 \circ (\xi_1, [\beta_1, \gamma_1]) \circ \Delta_1 \circ (\xi_2, [\beta_2, \gamma_2]) \circ \Delta_3 \circ \ldots \Delta_{m-1} \circ (\xi_m, [\beta_m, \gamma_m]) \circ \Delta_m\,.$$

Δ evidently forms a δ-fine partition of $[a, b]$ and therefore (1.27) holds for this partition. Hence

(1.30) $\|(\Phi(b))^{-1} P(V, \Delta) - I\| = \|(\Phi(b))^{-1} [P(V, \Delta) - \Phi(b)]\| < \|(\Phi(b))^{-1}\| \varepsilon .$

Further we have evidently

$$P(V, \Delta) = P(V, \Delta_m) V(\xi_m, [\beta_m, \gamma_m]) P(V, \Delta_{m-1}) \dots$$
$$\dots P(V, \Delta_1) V(\xi_1, [\beta_1, \gamma_1]) P(V, \Delta_0)$$

and

$$(\Phi(b))^{-1} P(V, \Delta) = (\Phi(b))^{-1} P(V, \Delta_m) V(\xi_m, [\beta_m, \gamma_m]) \dots$$
$$\dots P(V, \Delta_1) V(\xi_1, [\beta_1, \gamma_1]) P(V, \Delta_0) = (\Phi(\beta_{m+1}))^{-1} P(V, \Delta_m) \Phi(\gamma_m) .$$
$$. (\Phi(\gamma_m))^{-1} V(\xi_m, [\beta_m, \gamma_m]) \Phi(\beta_m) (\Phi(\beta_m))^{-1} P(V, \Delta_{m-1}) \Phi(\gamma_{m-1}) .$$
$$. (\Phi(\gamma_{m-1}))^{-1} \dots \Phi(\beta_2) (\Phi(\beta_2))^{-1} P(V, \Delta_1) \Phi(\gamma_1) (\Phi(\gamma_1))^{-1} .$$
$$. V(\xi_1, [\beta_1, \gamma_1]) \Phi(\beta_1) (\Phi(\beta_1))^{-1} P(V, \Delta_0) \Phi(\gamma_0) .$$

Denoting

$$(\Phi(\beta_{j+1}))^{-1} P(V, \Delta_j) \Phi(\gamma_j) = A_j + I , \quad j = 0, 1, \dots, m$$

and

$$(\Phi(\gamma_j))^{-1} V(\xi_j, [\beta_j, \gamma_j]) \Phi(\beta_j) = Z_j + I , \quad j = 1, 2, \dots, m$$

we obtain

$$(\Phi(b))^{-1} P(V, \Delta) =$$
$$= (I + A_m)(I + Z_m)(I + A_{m-1})(I + Z_{m-1}) \dots (I + A_1)(I + Z_1)(I + A_0)$$

and (1.30) can be rewritten in the form

(1.31) $\|(I + A_m)(I + Z_m)(I + A_{m-1}) \dots (I + A_1)(I + Z_1)(I + A_0) - I\| <$
$< \|(\Phi(b))^{-1}\| \varepsilon .$

By (1.29) we have

(1.32) $\|A_j\| = \|(\Phi(\beta_{j+1}))^{-1} P(V, \Delta_j) \Phi(\gamma_j) - I\| =$
$= \|(\Phi(\beta_{j+1}))^{-1} [P(V, \Delta_j) - \Phi(\beta_{j+1}) (\Phi(\gamma_j))^{-1}] \Phi(\gamma_j)\| \leqq K^2 \eta$

where K is the constant given by Theorem 1.7, $j = 0, 1, \dots, m$.

The estimate (1.32) easily gives the following:
for every $\vartheta > 0$ there is a $\eta > 0$ such that

$$\|(I + A_m)(I + Z_m)(I + A_{m-1}) \dots (I + A_1)(I + Z_1)(I + A_0) -$$
$$- (I + Z_m)(I + Z_{m-1}) \dots (I + Z_1)\| < \vartheta .$$

Hence by (1.31) we have

$$\|(I + Z_m)(I + Z_{m-1}) \dots (I + Z_1) - I\| \leqq$$
$$\leqq \|(I + A_m)(I + Z_m)(I + A_{m-1}) \dots (I + A_1)(I + Z_1)(I + A_0) -$$

$$- (I + Z_m)(I + Z_{m-1}) \dots (I + Z_1)\| +$$
$$+ \|(I + A_m)(I + Z_m)(I + A_{m-1}) \dots (I + Z_1)(I + A_0) - I\| < \vartheta +$$
$$+ \|(\Phi(b))^{-1}\| \, \varepsilon$$

where $\vartheta > 0$ is arbitrary and therefore

$$\|(I + Z_m)(I + Z_{m-1}) \dots (I + Z_1) - I\| \leqq \|(\Phi(b))^{-1}\| \, \varepsilon$$

and by the definition of Z_j, $j = 1, \dots, m$ we obtain (1.28).

1.9. Corollary. *Assume that $V: [a, b] \times J \to L(\mathbb{R}^n)$ is Perron product integrable over $[a, b]$ and that the condition $\mathscr{C}$ is satisfied.*

Then to every $\eta > 0$, $t \in [a, b]$ there exists a $\delta > 0$ such that

$$(1.33) \qquad \|(\Phi(\gamma))^{-1} \, V(t, [\beta, \gamma]) \, \Phi(\beta) - I\| < \eta$$

and

$$(1.34) \qquad \|V(t, [\beta, \gamma]) - \Phi(\gamma) \, (\Phi(\beta))^{-1}\| \leqq K^2 \eta$$

provided $\beta, \gamma \in [a, b]$, $t - \delta < \beta \leqq t \leqq t + \delta$, where $\Phi: [a, b] \to L(\mathbb{R}^n)$ is given by (1.26) and K is the constant from Theorem 1.7.

Proof. Taking $\varepsilon = \eta \|(\Phi(b))^{-1}\|^{-1} > 0$ we obtain (1.33) immediately from Lemma 1.8 when $\delta: [a, b] \to (0, +\infty)$ is the gauge on $[a, b]$ corresponding to this choice of ε.

Since we have

$$\|V(t, [\beta, \gamma]) - \Phi(\gamma) \, (\Phi(\beta))^{-1}\| =$$
$$= \|\Phi(\gamma) \, [\Phi(\gamma))^{-1} \, V(t, [\beta, \gamma]) \, \Phi(\beta) - I] \, (\Phi(\beta))^{-1}\| \leqq$$
$$\leqq \|\Phi(\gamma)\| \, . \, \|(\Phi(\beta))^{-1}\| \, . \, \|(\Phi(\gamma))^{-1} \, V(t, [\beta, \gamma]) \Phi(\beta) - I\| \, ,$$

we obtain (1.34) from (1.33) and from the inequalities $\|\Phi(t)\| \leqq K$, $\|(\Phi(t))^{-1}\| \leqq K$ which hold by Theorem 1.7 for every $t \in [a, b]$.

1.10. Lemma. *Assume that $A, A_k \in L(\mathbb{R}^n)$, $k = 1, 2, \dots$ are invertible such that*

$$(1.35) \qquad \lim_{k \to \infty} A_k = A \, .$$

Then

$$(1.36) \qquad \lim_{k \to \infty} (A_k)^{-1} = A^{-1} \, .$$

Proof. By (1.35) there is a $k_0 \in \mathbb{N}$ such that for $k > k_0$ we have $\|A - A_k\| < \|A^{-1}\|^{-1}$ and therefore

$$\|I - A_k A^{-1}\| = \|(A - A_k) \, A^{-1}\| \leqq \|A - A_k\| \, . \, \|A^{-1}\| < 1 \, .$$

Hence $A_k A^{-1}$ has an inverse given by

$$(A_k A^{-1})^{-1} = \sum_{l=1}^{\infty} (I - A_k A^{-1})^l = \sum_{l=1}^{\infty} ((A - A_k) A^{-1})^l = A A_k^{-1} .$$

Consequently

$$A_k^{-1} = A^{-1} \sum_{l=0}^{\infty} ((A - A_k) A^{-1})^l = A^{-1} + A^{-1} \sum_{l=0}^{\infty} ((A - A_k) A^{-1})^l ,$$

i.e.

$$A_k^{-1} - A^{-1} = A^{-1} \sum_{l=1}^{\infty} ((A - A_k) A^{-1})^l$$

and

$$\|A_k^{-1} - A^{-1}\| \leqq \|A^{-1}\| \sum_{l=1}^{\infty} (\|A - A_k\| . \|A^{-1}\|)^l \leqq$$

$$\leqq \|A^{-1}\| \frac{\|A - A_k\| . \|A^{-1}\|}{1 - \|A - A_k\| . \|A^{-1}\|}$$

for $k > k_0$.

Since $\|A - A_k\| \to 0$ for $k \to \infty$ we obtain from this estimate that

$$\|A_1^{-k} - A^{-1}\| \to 0 \quad \text{for} \quad k \to \infty , \quad \text{i.e. (1.36) holds} .$$

1.11. Lemma. *If* $V: [a, b] \times J \to L(\mathbb{R}^n)$ *satisfies the condition* $\mathscr{C}$ *and is Perron product integrable over* $[a, b]$ *then*

(1.37) $$\lim_{\beta \to t-} \Phi(\beta) = (V_-(t))^{-1} \Phi(t) \quad \text{for} \quad t \in (a, b]$$

and

(1.38) $$\lim_{\gamma \to t+} \Phi(\gamma) = V_+(t) \Phi(t) \quad \textit{for} \quad t \in [a, b) .$$

Proof. From Corollary 1.9 it follows immediately that

(1.39) $$\lim_{\beta \to t-} \|(\Phi(t))^{-1} V(t, [\beta, t]) \Phi(\beta) - I\| = 0 \quad \text{for} \quad t \in (a, b]$$

and

(1.40) $$\lim_{\gamma \to t+} \|(\Phi(\gamma))^{-1} V(t, [t, \gamma]) \Phi(t) - I\| = 0 \quad \text{for} \quad t \in [a, b) .$$

By (1.4) from the condition $\mathscr{C}$ we also have

(1.41) $$\lim_{\beta \to t-} \|V(t, [\beta, t]) - V_-(t)\| = 0 \quad \text{for} \quad t \in (a, b]$$

and

(1.42) $$\lim_{\gamma \to t+} \|V(t, [t, \gamma]) - V_+(t)\| = 0 \quad \text{for} \quad t \in [a, b)$$

where $V_-(t)$, $V_+(t) \in L(\mathbb{R}^n)$ are invertible. Since by Theorem 1.7 we have $\|\Phi(t)\| \leqq K$ $\|(\Phi(t))^{-1}\| \leqq K$, we get for $t \in (a, b]$, $\beta < t$ the inequality

$$\|(\Phi(\beta))^{-1} - (\Phi(t))^{-1} V_-(t)\| =$$
$$= \|(\Phi(\beta))^{-1} - (\Phi(t))^{-1} V(t, [\beta, t]) + (\Phi(t))^{-1} V(t, [\beta, t]) - (\Phi(t))^{-1} V_-(t))\| =$$
$$= \|[I-(\Phi(t))^{-1} V(t,[\beta,t]) \Phi(\beta)] (\Phi(\beta))^{-1} + (\Phi(t))^{-1} V(t, [\beta,t]) - (\Phi(t))^{-1} V_-(t)\| \leqq$$
$$\leqq K[\|I - (\Phi(t))^{-1} V(t, [\beta, t]) \Phi(\beta)\| + \|V(t, [\beta, t]) - V_-(t)\|].$$

This inequality together with (1.39) and (1.41) implies

$$\lim_{\beta \to t-} (\Phi(\beta))^{-1} = (\Phi(t))^{-1} V_-(t)$$

and by Lemma 1.10 we obtain immediately (1.37).

Similarly for $t \in [a, b)$, $\gamma > t$ we have

$$\|\Phi(\gamma) - V_+(t) \Phi(t)\| = \|\Phi(\gamma) - V(t, [t, \gamma]) \Phi(t) +$$
$$+ V(t, [t, \gamma]) \Phi(t) - V_+(t) \Phi(t)\| \leqq$$
$$\leqq \|\Phi(\gamma) [I - (\Phi(\gamma))^{-1} V(t, [t, \gamma]) \Phi(t)]\| +$$
$$+ \|[V(t, [t, \gamma]) - V_+(t)] \Phi(t)\| \leqq$$
$$\leqq K[\|I - (\Phi(\gamma))^{-1} V(t, [t, \gamma]) \Phi(t)\| + \|V(t, [t, \gamma]) - V_+(t)\|)]$$

and (1.40) with (1.42) imply (1.38).

1.12. Lemma. *Let* $Y_1, Y_2, \ldots, Y_k \in L(\mathbb{R}^n)$, $\sum_{i=1}^{k} \|Y_i\| \leqq 1, X = (I + Y_k)(I + Y_{k-1}) \ldots \ldots (I + Y_1) - I$, $Z = X - \sum_{i=1}^{k} Y_i$. *Then*

$$\|X\| \leqq 2 \sum_{i=1}^{k} \|Y_i\|$$

and

$$\|Z\| \leqq (\sum_{i=1}^{k} \|Y_i\|)^2 .$$

Proof. Put $\lambda_i = \|Y_i\|$, $i = 1, 2, \ldots, k$, $\lambda = \sum_{i=1}^{k} \lambda_i \leqq 1$.

We have

$$(1 + \lambda_k)(1 + \lambda_{k-1}) \ldots (1 + \lambda_1) = 1 + \sum_{j=1}^{k} \lambda_j + \sum_{j_2 > j_1} \lambda_{j_2} \lambda_{j_1} +$$
$$+ \sum_{j_3 > j_2 > j_1} \lambda_{j_3} \lambda_{j_2} \lambda_{j_1} + \ldots + \lambda_k \lambda_{k-1} \ldots \lambda_1 \leqq e^{\lambda_k} e^{\lambda_{k-1}} \ldots e^{\lambda_1} = e^{\lambda} .$$

Hence

$$\sum_{j=1}^{k} \lambda_j + \sum_{j_2 > j_1} \lambda_{j_2} \lambda_{j_1} + \ldots + \lambda_k \lambda_{k-1} \ldots \lambda_1 \leqq e^{\lambda} - 1 < 2\lambda$$

and

$$\sum_{j_2 > j_1} \lambda_{j_2} \lambda_{j_1} + \sum_{j_3 > j_2 > j_1} \lambda_{j_3} \lambda_{j_2} \lambda_{j_1} + \ldots + \lambda_k \lambda_{k-1} \ldots \lambda_1 \leqq e^{\lambda} - 1 - \lambda \leqq \lambda^2$$

because $\lambda \leqq 1$. We have evidently

$$X = \sum_{j=1}^{k} Y_j + \sum_{j_2 > j_1} Y_{j_2} Y_{j_1} + \ldots + Y_k Y_{k-1} \ldots Y_1$$

and

$$Z = \sum_{j_2 > j_1} Y_{j_2} Y_{j_1} + \sum_{j_3 > j_2 > j_1} Y_{j_3} Y_{j_2} Y_{j_1} + \ldots + Y_k Y_{k-1} \ldots Y_1 .$$

Hence

$$\|X\| \leqq \sum_{j=1}^{k} \|Y_j\| + \sum_{j_2 > j_1} \|Y_{j_2}\| \cdot \|Y_{j_1}\| + \ldots + \|Y_k\| \cdot \|Y_{k-1}\| \ldots \| \|Y_1\| =$$

$$= \sum_{j=1}^{k} \lambda_j + \sum_{j_2 > j_1} \lambda_{j_2} \lambda_{j_1} + \ldots + \lambda_k \lambda_{k-1} \ldots \lambda_1 < 2\lambda = 2 \sum_{j=1}^{k} \|Y_j\|$$

and similarly also

$$\|Z\| = \sum_{j_2 > j_1} \lambda_{j_2} \lambda_{j_1} + \sum_{j_3 > j_2 > j_1} \lambda_{j_3} \lambda_{j_2} \lambda_{j_1} + \ldots + \lambda_k \lambda_{k-1} \ldots \lambda_1 < \lambda^2 =$$

$$= \left(\sum_{j=1}^{k} \|Y_j\| \right)^2 .$$

1.13. Theorem. *Assume that $V: [a, b] \times J \to L(\mathbb{R}^n)$ satisfies the condition $\mathscr{C}$ and that for every $c \in [a, b)$ the Perron product integral $\prod_a^c V(t, \mathrm{d}t)$ exists.*

Let the limit

$$(1.43) \qquad \lim_{c \to b-} V(b, [c, b]) \prod_a^c V(t, \mathrm{d}t) = Q$$

exists, where $Q \in L(\mathbb{R}^n)$ is invertible.

Then V is Perron product integrable over $[a, b]$ and

$$(1.44) \qquad \prod_a^b V(t, \mathrm{d}t) = Q .$$

Proof. Let $\varepsilon \in (0, 1)$ be given. Since the limit (1.43) exists, there is a $B \in [a, b)$ such that for every $c \in [B, b)$ we have

$$(1.45) \qquad \left\| V(b, [c, b]) \prod_a^c V(t, \mathrm{d}t) - Q \right\| < \varepsilon .$$

Let us have a sequence $a = c_0 < c_1 < \ldots$, $\lim_{p \to \infty} c_p = b$. Since V is Perron product integrable over every $[a, c_p]$, $p = 1, 2 \ldots$, there exists a gauge $\delta_p: [0, c_p] \to (0, +\infty)$, $p = 1, 2, \ldots$ such that for every δ_p-fine partition Δ of $[a, c_p]$ we have

$$(1.46) \qquad \left\| P(V, \Delta) - \prod_a^{c_p} V(t, \mathrm{d}t) \right\| \leqq \frac{\varepsilon}{\left\| \left(\prod_a^{c_p} V(t, \mathrm{d}t) \right)^{-1} \right\| \cdot 2^{p+1}} , \quad p = 1, 2, \ldots .$$

For every $t \in [a, b)$ there is exactly one $p(t) \in \mathbb{N}$ such that $t \in [c_{p-1}, c_p)$. For $t \in [a, b)$ let us choose $\delta^0(t) > 0$ such that $\delta^0(t) \leqq \delta_{p(t)}$ and $[t - \delta^0(t), t + \delta^0(t)] \cap \cap [a, b) \subset [a, c_{p(t)}]$.

If $c \in [a, b)$ and $\Delta^- = \{\alpha_0, t_1, \alpha_1, \ldots, \alpha_{k-2}, t_{k-1}, \alpha_{k-1}\}$ is a δ^0-fine partition of $[a, c]$, then if $p(t_j) = p$, we have

$$[\alpha_{j-1}, \alpha_j] \subset (t_j - \delta^0(t_j),\ t_j + \delta^0(t_j)) \subset [a, c_p]$$

and also

$$(1.47) \qquad [\alpha_{j-1}, \alpha_j] \subset (t_j - \delta_p(t_j), t_j + \delta_p(t_j)) .$$

For the partition Δ^- we have

$$P(V, \Delta^-) = V(t_{k-1}, [\alpha_{k-2}, \alpha_{k-1}])\, V(t_{k-2}, [\alpha_{k-3}, \alpha_{k-2}]) \ldots$$
$$\ldots V(t_1, [\alpha_0, \alpha_1]) = A_m A_{m-1} \ldots A_1$$

where A_j, $j = 1, 2, \ldots, m$ is the ordered product of all factors $V(t_1, [\alpha_{l-1}, \alpha_l])$, $1 \leqq l \leqq k - 1$ with $t_l \in [c_{p_{j-1}}, c_{p_j}]$, i.e.

$$A_j = V(t_{r_j+s_j}, [\alpha_{r_j+s_j-1}, \alpha_{r_j+s_j}])\, V(t_{r_j+s_j-1}, [\alpha_{r_j+s_j-2}, \alpha_{r_j+s_j-1}]) \ldots$$
$$\ldots V(t_{r_j}, [\alpha_{r_j-1}, \alpha_{r_j}])$$

and $t_{r_j}, t_{r_{j+1}}, \ldots, t_{r_j+s_j} \in [c_{p_{j-1}}, c_{p_j}]$ with $1 \leqq r_j \leqq r_j + s_j \leqq k - 1$. By the property (1.47) of the partition Δ^- we also have

$$[\alpha_{i-1}, \alpha_i] \subset (t_i - \delta_{p_j}(t_i), t_i + \delta_{p_j}(t_i)) , \quad i = r_j, r_j + 1, \ldots, r_j + s_j .$$

Using (1.46) and Lemma 1.8 we obtain

$$\|(\textstyle\prod_a^{t_{r_j+s_j}} V(t, dt))^{-1}\, V(t_{r_j+s_j}, [\alpha_{r_j+s_j-1}, \alpha_{r_j+s_j}]) \ldots$$
$$\ldots V(t_{r_j}, [\alpha_{r_j-1}, \alpha_{r_j}]) \textstyle\prod_a^{t_{r_j}} V(t, dt) - I\| =$$
$$= \|(\textstyle\prod_a^{t_{r_j+s_j}} V(t, dt))^{-1} A_j \prod_a^{t_{r_j}} V(t, dt) - I\| \leqq$$
$$\leqq \frac{\varepsilon \|(\prod_a^{c_{p_j}} V(t, dt))^{-1}\|}{2^{p_j+1} \|(\prod_a^{c_{p_j}} V(t, dt))^{-1}\|} = \frac{\varepsilon}{2^{p_j+1}}$$

for every $j = 1, 2, \ldots, m$. Hence

$$(1.48) \qquad \sum_{j=1}^{m} \|(\textstyle\prod_a^{t_{r_j+s_j}} V(t, dt))^{-1} A_j \prod_a^{t_{r_j}} V(t, dt) - I\| \leqq \displaystyle\sum_{j=1}^{m} \frac{\varepsilon}{2^{p_j+1}} < \varepsilon .$$

Denoting $Y_j = (\prod_a^{t_{r_j+s_j}} V(t, dt))^{-1} A_j \prod_a^{t_{r_j}} V(t, dt) - I$, $j = 1, 2, \ldots, m$ we have by (1.48) $\sum_{j=1}^{m} \|Y_j\| < \varepsilon < 1$ and for

$$X = (I + Y_m)(I + Y_{m-1}) \ldots (I + Y_1) - I =$$
$$= (\textstyle\prod_a^{\alpha_{k-1}} V(t, dt))^{-1} A_m A_{m-1} \ldots A_1 \prod_a^a V(t, dt) - I =$$
$$= (\textstyle\prod_a^{\alpha_{k-1}} V(t, dt))^{-1} A_m A_{m-1} \ldots A_1 - I = (\prod_a^c V(t, dt)^{-1} P(V, \Delta^-) - I$$

we obtain by Lemma 1.12 the estimate

$$(1.49)\qquad \|X\| = \|(\textstyle\prod_a^c V(t, \mathrm{d}t))^{-1} P(V, \Delta^-) - I\| \leqq 2 \sum_{j=1}^{m} \|Y_j\| < 2\varepsilon\,,$$

which does not depend on $c \in [a, b)$.

Define now a gauge δ on $[a, b]$ as follows. For $t \in [a, b)$ put

$$0 < \delta(t) < \min(b - t, \delta^0(t))$$

and

$$0 < \delta(b) < b - B\,.$$

If $\Delta = \{\alpha_0, t_1, \alpha_1, \ldots, \alpha_{k-1}, t_k, \alpha_k\}$ is an arbitrary δ-fine partition of $[a, b]$ then by the choice of the gauge δ we have necessarily $t_k = \alpha_k = b$ and $\alpha_{k-1} \in (B, b)$. We have also $\Delta = \Delta^- \circ (b, [\alpha_{k-1}, b])$ where

$$\Delta^- = \{\alpha_0, t_1, \alpha_1, \ldots, \alpha_{k-2}, t_{k-1}, \alpha_{k-1}\}$$

and $P(V, \Delta) = V(b, [\alpha_{k-1}, b])\, P(V, \Delta^-)$. Hence we have

$$\begin{aligned}(1.50)\qquad \|P(V, \Delta) - Q\| &= \|V(b, [\alpha_{k-1}, b])\, P(V, \Delta^-) - Q\| = \\ &= \|V(b, [\alpha_{k-1}, b]) \textstyle\prod_a^{\alpha_{k-1}} V(t, \mathrm{d}t)\, (\prod_a^{\alpha_{k-1}} V(t, \mathrm{d}t))^{-1} P(V, \Delta^-) - Q\| = \\ &= \|V(b, [\alpha_{k-1}, b]) \textstyle\prod_a^{\alpha_{k-1}} V(t, \mathrm{d}t)\, [(\prod_a^{\alpha_{k-1}} V(t, \mathrm{d}t))^{-1} P(V, \Delta^-) - I] + \\ &+ V(b, [\alpha_{k-1}, b]) \textstyle\prod_a^{\alpha_{k-1}} V(t, \mathrm{d}t) - Q\| \leqq \\ &\leqq [\|V(b, [\alpha_{k-1}, b] \textstyle\prod_a^{\alpha_{k-1}} V(t, \mathrm{d}t) - Q\| + \|Q\|]\,. \\ &.\ \|(\textstyle\prod_a^{\alpha_{k-1}} V(t, \mathrm{d}t))^{-1} P(V, \Delta^-) - I\| + \\ &+ \|V(b, [\alpha_{k-1}, b]) \textstyle\prod_a^{\alpha_{k-1}} V(t, \mathrm{d}t) - Q\|\,.\end{aligned}$$

Since $B < \alpha_{k-1} < b$ we have by (1.45)

$$\|V(b, [\alpha_{k-1}, b]) \textstyle\prod_a^{\alpha_{k-1}} V(t, \mathrm{d}t) - Q\| < \varepsilon$$

and by (1.49) we get

$$\|(\textstyle\prod_a^{\alpha_{k-1}} V(t, \mathrm{d}t))^{-1} P(V, \Delta^-) - I\| < 2\varepsilon\,.$$

Hence (1.50) yields

$$\|P(V, \Delta) - Q\| < (\varepsilon + \|Q\|)\,.\,2\varepsilon + \varepsilon = \varepsilon(2\varepsilon + 1 + 2\|Q\|)$$

for an arbitrary δ-fine partition Δ of $[a, b]$, i.e. the Perron product integral $\prod_a^b V(t, \mathrm{d}t)$ exists and its value is Q by definition.

In a completely similar way also the following result can be proved.

1.14. Theorem. *Assume that $V: [a, b] \times J \to L(\mathbb{R}^n)$ satisfies the condition $\mathscr{C}$. Assume further that for every $c \in (a, b]$ the Perron product integral $\prod_c^b V(t, \mathrm{d}t)$ exists. Let the limit*

$$\lim_{c \to a+} \prod_c^b V(t, dt)\, V(a, [a, c]) = Q$$

exists, where $Q \in L(\mathbb{R}^n)$ is invertible.

Then V is Perron product integrable over $[a, b]$ and

$$\prod_a^b V(t, dt) = Q\,.$$

Remark. It is not difficult to check, that if $V: [a, b] \times J \to L(\mathbb{R}^n)$ satisfies condition $\mathscr{C}$ and if V is Perron product integrable over $[a, b]$, then for every $d \in (a, b]$ we have

$$\lim_{c \to d-} \prod_a^c V(t, dt) = (V_-(d))^{-1} \prod_a^d V(t, dt)$$

and similarly for $d \in [a, b]$

$$\lim_{c \to d+} \prod_c^b V(t, dt) = \prod_d^b V(t, dt)\,(V_+(d))^{-1}\,.$$

If $d \in (a, b)$ then

$$\prod_a^b V(t, dt) = \lim_{c \to d+} \prod_c^b V(t, dt)\, V_+(d)\, V_-(d) \lim_{c \to d-} \prod_a^c V(t, dt)\,.$$

In [2] the following was proved.

1.15. Lemma. *Assume that $L \geqq 1$ is such a constant that for every $Z \in L(\mathbb{R}^n)$, $Z = (Z_{l,m})_{l,m=1,\dots,n}$ the inequality*

$$L^{-1} \max_{l,m} |Z_{l,m}| \leqq \|Z\| \leqq L \max_{l,m} |Z_{l,m}|$$

holds. Let $0 < \vartheta < \frac{1}{9}L^{-4}$, $Z_1, Z_2, \dots, Z_r \in L(\mathbb{R}^n)$ and assume that for every p-tuple $\{j_1, j_2, \dots, j_p\} \subset \{1, 2, \dots, r\}$, $j_1 < j_2 < \dots < j_p$ the inequality

$$(1.51) \qquad \|(I + Z_{j_p})(I + Z_{j_{p-1}}) \dots (I + Z_{j_1}) - I\| \leqq \vartheta$$

holds. Then

$$(1.52) \qquad \sum_{j=1}^r \|Z_j\| \leqq M\vartheta\,,$$

where $M = 4n^2L^2$.

The following result is a consequence of Lemma 1.15 and Lemma 1.8.

1.16. Theorem. *Assume that $V: [a, b] \times J \to L(\mathbb{R}^n)$ satisfies the condition $\mathscr{C}$ and that the Perron product integral $\prod_a^b V(t, dt) = Q$ exists. Let $\Phi: [a, b] \to L(\mathbb{R}^n)$ be given by (1.26).*

Let $\varepsilon \in (0, \frac{1}{9}L^{-1}\|(\Phi(b))^{-1}\|^{-1})$, where L is the constant from Lemma 1.15 and let $\delta: [a, b] \to (0, +\infty)$ be such a gauge on $[a, b]$ that

$$\|P(V, \Delta) - \Phi(b)\| < \varepsilon$$

for every δ-fine partition Δ of $[a, b]$.

If

$$a \leqq \beta_1 \leqq \xi_1 \leqq \gamma_1 \leqq \beta_2 \leqq \xi_2 \leqq \gamma_2 \leqq \dots \leqq \beta_m \leqq \xi_m \leqq \gamma_m \leqq b\,,$$

where

$$\xi_j - \delta(\xi_j) < \beta_j \leqq \xi_j \leqq \gamma_j < \xi_j + \delta(\xi_j)\,, \quad j = 1, 2, \dots, m\,,$$

then

$$(1.53) \qquad \sum_{j=1}^{m} \|(\Phi(\gamma_j))^{-1}\, V(\xi_j, [\beta_j, \gamma_j])\, \Phi(\beta_j) - I\| \leqq M\|(\Phi(b))^{-1}\|\, \varepsilon$$

where M is the constant from Lemma 1.15 *and*

$$(1.54) \qquad \sum_{j=1}^{m} \|V(\xi_j, [\beta_j, \gamma_j]) - \textstyle\prod_{\beta_j}^{\gamma_j} V(t, \mathrm{d}t)\| \leqq K^2 M\|(\Phi(b))^{-1}\|\, \varepsilon\,,$$

where K is the constant given in Theorem 1.7.

The proof follows exactly the lines of the proof of an analogous statement given in [2, Theorem 2.4].

Let us set

$$Z_j = (\Phi(\gamma_j))^{-1}\, V(\xi_j, [\beta_j, \gamma_j])\, \Phi(\beta_j) - I\,, \quad j = 1, \dots, m\,.$$

Since all the the assumptions of Lemma 1.8 are satisfied, we obtain by (1.28) the inequalities

$$\|(I + Z_{j_p})(I + Z_{j_{p-1}}) \dots (I + Z_{j_1}) - I\| \leqq \|(\Phi(b))^{-1}\|\, \varepsilon$$

for every p-tuple $\{j_1, \dots, j_p\} \subset \{1, 2, \dots, m\}$, $j_1 < j_2 < \dots < j_p$ and by the choice of $\varepsilon > 0$ we also have $\|(\Phi(b))^{-1}\|\, \varepsilon < (1/a)\, L^{-1}$. Hence Lemma 1.15 yields

$$(1.55) \qquad \sum_{j=1}^{m} \|Z_j\| \leqq M\|(\Phi(b))^{-1}\|\, \varepsilon$$

and (1.53) is satisfied.

Since $\prod_{\beta_j}^{\gamma_j} V(t, \mathrm{d}t) = \Phi(\gamma_j)\,(\Phi(\beta_j))^{-1}$, $j = 1, \dots, m$ and therefore also

$$\begin{aligned} &V(\xi_j, [\beta_j, \gamma_j]) - \textstyle\prod_{\beta_j}^{\gamma_j} V(t, \mathrm{d}t) = \\ &= \Phi(\gamma_j)\,[(\Phi(\gamma_j))^{-1}\, V(\xi_j, [\beta_j \gamma_j])\, \Phi(\beta_j) - I]\,(\Phi(\beta_j))^{-1} = \\ &= \Phi(\gamma_j)\, Z_j (\Phi(\beta_j))^{-1}\,, \end{aligned}$$

for $j = 1, \dots, m$, we obtain by Theorem 1.7 the estimate

$$\|V(\xi_j, [\beta_j, \gamma_j]) - \textstyle\prod_{\beta_j}^{\gamma_j} V(t, \mathrm{d}t)\| \leqq K^2 \|Z_j\|\,, \quad j = 1, \dots, m$$

which together with (1.55) implies (1.54).

1.17. Remark. Lemma 1.15 and also its proof given in [2] is strictly based on the structure of matrices which represent the operators from $L(\mathbb{R}^n)$. It is easy to observe that all the statements given before Lemma 1.15 do not use the structure of $\mathbb{R}^n$ and $L(\mathbb{R}^n)$ and that in all of them we can replace $L(\mathbb{R}^n)$ by $L(X)$, where X is an arbitrary Banach space and $L(X)$ is the Banach space of all bounded linear operators on X equipped with the corresponding operator norm.

In this connection it is natural to ask whether an analog of Lemma 1.15 holds also for infinitedimensional Banach spaces. The following example shows that the answer to this question is negative.

Example (J. Kurzweil). Let $X = c_0$, where c_0 is the Banach space of all bounded real sequences $x = (\alpha_n)_{n=1}^{\infty}$ such that $\lim_{n\to\infty} \alpha_n = 0$ with the norm

$$\|x\| = \sup\{|\alpha_j|;\ j \in \mathbb{N}\}\,, \quad x \in X\,.$$

For every $i \in \mathbb{N}$ define the operator $E_i\colon X \to X$ as follows:

$$E_i x = y = (\beta_j)_{j=1}^{\infty}\,, \quad \text{where} \quad x = (\alpha_j)_{j=1}^{\infty} \quad \text{and} \quad \beta_j = 0\,, \quad j \in \mathbb{N}\,,$$
$$j \neq 2_{i-1}\,, \quad \beta_{2_{i-1}} = \alpha_{2_i}\,.$$

The operator E_i shifts the element α_{2i} of the sequence x to the $2i$-1-th position and sets all the other elements of the resulting sequence to zero.

It is evident E_i, $i = 1, 2, \ldots$ are linear operators and that

$$(1.56) \qquad \|E_i\| = \sup_{\|x\| \leq 1} \|E_i x\| = \sup_{\|x\| \leq 1} |\beta_j| = \sup_{\|x\| \leq 1} |\alpha_{2i}| = 1$$

for every $i = 1, 2, \ldots$, i.e. $E_i \in L(X)$.

Further it is easy to see that

$$(1.57) \qquad E_i E_j = 0 \quad \text{for all} \quad i, j \in \mathbb{N}\,.$$

Assume that $\eta > 0$ is given and define

$$Z_i = \eta E_i\,, \quad i \in \mathbb{N}\,.$$

Let $j_1, j_2, \ldots, j_p \in \mathbb{N}$ be an arbitrary p-tuple such that $j_1 < j_2 < \ldots < j_p$. Then by (1.57) we have

$$(I + Z_{j_p})(I + Z_{j_{p-1}}) \ldots (I + Z_{j_1}) = I + \sum_{k=1}^{p} Z_{j_k} = I + \eta \sum_{k=1}^{p} E_{j_k}$$

and

$$(1.58) \qquad (I + Z_{j_p})(I + Z_{j_{p-1}}) \ldots (I + Z_{j_1}) - I = \eta \sum_{k=1}^{p} E_{j_k}\,.$$

Since by the definition of E_i, $i \in \mathbb{N}$ we have for $x = (\alpha_j)_{j=1}^{\infty} \in X$

$$\left(\sum_{k=1}^{p} E_{j_k}\right) x = \sum_{k=1}^{p} E_{j_k} x = y = (\beta_j)_{j=1}^{\infty}$$

where $\beta_j = 0$ for $j \neq 2_{j_k - 1}$, $k = 1, \ldots, p$ and

$$\beta_{2 j_k - 1} = \alpha_{2 j_k}\,, \quad k = 1, 2, \ldots, p$$

we obtain

$$\Big\|\sum_{k=1}^{p} E_{j_k}\Big\| = \sup_{\|x\|\leq 1}\Big\|\sum_{k=1}^{p} E_{j_k}x\Big\| = \sup_{j}|\beta_j| = \sup_{k}|\alpha_{2j_k}| = 1$$

and therefore by (1.58) we have

$$(1.59) \qquad \|(I + Z_{j_p})(I + Z_{j_{p-1}})\dots(I + Z_{j_1}) - I\| = \eta\Big\|\sum_{k=1}^{p} E_{j_k}\Big\| = \eta\,.$$

If we take an arbitrarily large $M > 0$ and if $r \in \mathbb{N}$ is such that $r > 2(M + 1)$, than we can take r operators of the form $Z_i = \eta E_i$ (e.g. $Z_1, Z_2, \dots, Z_r$) and by (1.59) we have

$$\|(I + Z_{j_p})(I + Z_{j_{p-1}})\dots(I + Z_{j_1}) - I\| = \eta$$

for every p-tuple $j_1, \dots, j_p \in \{1, 2, \dots, r\}$, $j_1 < j_2 < \dots < j_p$ and (1.56) yields

$$(1.60) \qquad \sum_{j=1}^{r}\|Z_j\| = \sum_{j=1}^{r}\eta\|E_j\| = r\eta > 2(M + 1)\,\eta\,.$$

Taking now e.g. $\eta = \vartheta/2$ then the assumption (1.51) of Lemma 1.15 is satisfied but we have by (1.60) the inequality

$$\sum_{j=1}^{r}\|Z_j\| > 2(M + 1)\frac{\vartheta}{2} > M\vartheta$$

and this inequality shows that Lemma 1.15 cannot hold for infinite-dimensional spaces, because M can be choosen arbitrarily large.

2. THE CONDITION $\mathscr{C}^+$ AND GENERALIZED LINEAR DIFFERENTIAL EQUATIONS

Let us introduce the following condition for functions $V\colon [a, b] \times J \to L(\mathbb{R}^n)$.

Condition $\mathscr{C}^+$.

There exists a nondecreasing function $g\colon [a, b] \to \mathbb{R}$ *such that for every* $t \in [a, b]$ *there is a* $\varrho = \varrho(t) > 0$ *such that*

$$(2.1) \qquad \|V(t, [x, y]) - I\| \leqq g(y) - g(x)$$

for all $x, y \in [a, b]$, $t - \varrho < x \leqq t \leqq y < t + \varrho$.

2.1. Remark. It is easy to see that if $V\colon [a, b] \times J \to L(\mathbb{R}^n)$ satisfies the condition $\mathscr{C}^+$ with a continuous nondecreasing function $g\colon [a, b] \to \mathbb{R}$ then V satisfies (1.5), i.e. the condition given by Jarník and Kurzweil in [2] is fulfilled.

The following type of a function V motivates the introduction of the condition $\mathscr{C}^+$.

Let $A\colon [a, b] \to L(\mathbb{R}^n)$ be given such that $A \in BV([a, b]; L(\mathbb{R}^n))$. Put

$$(2.2) \qquad V_1(t, [x, y]) = I + A(y) - A(x)$$

for $x, y \in [a, b]$, $x \leqq t \leqq y$.

If in addition $M: [a, b] \to L(\mathbb{R}^n)$ is bounded, i.e. $\|M(t)\| \leqq L$ for $t \in [a, b]$, then put

$$(2.3) \qquad V_1^M(t, [x, y]) = I + M(t)[A(y) - A(x)]$$

for $x, y \in [a, b]$, $x \leqq t \leqq y$.

We have

$$\|V_1(t, [x, y]) - I\| = \|A(y) - A(x)\| \leqq \operatorname{var}_a^y A - \operatorname{var}_a^x A$$

and therefore V_1 evidently satisfies the condition $\mathscr{C}^+$ with $g(s) = \operatorname{var}_a^s A$, $s \in [a, b]$. Similarly

$$\|V_1^M(t, [x, y]) - I = \|M(t)(A(y) - A(x))\| \leqq L\|A(y) - A(x)\| \leqq$$
$$\leqq L(\operatorname{var}_a^y A - \operatorname{var}_a^x A)$$

and V_1^M satisfies the condition $\mathscr{C}^+$ with $g(s) = L \operatorname{var}_a^s A$, $s \in [a, b]$.

If $V: [a, b] \times J \to L(\mathbb{R}^n)$ is such that

$$(2.4) \qquad V(t, [x, y]) = V(t, [x, t]) + V(t, [t, y]) - I$$

for $a \leqq x \leqq t \leqq y \leqq b$ then

$$(2.5) \qquad V(t, [x, y]) - V(t, [t, y]) V(t, [x, t]) = (V(t, [t, y]) - I)(V(t, [x, t]) - I)$$

because evidently

$$(V(t, [t, y]) - I)(V(t, [x, t] - I) =$$
$$= V(t, [t, y]) V(t, [x, t]) - V(t, [x, t]) - V(t, [t, y]) + I .$$

It is easy to see that V_1, V_1^M given in (2.2), (2.3) respectively, satisfy (2.4).

If $V: [a, b] \times J \to L(\mathbb{R}^n)$ satisfies the condition $\mathscr{C}^+$ and (2.4) then by (2.1) and (2.5) we have

$$\|V(t, [x, y]) - V(t, [t, y]) V(t, [x, t])\| \leqq$$
$$\leqq \|V(t, [t, y]) - I\| . \|V(t, [x, t]) - I\| \leqq (g(y) - g(t))(g(t) - g(x)) .$$

If in this situation for any $t \in [a, b]$ either $\lim_{y \to t+} g(y) = g(t+) = g(t)$ or $\lim_{x \to t-} g(x) = g(t-) = g(t)$ then it is not difficult to check that V satisfies (1.3) from the condition $\mathscr{C}$.

For V_1 given in (2.2) we have

$$\|V_1(t, [x, y]) - V_1(t, [t, y]) V_1(t, [x, t])\| =$$
$$= \|(A(y) - A(t))(A(t) - A(x)\| .$$

Since $A \in BV([a, b]; L(\mathbb{R}^n))$ the limits $\lim_{x \to t-} A(x) = A(t-)$ and $\lim_{y \to t+} A(y) = A(t+)$

exist. Denote $\Delta^+ A(t) = A(t+) - A(t)$ and $\Delta^- A(t) = A(t) - A(t-)$. Hence V_1 satisfies (1.3) from the condition $\mathscr{C}$ if and only if $\Delta^+ A(t)\,\Delta^- A(t) = 0$, $t \in [a, b]$.

Similarly for V_1^M given by (2.3) we get

$$\|V_1^M(t, [x, y]) - V_1^M(t, [t, y])\, V_1^M(t, [x, t])\| =$$
$$= \|M(t)\,(A(y) - A(t))\,.\,(A(t) - A(x))\,M(t)\|$$

and again the condition $\Delta^+ A(t)\,\Delta^- A(t) = 0$, $t \in [a, b]$ is necessary and sufficient for V_1^M to satisfy (1.3) from the condition $\mathscr{C}$ because M is bounded.

It is easy to see that V_1, V_1^M given above satisfy also (1.2) from condition $\mathscr{C}$.

Since $\lim_{y \to t+} V_1(t, [t, y]) = I + \Delta^+ A(t)$, $t \in [a, b)$ and $\lim_{x \to t-} V_1(t, [x, t]) = I + \Delta^- A(t)$, $t \in (a, b]$ we obtain that V_1 satisfies (1.4) from the condition $\mathscr{C}$ if and only if $I + \Delta^+ A(t)$, $t \in [a, b)$ and $I + \Delta^- A(t)$, $t \in (a, b]$ are invertible.

Similarly V_1^M satisfies (1.4) if and only if $I + M(t)\,\Delta^+ A(t)$, $t \in [a, b)$ and $I + M(t)\,\Delta^- A(t)$, $t \in (a, b]$ are invertible.

2.2. Lemma. *Assume that $V: [a, b] \times J \to L(\mathbb{R}^n)$ is Perron product integrable over $[a, b]$ and that the conditions $\mathscr{C}$ and $\mathscr{C}^+$ are satisfied.*

Then for the function $\Phi: [a, b] \to L(\mathbb{R}^n)$ given by

$$\Phi(a) = I\,, \quad \Phi(s) = \prod_a^s V(t, \mathrm{d}t)\,, \quad s \in (a, b] \tag{2.6}$$

we have $\Phi \in BV([a, b]; L(\mathbb{R}^n))$, $\Phi^{-1} \in BV([a, b]; L(\mathbb{R}^n))$.

Proof. Assume that $x, y \in [a, b]$, $x \leqq y$. Then if $t \in [x, y]$, we have

$$\Phi(y) - \Phi(x) = (\Phi(y)\,(\Phi(x))^{-1} - I)\,\Phi(x) = (\prod_x^y V(t, \mathrm{d}t) - I)\,\Phi(x) =$$
$$= (\prod_x^y V(t, \mathrm{d}t) - V(t, [x, y]))\,\Phi(x) + (V(t, [x, y]) - I)\,\Phi(x)\,.$$

By Theorem 1.7 and by the condition $\mathscr{C}^+$ we therefore have

$$\|\Phi(y) - \Phi(x)\| \leqq K[\|\prod_x^y V(t, \mathrm{d}t) - V(t, [x, y])\| + g(y) - g(x)] \tag{2.7}$$
$$\text{provided} \quad t - \varrho(t) < x \leqq t \leqq y < t + \varrho(t)\,.$$

Assume further that $\varepsilon > 0$ is given and that $\delta: [a, b] \to (0, +\infty)$ is such a gauge on $[a, b]$ that

$$\|P(V, \Delta) - \Phi(b)\| < \varepsilon$$

holds for every δ-fine partition Δ on $[a, b]$ and that $\delta(t) < \varrho(t)$ for $t \in [a, b]$, where $\varrho(t) > 0$ is given in condition $\mathscr{C}^+$.

Let now $a = s_0 < s_1 < \ldots < s_m = b$ be given and let

$$\Delta^p = \{\alpha_0^p, t_1^p, \alpha_1^p, \ldots, t_{k_p}^p, \alpha_{k_p}^p\}$$

be an arbitrary δ-fine partition of $[s_{p-1}, s_p]$, $p = 1, \ldots, m$. Then by (2.7) we have

$$\|\Phi(s_p) - \Phi(s_{p-1})\| \leqq \sum_{j=1}^{k_p} \|\Phi(\alpha_j^p) - \Phi(\alpha_{j-1}^p)\| \leqq$$

$$\leqq K \sum_{j=1}^{k_p} (\|\prod_{\alpha^p_{j-1}}^{\alpha_j^p} V(t, \mathrm{d}t) - V(t_j^p, [\alpha_{j-1}^p, \alpha_j^p])\| + g(\alpha_j^p) - g(\alpha_{j-1}^p)) =$$

$$= K \sum_{j=1}^{k_p} \|\prod_{\alpha_{j-1}^p}^{\alpha_j^p} V(t, \mathrm{d}t) - V_j^p(t, [\alpha_{j-1}^p, \alpha_j^p])\| + K(g(s_p) - g(s_{p-1}))$$

for every $p = 1, 2, \ldots, m$ and henceforth

$$(2.8) \qquad \sum_{p=1}^{m} \|\Phi(s_p) - \Phi(s_{p-1})\| \leqq$$

$$\leqq K \sum_{p=1}^{m} \sum_{j=1}^{k_p} \|\prod_{\alpha^p_{j-1}}^{\alpha_j^p} V(t, \mathrm{d}t) - V_j^p(t, [\alpha_{j-1}^p, \alpha_j^p])\| + K(g(b) - g(a))\,.$$

Using Theorem 1.16 we obtain the estimate

$$\sum_{p=1}^{m} \sum_{j=1}^{k_p} \|\prod_{\alpha^p_{j-1}}^{\alpha_j^p} V(t, \mathrm{d}t) - V(t_j^p, [\alpha_{j-1}^p, \alpha_j^p])\| \leqq K^2 M \|(\Phi(b))^{-1}\| \,\varepsilon$$

because evidently $\Delta = \Delta^1 \circ \Delta^2 \circ \ldots \circ \Delta^m$ is a δ-fine partition of $[a, b]$. Therefore by (2.8) we have

$$\sum_{p=1}^{m} \|\Phi(s_p) - \Phi(s_{p-1})\| \leqq K^3 M \|(\Phi(b))^{-1}\| \,\varepsilon + K(g(b) - g(a))$$

for an arbitrary choice of points $a = s_0 < s_1 < \ldots < s_m = b$ and consequently also

$$(2.9) \qquad \operatorname{var}_a^b \Phi \leqq K^3 M \|(\Phi(b))^{-1}\| \,\varepsilon + K(g(b) - g(a)) < \infty\,,$$

i.e. $\Phi \in BV([a, b]; L(\mathbb{R}^n))$.

It can be observed easily that (2.9) yields the inequality

$$\operatorname{var}_a^b \Phi \leqq K(g(b) - g(a))$$

because $\varepsilon > 0$ in (2.9) can be taken arbitrarily small.

Since $(\Phi(s))^{-1} = (\Phi(b))^{-1} \prod_s^b V(t, \mathrm{d}t)$, the boundedness of $\operatorname{var}_a^b \Phi^{-1}$ can be shown similarly.

2.3. Lemma. *Suppose the assumptions of Lemma 2.2 are satisfied. Then for every $t \in [a, b]$ the Perron-Stieltjes integral*

$$(2.10) \qquad \int_a^t \mathrm{d}[\Phi(r)]\, (\Phi(r))^{-1} = \tilde{A}(t) \in L(\mathbb{R}^n)$$

exists. For $\tilde{A}$ given by (2.10) we have $\tilde{A} \in BV([a, b]; L(\mathbb{R}^n))$ and $[I - \Delta^- \tilde{A}(t)]^{-1}$, $t \in (a, b]$, $[I + \Delta^+ \tilde{A}(t)]^{-1}$, $t \in [a, b)$ exist.

Proof. By Lemma 2.2 Φ and Φ^{-1} are of bounded variation. Therefore the Perron-Stieltjes integral in (2.10) exists (see e.g. [4] or [3]). $\tilde{A} \in BV([a, b]; L(\mathbb{R}^n))$ follows from the fact that $\Phi \in BV([a, b]; L(\mathbb{R}^n))$.

For every $\delta > 0$ we have

$$\tilde{A}(t) - \tilde{A}(t - \delta) = \int_{t-\delta}^{t} \mathrm{d}[\Phi(r)] (\Phi(r))^{-1}$$

and therefore

$$\Delta^- \tilde{A}(t) = \lim_{\delta \to 0+} \int_{t-\delta}^{t} \mathrm{d}[\Phi(r)] (\Phi^{-1}(r)) = \lim_{\delta \to 0+} (\Phi(t) - \Phi(t-)) (\Phi(t))^{-1} =$$

$$= (\Phi(t) - \Phi(t-)) (\Phi(t))^{-1} = I - \Phi(t-) (\Phi(t))^{-1}$$

(see again [4] for the calculation of this limit). By (1.37) in Lemma 1.11 we have $\Phi(t-) = (V_-(t))^{-1} \Phi(t)$, i.e.

$$I - \Delta^- \tilde{A}(t) = \Phi(t-) (\Phi(t))^{-1} = (V_-(t))^{-1} \Phi(t) (\Phi(t))^{-1} = (V_-(t))^{-1}$$

for $t \in (a, b]$, where $V_-(t)$ is invertible by (1.4) from condition $\mathscr{C}$.

In a completely analogous way we obtain also

$$I + \Delta^+ \tilde{A}(t) = V_+(t)$$

for $t \in [a, b)$, where $V_+(t)$ is invertible by (1.4).

2.4. Theorem. *Suppose the assumptions of Lemma 2.2 are satisfied. Then the relation*

$$(2.11) \qquad \Phi(s) = \Phi(a) + \int_a^b \mathrm{d}[\tilde{A}(t)] \Phi(t), \quad s \in [a, b]$$

holds, where $\Phi: [a, b] \to L(\mathbb{R}^n)$ *is given by* (2.6) *and* $\tilde{A}: [a, b] \to L(\mathbb{R}^n)$ *is defined by* (2.10).

Proof. Using the substitution theorem for Perron-Stieltjes integrals (see e.g. [3, I.4.25]) we have by the definition of $\tilde{A}$

$$\int_a^s \mathrm{d}[\tilde{A}(t)] \Phi(t) = \int_a^s \mathrm{d}[\int_a^t \mathrm{d}[\Phi(r)] (\Phi(r))^{-1}] \Phi(t) =$$

$$= \int_a^s \mathrm{d}[\Phi(r) (\Phi(r))^{-1} \Phi(r) = \int_a^s \mathrm{d}[\Phi(r)] = \Phi(s) - \Phi(a)$$

for every $s \in [a, b]$, i.e. (2.11) holds.

In [3] a theory of generalized linear differential equations of the form

$$(2.12) \qquad \mathrm{d}x = \mathrm{d}[A]x + \mathrm{d}g$$

was developed in the case when $A: [a, b] \to L(\mathbb{R}^n)$, $A \in BV([a, b]; L(\mathbb{R}^n))$, $g: [a, b] \to \mathbb{R}^n$, $g \in BV([a, b]; \mathbb{R}^n)$.

A function $x: [\alpha, \beta] \to \mathbb{R}^n$, is said to be a solution of (2.12) on the interval $[\alpha, \beta] \subset [a, b]$ if for every $t, t_0 \in [\alpha, \beta]$ the equality

$$(2.13) \qquad x(t) = x(t_0) + \int_{t_0}^{t} \mathrm{d}[A(r)] x(r) + g(t) - g(t_0)$$

is satisfied, where the integral in this relation is taken in the Perron-Stieltjes sense (see also [4] for this matter).

The following results are known for equations of the form (2.12).

2.5. Theorem. I. *If $A \in BV([a, b]; L(\mathbb{R}^n))$ then the initial value problem*

$$(2.14) \qquad dx = d[A]\, x + dg\,, \quad x(t_0) = \tilde{x} \in R^n\,, \quad t_0 \in [a, b]$$

has a unique solution $x\colon [a, b] \to \mathbb{R}^n$ on $[a, b]$ for any choice of $g \in BV([a, b]; \mathbb{R}^n)$, $t_0 \in [a, b]$, $\tilde{x} \in \mathbb{R}^n$ if and only if $I + \Delta^+ A(t) \in L(\mathbb{R}^n)$ is invertible for every $t \in [a, b)$, $I - \Delta^- A(t) \in L(\mathbb{R}^n)$ is invertible for every $t \in (a, b]$. (See Theorem III.1.4 in [3].)

Assume that $A \in BV([a, b]; L(\mathbb{R}^n))$ satisfies

$$(2.15) \qquad \begin{aligned} &[I + \Delta^+ A(t)]^{-1} \quad \text{exists for every} \quad t \in [a, b)\,, \\ &[I - \Delta^- A(t)]^{-1} \quad \text{exists for every} \quad t \in (a, b]\,. \end{aligned}$$

II. *There exists a uniquely determined $\Psi\colon [a, b] \to L(\mathbb{R}^n)$ called the fundamental matrix of (2.12) such that*

$$(2.16) \qquad \Psi(t) = I + \int_a^t d[A(r)]\, \Psi(r)\,, \quad t \in [a, b]\,.$$

$\Psi(t) \in L(\mathbb{R}^n)$ is invertible for every $t \in [a, b]$, there exists a constant $M > 0$ such that

$$(2.17) \qquad \|\Psi(t)\, (\Psi(s))^{-1}\| \leqq M\,, \quad s, t \in [a, b]\,.$$

(See III.2.2 and III.2.3 in [3].)

III. *The unique solution $x(t)\colon [a, b] \to L(\mathbb{R}^n)$ of (2.14) is given by the variation of constants formula*

$$\begin{aligned} x(t) &= \Psi(t)\, (\Psi(t_0))^{-1}\, \tilde{x} + g(t) - g(t_0) - \\ &- \int_{t_0}^t d_s[\Psi(t)\, \Psi^{-1}(s)]\, (g(s) - g(t_0)) = \\ &= g(t) + \Psi(t)\, (\Psi(t_0))^{-1}\, (\tilde{x} - g(t_0)) - \\ &- \Psi(t) \int_{t_0}^t d[\Psi^{-1}(s)]\, g(s)\,, \quad t \in [a, b]\,. \end{aligned}$$

(See III.2.13 in [3].)

Using the concept of the generalized linear differential equation (2.12) we can reformulate the results of Theorem 2.4 and Lemma 2.3 as follows.

2.6. Theorem. *If $V\colon [a, b] \times J \to L(\mathbb{R}^n)$ is Perron product integrable over $[a, b]$ and if it satisfies the conditions $\mathscr{C}$ and $\mathscr{C}^+$, then there exists a $\tilde{A} \in BV([a, b]; L(\mathbb{R}^n))$ which satisfies (2.15) with $A = \tilde{A}$ such that the function $\Phi\colon [a, b] \to L(\mathbb{R}^n)$ given in (2.6) is the fundamental matrix of the generalized linear differential equation (2.12) with $A = \tilde{A}$.*

Theorem 2.6 naturally suggests the following problem.

Given $A \in BV([a, b]; L(\mathbb{R}^n))$ such that (2.15) holds. Construct a function $V\colon [a, b] \times J \to L(\mathbb{R}^n)$ which is Perron product integrable over $[a, b]$, for which the conditions $\mathscr{C}$ and $\mathscr{C}^+$ are fulfilled such that for the function $\Phi: [a, b] \to L(\mathbb{R}^n)$ given by (2.6) the equality

$$\Phi(s) = I + \int_a^s \mathrm{d}[A(r)]\, \Phi(r), \quad s \in [a, b]$$

holds.

Since by I. from Theorem 2.5 the solution of (2.16) is unique, we are in fact asking for a Perron product integral representation of the fundamental matrix Ψ of the equation (2.12).

The problem has a positive answer in the case when $A: [a, b] \to L(\mathbb{R}^n)$ is such that

$$(2.18) \qquad A \in BV([a, b]; L(\mathbb{R}^n)),\ A(t-) = A(t) \text{ for every } t \in (a, b],$$
$$[I + \Delta^+ A(t)]^{-1} \text{ exists for every } t \in [a, b).$$

For A satisfying (2.18) define

$$(2.19) \qquad V_1(t, [x, y]) = I + A(y) - A(x), \quad x, y \in [a, b], \quad x \leqq t \leqq y.$$

Using the facts listed in Remark 2.1 it is easy to see that $V_1: [a, b] \times J \to L(\mathbb{R}^n)$ satisfies the conditions $\mathscr{C}$ and $\mathscr{C}^+$.

2.7. Lemma. *Assume that A satisfies (2.18). If $\Psi: [a, b] \to L(\mathbb{R}^n)$ is the fundamental matrix of (2.12) (see II. in Theorem 2.5), then for every $\vartheta > 0$ there is a gauge δ on $[a, b]$ such that*

$$(2.20) \qquad \sum_{j=1}^{k} \| V_1(t_j, [\alpha_{j-1}, \alpha_j]) - \Psi(\alpha_j)(\Psi(\alpha_{j-1}))^{-1} \| < \vartheta$$

for every δ-fine partition $\Delta = \{\alpha_0, t_1, \alpha_1, \ldots, \alpha_{k-1}, t_k, \alpha_k\}$ of $[a, b]$.

Proof. Let $\varepsilon > 0$ be given. Since A is continuous from the left, for every $t \in [a, b]$ there is a $\delta_1(t) > 0$ such that

$$(2.21) \qquad \operatorname{var}_x^t A < \varepsilon$$

for $x \in [a, b]$, $t - \delta_1(t) < x \leqq t$.

Since the integral $\int_a^b \mathrm{d}[A(r)]\, \Psi(r) = I - \Psi(b)$ exists, by the Saks-Henstock lemma (see e.g. [4]) there is a gauge δ on $[a, b]$, $\delta(t) < \delta_1(t)$, $t \in [a, b]$ such that for every δ-fine partition Δ of $[a, b]$ we have

$$(2.22) \qquad \sum_{j=1}^{k} \| (A(\alpha_j) - A(\alpha_{j-1}))\, \Psi(t_j) - \int_{\alpha_{j-1}}^{\alpha_j} \mathrm{d}[A(r)]\, \Psi(r) \| < \varepsilon.$$

For any δ-fine partition Δ (2.21) implies

$$(2.23) \qquad \operatorname{var}_{\alpha_{j-1}}^{t_j} A < \varepsilon, \quad j = 1, 2, \ldots, k.$$

Moreover we have

$$(2.24) \qquad \Psi(\alpha_j)(\Psi(\alpha_{j-1}))^{-1} = I + \int_{\alpha_{j-1}}^{\alpha_j} \mathrm{d}[A(r)]\, \Psi(r)(\Psi(\alpha_{j-1}))^{-1}, \quad j = 1, 2, \ldots, k$$

and for every $j = 1, 2, \ldots, k$ we have

(2.25) $$V_1(t_j,]\alpha_{j-1}, \alpha_j]) - \Psi(\alpha_j)(\Psi(\alpha_{j-1}))^{-1} =$$
$$= I + A(\alpha_j) - A(\alpha_{j-1}) - I - \int_{\alpha_{j-1}}^{\alpha_j} \mathrm{d}[A(r)]\, \Psi(r)(\Psi(\alpha_{j-1}))^{-1} =$$
$$= (A(\alpha_j) - A(\alpha_{j-1}))(I - \Psi(t_j)(\Psi(\alpha_{j-1}))^{-1}) +$$
$$+ [(A(\alpha_j) - A(\alpha_{j-1}))\, \Psi(t_j) - \int_{\alpha_{j-1}}^{\alpha_j} \mathrm{d}[A(r)]\, \Psi(r)(\Psi(\alpha_{j-1}))^{-1}\,.$$

Using (2.23) and (2.17) we have

$$\|\Psi(t_j)(\Psi(\alpha_{j-1}))^{-1} - I\| = \|\int_{\alpha_{j-1}}^{t_j} \mathrm{d}[A(r)]\, \Psi(r)(\Psi(\alpha_{j-1}))^{-1}\| \leqq$$
$$\leqq \operatorname{var}_{\alpha_{j-1}}^{t_j} A M < \varepsilon M\,.$$

Hence using (2.22) and (2.17) we get by (2.25)

$$\sum_{j=1}^{k} \|V_1(t_j, [\alpha_{j-1}, \alpha_j]) - \Psi(\alpha_j)(\Psi(\alpha_{j-1}))^{-1}\| < \sum_{j=1}^{k} \|A(\alpha_j) - A(\alpha_{j-1})\|\, \varepsilon M +$$
$$+ \sum_{j=1}^{k} \|(A(\alpha_j) - A(\alpha_{j-1}))\, \Psi(t_j) -$$
$$- \int_{\alpha_{j-1}}^{\alpha_j} \mathrm{d}[A(r)]\, \Psi(r)\| \,.\, \|(\Psi(\alpha_{j-1}))^{-1}\| \leqq$$
$$\leqq \varepsilon M \operatorname{var}_a^b A + \varepsilon M = \varepsilon M(1 + \operatorname{var}_a^b A)\,.$$

Taking $\varepsilon = \vartheta/[(M + 1)(1 + \operatorname{var}_a^b A)]$ for an arbitrary $\vartheta > 0$ we obtain immediately (2.20).

Theorem 2.7 in [2] states the following

2.8. Theorem. *Assume that* $W: [a, b] \to L(\mathbb{R}^n)$ *is such that*

$$\max\{\|W(t)\|, \|(W(t))^{-1}\|\} \leqq M$$

where $M > 0$ *is a constant. Let* $V: [a, b] \times J \to L(\mathbb{R}^n)$ *be such that for every* $\vartheta > 0$ *there exists a gauge* δ *on* $[a, b]$ *such that*

$$\sum_{j=1}^{k} \|V(t_j, [\alpha_{j-1}, \alpha_j]) - W(\alpha_j)(W(\alpha_{j-1}))^{-1}\| < \vartheta$$

provided Δ *is a* δ*-fine partition of* $[a, b]$.

Then the Perron product integral $\prod_a^b V(t, \mathrm{d}t)$ *exists and is equal to* $W(b)(W(a))^{-1}$.

Using this result and Lemma 2.7 we obtain the following.

2.9. Theorem. *Assume that* A *satisfies* (2.18). *Then the function* $V_1: [a, b] \times J \to L(\mathbb{R}^n)$ *given by* (2.19) *is Perron product integrable over* $[a, b]$ *and for every* $s \in [a, b]$ *we have*

$$\Psi(s) = \prod_a^s V_1(t, \mathrm{d}t)$$

where $\Psi: [a, b] \to L(\mathbb{R}^n)$ *is the fundamental matrix of* (2.12).

Let us now replace (2.18) by the following assumption

(2.26) $\quad A \in BV([a, b]; L(\mathbb{R}^n))$, *A is continuous at every point* $t \in [a, b]$.

For $A: [a, b] \to L(\mathbb{R}^n)$ satisfying (2.26) let us define

$$(2.27) \qquad V_2(t, [x, y]) = \exp(A(y) - A(x)) = \sum_{k=0}^{\infty} \frac{(A(y) - A(x))^k}{k!}.$$

2.10. Lemma. *If* $A: [a, b] \to L(\mathbb{R}^n)$ *satisfies* (2.26) *then to every* $\eta > 0$ *there exists a gauge* δ *on* $[a, b]$ *such that*

$$(2.28) \qquad \sum_{j=1}^{k} \|V_1(t_j, [\alpha_{j-1}, \alpha_j]) - V_2(t_j, [\alpha_{j-1}, \alpha_j])\| < \eta$$

for every δ*-fine partition* $\Delta = \{\alpha_0, t_1, \alpha_1, \ldots, \alpha_{k-1}, t_k, \alpha\}\}$ *of* $[a, b]$, *where* V_1, V_2 *is given by* (2.19), (2.27) *respectively.*

Proof. Since A is assumed to be continuous in $[a, b]$, for every $\varepsilon \in (0, 1)$ and $t \in [a, b]$ there is a $\delta(t) > 0$ such that

$$(2.29) \qquad \|A(y) - A(x)\| < \varepsilon$$

for every $x, y \in [a, b]$, $t - \delta(t) < x \leqq t \leqq y < t + \delta(t)$. For such $x, y \in [a, b]$ we have

$$V_2(t, [x, y]) - V_1(t, [x, y]) = \exp(A(y) - A(x)) - I - (A(y) - A(x)) = \sum_{k=2}^{\infty} \frac{(A(y) - A(x))^k}{k!}$$

and also

$$\|V_1(t, [x, y]) - V_2(t, {}^2(t, [x, y])\| \leqq \sum_{k=2}^{\infty} \frac{\|A(y) - A(x)\|^k}{k!}.$$

Denoting $\|A(y) - A(x)\| = \lambda$ we have $\lambda < \varepsilon < 1$ and this yields

$$(2.30) \qquad \|V_1(t, [x, y]) - V_2(t, [x, y[\,[x, y])\| \leqq \sum_{k=2} \frac{\lambda^k}{k!} = $$
$$= e^{\lambda} - 1 - \lambda < \lambda^2 = \|A(y) - A(x)\|^2 \leqq \varepsilon\|A(y) - A(x)\|$$

by (2.29) for $x, y \in [a, b]$, $t - \delta(t) < x \leqq t \leqq y < t + \delta(t)$.

Given $\eta > 0$; let us choose

$$\varepsilon \in \left(0, \min\left\{1, \frac{\eta}{\operatorname{var}_a^b A + 1}\right\}\right)$$

and let $\delta(t) > 0$, $t \in [a, b]$ be such that (2.29) holds for this choice of $\varepsilon > 0$. In this case (2.30) has the form

$$(2.31) \quad \|V_1(t, [x, y]) - V_2(t, [x, y])\| \leqq \frac{\eta}{1 + \operatorname{var}_a^b A} \|A(y) - A(x)\|$$

for $x, y \in [a, b]$; $t - \delta(t) < x \leqq t \leqq y < t + \delta(t)$.

Let now $\Delta = \{\alpha_0, t_1, \alpha_1, \ldots, \alpha_{k-1}, t_k, \alpha_k\}$ be an arbitrary δ-fine partition of $[a, b]$. Then we have by (2.31)

$$\|V_1(t_j, [\alpha_{j-1}, \alpha_j]) - V_2(t_j, [\alpha_{j-1}, \alpha_j])\| \leqq$$
$$\leqq \frac{\eta}{1 + \operatorname{var}_a^b A} \|A(\alpha_j) - A(\alpha_{j-1})\|$$

and consequently

$$\sum_{j=1}^{k} \|V_1(t_j, [\alpha_{j-1}, \alpha_j]) - V_2(t_j, [\alpha_{j-1}, \alpha_j])\| \leqq$$
$$\leqq \frac{\eta}{1 + \operatorname{var}_a^b A} \sum_{j=1}^{k} \|A(\alpha_j) - A(\alpha_{j-1})\| \leqq \frac{\eta \operatorname{var}_a^b A}{1 + \operatorname{var}_a^b A} \leqq \eta .$$

In [2] the following definition is given.

The functions V_1, V_2: $[a, b] \times J \to L(\mathbb{R}^n)$ are called equivalent if for every $\eta > 0$ there is a gauge δ on $[a, b]$ such that (2.28) holds for every δ-fine partition Δ of $[a, b]$.

In the sense of this definition the functions V_1, V_2 from (2.19) and (2.27) are equivalent by the Lemma 2.9.

The following analog of Theorem 2.9 in [2] is true

2.11. Theorem *Let the function V: $[a, b] \times J \to L(\mathbb{R}^n)$ is Perron product integrable over $[a, b]$ and let the condition $\mathscr{C}$ be satisfied. If V_2: $[a, b] \times J \to L(\mathbb{R}^n)$ is equivalent to V, then the Perron product integral $\prod_a^b V_2(t, dt)$ exists and*

$$\prod_a^b V_2(t, dt) = \prod_a^b V(t, dt) .$$

Proof. By (1.54) from Theorem 1.16 and by the equivalence of V_2 and V we obtain that for every $\eta > 0$ there is a gauge δ on $[a, b]$ such that for every δ-fine partition Δ of $[a, b]$ we have

$$\sum_{j=1}^{k} \|V(t_j, [\alpha_{j-1}, \alpha_j]) - \prod_{\alpha_{j-1}}^{\alpha_j} V(t, dt)\| =$$
$$= \sum_{j=1}^{k} \|V(t_j, [\alpha_{j-1}, \alpha_j]) - \Phi(\alpha_j) (\Phi(\alpha_{j-1}))^{-1}\| < \eta$$

and

$$\sum_{j=1}^{k} \|V_2(t_j, [\alpha_{j-1}, \alpha_j]) - V(t_j, [\alpha_{j-1}, \alpha_j])\| < \eta ,$$

where $\Phi(s) = \prod_a^s V(t, dt)$, $s \in [a, b]$.

Therefore

$$\sum_{j=1}^{k} \|V_2(t_j, [\alpha_{j-1}, \alpha_j]) - \Phi(\alpha_j) (\Phi(\alpha_{j-1}))^{-1}\| < 2\eta$$

and Theorem 2.8 yields the existence of $\prod_a^b V_2(t, \mathrm{d}t)$ as well as the equality $\prod_a^b V_2(t, \mathrm{d}t) = \Phi(b)(\Phi(a))^{-1} = \prod_a^b V(t, \mathrm{d}t)$.

2.12. Theorem. *If $A: [a, b] \to L(\mathbb{R}^n)$ satisfies (2.26) then the functions V_1, V_2: $:[a, b] \times J \to L(\mathbb{R}^n)$ given by (2.19), (2.27) respectively are both Perron product integrable and*

$$\prod_a^s V_1(t, \mathrm{d}t) = \prod_a^s V_2(t, \mathrm{d}t)$$

for every $s \in [a, b]$.

Proof. The result follows immediately from the fact that V_1 is Perron product integrable over $[a, b]$ by Theorem 2.9, because if (2.26) is satisfied, then (2.18) holds too. V_1 and V_2 are equivalent by Lemma 2.10 and therefore by Theorem 2.11 also V_2 is Perron product integrable and both integrals have the same value.

2.13. Remark. Theorem 2.12 gives another representation of the fundamental matrix of the equation (2.12), i.e. we have also

$$\Psi(s) = \prod_a^s V_2(t, \mathrm{d}t), \quad s \in [a, b]$$

for the fundamental matrix Ψ of (2.12), when A satisfies (2.26) (c.f. Theorem 2.9).

Let us now consider the general case of $A: [a, b] \to L(\mathbb{R}^n)$, i.e. the case described in Theorem 2.5, which assures the existence of a unique fundamental matrix of the system (2.12)

Assume that $A: [a, b] \to L(\mathbb{R}^n)$ satisfies

$$A \in BV([a, b]; L(\mathbb{R}^n)), \tag{2.32}$$
$$[I + \Delta^+ A(t)]^{-1} \quad \text{exists for every} \quad t \in [a, b),$$
$$[I - \Delta^- A(t)]^{-1} \quad \text{exists for every} \quad t \in (a, b].$$

For $x, y, t \in [a, b]$, $x \leqq t \leqq y$ define

$$W(t, [x, y]) = [I + A(y) - A(t)][I + A(x) - A(t)]^{-1}. \tag{2.33}$$

If A satisfies (2.32) then we have $\|\Delta^- A(t)\| < \frac{1}{2}$ except a finite set of points $t_1, t_2, \ldots$ $\ldots, t_l \in (a, b]$. We have then for $t \neq t_1, \ldots, t_l$

$$[I - \Delta^- A(t)]^{-1} = \sum_{k=0}^{\infty} (\Delta^- A(t))^k$$

and also

$$\|[I - \Delta^- A(t)]^{-1}\| < \sum_{k=0}^{\infty} \|\Delta^- A(t)\|^k < 2.$$

Taking $\tilde{K} = \max\{2; \|[I - \Delta^- A(t_1)]^{-1}\|, \ldots, \|[I - \Delta^- A(t_l)]^{-1}\|\}$ we have

$$\|[I - \Delta^- A(t)]^{-1}\| \leqq \tilde{K} \quad \text{for every} \quad t \in (a, b]$$

and similarly it can be shown also that

$$\|[I + \Delta^+ A(t)]^{-1}\| < \tilde{K}^* \quad \text{for every} \quad t \in [a, b)$$

where $\tilde{K}^*$ is a constant.

Since the onesided limits of A exist in $[a, b]$ we can easily state that there is a constant $L > 0$ such that for every $t \in [a, b]$ there is a $\delta_1(t) > 0$ such that

$$[I + A(x) - A(t)]^{-1}, \; [I + A(y) - A(t)]^{-1} \quad \text{exist}$$

and

$$\|[I + A(x) - A(t)]^{-1}\| \leqq L, \quad \|[I + A(y) - A(t)^{-1}\| \leqq L \tag{2.34}$$

provided $x, y \in [a, b]$, $t - \delta_1(t) < x \leqq t \leqq y < t + \delta_1(t)$.

For $W: [a, b] \times J \to L(\mathbb{R}^n)$ the following hold:

$$W(t, [t, t]) = I, \quad t \in [a, b],$$

$$W(t, [x, t]) = [I + A(x) - A(t)]^{-1}, \quad W(t, [t, y]) = I + A(y) - A(t)$$

and consequently

$$W(t, [x, y]) = W(t, [t, y])\, W(t, [x, t]),$$

provided $x, y \in [a, b]$, $t - \delta_1(t) < x \leqq t \leqq y < t + \delta_1(t)$; finally we have

$$\lim_{y \to t+} W(t, [t, y]) = \lim_{y \to t+} I + A(y) - A(t) = I + \Delta^+ A(t), \quad t \in [a, b)$$

and

$$\lim_{x \to t-} W(t, [x, t]) = \lim_{x \to t-} [I + A(x) - A(t)]^{-1} = [I - \Delta^- A(t)]^{-1}, \quad t \in (a, b]$$

by Lemma 1.10. Hence we have verified that W given in (2.33) satisfies the condition $\mathscr{C}$. Moreover we have by (2.34)

$$\|W(t, [x, y]) - I\| = \|[I + A(y) - A(t)]\, [I + A(x) - A(t)]^{-1} - I\| =$$
$$= \|[I + A(y) - A(t) - (I + A(x) - A(t))]\, [I + A(x) - A(t)]^{-1}\| \leqq$$
$$\leqq \|A(y) - A(x)\|\, L \leqq L(\operatorname{var}_a^y A - \operatorname{var}_a^x A)$$

provided $x, y \in [a, b]$, $t - \delta_1(t) < x \leqq t \leqq y < t + \delta_1(t)$ and therefore we can see that W from (2.33) satisfies also the condition $\mathscr{C}^+$ with the nondecreasing function $g: [a, b] \to \mathbb{R}$ defined by $g(s) = L \operatorname{var}_a^s A$, $s \in [a, b]$.

Let now $\Psi: [a, b \to L(\mathbb{R}^n)$ be the fundamental matrix of (2.12), see Theorem 2.5. Since the Perron-Stieltjes integral $\int_a^b \mathrm{d}[A(r)]\, \Psi(r)$ exists, the Saks-Henstock lemma for sum integrals (see e.g. [4]) yields the following:

(2.35) For every $\varepsilon > 0$ there is a gauge δ_2 on $[a, b]$, $\delta_2(t) \leqq \delta_1(t)$, $t \in [a, b]$ such that if

$$a \leqq \beta_1 \leqq \xi_1 \leqq \gamma_1 \leqq \beta_2 \leqq \xi_2 \leqq \gamma_2 \leqq \dots \leqq \beta_m \leqq \xi_m \leqq \gamma_m \leqq b\,,$$

$$\xi_j - \delta_2(\xi_j) < \beta_j \leqq \xi_j \leqq \gamma_j < \xi_j + \delta_2(\xi_j)\,, \quad j = 1, \dots, m$$

then

$$\sum_{j=1}^{m} \|(A(\gamma_j) - A(\beta_j))\, \Psi(\xi_j) - \int_{\beta_j}^{\gamma_j} \mathrm{d}[A(r)]\, \Psi(r)\| < \varepsilon\,.$$

2.14. Lemma. *Assume that* $A: [a, b] \to L(\mathbb{R}^n)$ *satisfies* (2.32). *Let* $\Psi: [a, b] \to L(\mathbb{R}^n)$ *be the fundamental matrix of* (2.12) (*see II. in Theorem* 2.5).

Then for every $\vartheta > 0$ *there is a gauge* δ *on* $[a, b]$ *such that*

$$\sum_{j=1}^{k} \|W(t_j, [\alpha_{j-1}, \alpha_j]) - \Psi(\alpha_j)\,(\Psi(\alpha_{j-1}))^{-1}\| < \vartheta \tag{2.36}$$

for every δ-*fine partition* $\Delta = \{\alpha_0, t_1, \alpha_1, \dots, \alpha_{k-1}, t_k, \alpha_k\}$ *of* $[a, b]$.

Proof. Let $\varepsilon > 0$ be arbitrary and let δ be a gauge on $[a, b]$ such that $\delta(t) < \delta_2(t)$, $t \in [a, b]$, where δ_2 is given in (2.35). If Δ is a δ-fine partition of $[a, b]$, then $W(t_j, [\alpha_{j-1}, \alpha_j])$ is well defined (see (2.34)) for $j = 1, 2, \dots, k$ and we have by definition and by (2.34), (2.17)

$$\begin{aligned}
&\|W(t_j, [\alpha_{j-1}, \alpha_j]) - \Psi(\alpha_j)\,(\Psi(\alpha_{j-1}))^{-1}\| = \\
&= \|[I + A(\alpha_j) - A(t_j)]\,.\,[I + A(\alpha_{j-1}) - A(t\,)]^{-1} - \\
&- \Phi(\alpha_j)\,(\Phi(\alpha_{j-1}))^{-1}\| = \\
&= \|[I + A(\alpha_j) - A(t_j) - \Psi(\alpha_j)\,(\Psi(t_j))^{-1}]\,.\,[I + A(\alpha_{j-1}) - A(t_j)]^{-1} + \\
&+ \Psi(\alpha_j)\,(\Psi(t_j))^{-1}([I + A(\alpha_{j-1}) - A(t_j)]^{-1} - \Psi(t_j)\,(\Psi(\alpha_{j-1}))^{-1}) \leqq \\
&\leqq L\|I + A(\alpha_j) - A(t_j) - \Psi(\alpha_j)\,(\Psi(t_j))^{-1}\| + \\
&+ M\|[I + A(\alpha_{j-1}) - A(t_j)]^{-1}\|\,.\,\|[\Psi(\alpha_{j-1})\,(\Psi(t_j))^{-1} - \\
&- [I + A(\alpha_{j-1}) - A(t_j)]]\,\Psi(t_j)\,(\Psi(\alpha_{j-1}))^{-1}\| \leqq \\
&\leqq L\|[\Psi(t_j) + (A(\alpha_j) - A(t_j))\,\Psi(t_j) - \Psi(\alpha_j)]\,(\Psi(t_j))^{-1}\| + \\
&+ ML\|[\Psi(\alpha_{j-1}) - \Psi(t_j) - (A(\alpha_{j-1}) - A(t_j))\,\Psi(t_j)]\,(\Psi(t_j))^{-1}\,\Psi(t_j)\,. \\
&.\,(\Psi(\alpha_{j-1}))^{-1}\| \leqq LM\|(A(\alpha_j) - A(t_j))\,\Psi(t_j) - \int_{t_j}^{\alpha_j} \mathrm{d}[A(r)]\,\Psi(r)\| + \\
&+ LM^2\|(A(t_j) - A(\alpha_{j-1}))\,\Psi(t_j) - \int_{\alpha_{j-1}}^{t_j} \mathrm{d}[A(r)]\,\Psi(r)\|
\end{aligned}$$

for every $j = 1, 2, \dots, k$.

Using (2.35) and the fact, that Δ is a δ-fine partition, we obtain from the estimate given above the following

$$\sum_{j=1}^{k} \|W(t_j, [\alpha_{j-1}, \alpha_j]) - \Psi(\alpha_j)(\Psi(\alpha_{j-1}))^{-1}\| \leqq$$

$$\leqq LM \sum_{j=1}^{k} \|(A(\alpha_j) - A(t_j))\,\Psi(t_j) - \int_{t_j}^{\alpha_j} \mathrm{d}[A(r)]\,\Psi(r)\| +$$

$$+ LM^2 \sum_{j=1}^{k} \|(A(t_j) - A(\alpha_{j-1}))\,\Psi(t_j) -$$

$$- \int_{\alpha_{j-1}}^{t_j} \mathrm{d}[A(r)]\,\Psi(r)\| < \varepsilon LM(M + 1)\,.$$

Taking now $0 < \varepsilon < \vartheta/(LM(M + 1) + 1)$ for an arbitrary $\vartheta > 0$ we obtain (2.36) for δ-fine partitions Δ which correspond to this choice of $\varepsilon > 0$ by (2.35).

By the result given in Lemma 2.14 and by Theorem 2.8 we immediately obtain the following theorem.

2.15. Theorem. *Assume that* $A\colon [a, b] \to L(\mathbb{R}^n)$ *satisfies* (2.32). *Then the function* $W\colon [a, b] \times J \to L(\mathbb{R}^n)$ *given by* (2.33) *is Perron-product integrable over* $[a, b]$ *and for every* $s \in [a, b]$ *we have*

$$\Psi(s) = \prod_a^s W(t, \mathrm{d}t) \tag{2.37}$$

where $\Psi\colon [a, b \to L(\mathbb{R}^n)$ *is the uniquely determined fundamental matrix of* (2.12), *which satisfies the equation*

$$\Psi(s) = I + \int_a^s \mathrm{d}[A(r)]\,\Psi(r)\,, \quad s \in [a, b]\,.$$

Remark. Taking into account the results in Theorem 2.6 and in Theorem 2.15 we can see that there is a one-to-one correspondence between the „indefinite" Perron product integral $\prod_a^s V(t, \mathrm{d}t)$ of a function $V\colon [a, b] \times J \to L(\mathbb{R}^n)$ which fullfils the conditions $\mathscr{C}$ and $\mathscr{C}^+$ and the fundamental matrices of generalized linear differential equations (2.12) with $A\colon [a, b] \to L(\mathbb{R}^n)$ satisfying (2.32).

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Souhrn

PERRONŮV SOUČINOVÝ INTEGRÁL A ZOBECNĚNÉ LINEÁRNÍ DIFERENCIÁLNÍ ROVNICE

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Vyšetřuje se pojem Perronova součinového integrálu, který zavedli J. Jarník a J. Kurzweil. Je rozšířena třída perronovsky součinově integrovatelných funkcí definovaných pro body a intervaly a ukazuje se, že tato třída je vhodná pro reprezentaci fundamentální matice zobecněných lineárních diferenciálních rovnic.

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