Ondřej Došlý; Jan Osička

On oscillation criteria for self-adjoint linear differential equation of the fourth order

Časopis pro pěstování matematiky, Vol. 115 (1990), No. 4, 351--359

Persistent URL: `http://dml.cz/dmlcz/118413`

115 (1990) ČASOPIS PRO PĚSTOVÁNÍ MATEMATIKY No. 4, 351—359

ON OSCILLATION CRITERIA FOR SELF-ADJOINT LINEAR DIFFERENTIAL EQUATION OF THE FOURTH ORDER

ONDŘEJ DOŠLÝ, JAN OSIČKA, Brno

(Received October 20, 1988)

Summary. The paper presents sufficient conditions on the coefficients of the fourth order differential equation $(py'')'' + qy = 0$ for this equation to be oscillatory at a finite or infinite singular point. No sign restrictions on the function q are imposed.

Key words. Self-adjoint equation, conjugate points, principal solution.

AMS Classification. 34C10.

1. INTRODUCTION

We consider the self-adjoint linear differential equation of the fourth order

$$(1) \qquad (p(x)\,y'')'' + q(x)\,y = 0\,,$$

where $p(x) \in C^2(I)$, $q(x) \in C(I)$, $p(x) > 0$, $x \in I = (a, b)$, $-\infty \leqq a < b \leqq \infty$. There exists extensive literature dealing with the oscillation properties of the fourth order equations. Recall the books [3, 4, 6, 9, 10], the papers [2, 5, 7] and the references given there.

Recently, Müller-Pfeiffer [8] has proved that equation (1) is oscillatory at ∞ if either

i) $\int_c^\infty p^{-1}(x)\,\mathrm{d}x = \infty$ and there exists a number $x_0 \in R$ such that

$$\int_c^\infty q(x)\,(x - x_0)^2\,\mathrm{d}x = -\infty, \; c \in I,$$

or

ii) $\int_c^\infty x^2 p^{-1}(x)\,\mathrm{d}x = \infty$ and $\int_c^\infty q(x)\,\mathrm{d}x = -\infty$.

Note that these criteria require no sign restrictions on the function $q(x)$. Their proof is based on the application of the Courant variation principle to the quadratic functional

$$(2) \qquad I(y; a, b) = \int_a^b (p(x)\,y''^2(x) + q(x)\,y^2(x))\,\mathrm{d}x\,,$$

which corresponds to (1).

The principal concern of this paper is to generalize these criteria using the concept of the principal solution of the linear Hamiltonian system corresponding to (1).

The method used offers a unified approach to the study of oscillation properties of (1) near a finite or infinite singularity.

Two points $x_1, x_2 \in I$ are said to be conjugate relative to (1) if there exists a nontrivial solution $y(x)$ of this equation for which $y^{(i)}(x_1) = 0 = y^{(i)}(x_2)$, $i = 0, 1$. Equation (1) is said to be oscillatory at b if for every $c \in I$ there exist $x_1, x_2 \in (c, b)$ which are conjugate relative to (1). In the opposite case (1) is said to be nonoscillatory at b. Oscillation and nonoscillation of (1) at a is defined in a similar manner.

Self-adjoint equations of even orders are closely related to linear Hamiltonian systems (LHS). If y is a solution of (1) then the 2-dimensional vectors $u = (y, y')$, $v = (-(py'')', py'')$ are solutions of the LHS

$$u' = Au + B(x)\, v\,, \quad v' = C(x)\, u - A^{\mathrm{T}} v\,, \tag{3}$$

where

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & p^{-1} \end{pmatrix}, \quad C = \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}. \tag{4}$$

We shall say that the solution (u, v) of (3) is generated by the solution y of (1). Simultaneously with (3) we shall consider the matrix system

$$U' = AU + B(x)\, V, \quad V' = C(x)\, U - A^{\mathrm{T}} V, \tag{5}$$

where U, V are 2×2 matrices. A self-conjugate solution (U_b, V_b) of (5) (i.e. $U_b^{\mathrm{T}}(x)\, V_b(x) \equiv V_b^{\mathrm{T}}(x)\, U_b(x)$) is said to be principal at b if the matrix $U_b(x)$ is nonsingular in a left neighbourhood of b and $\lim\limits_{x \to b^-} [\int^x U_b^{-1}(s)\, B(s)\, U_b^{\mathrm{T}-1}(s)\, \mathrm{d}s]^{-1} = 0$. Two solutions y_1, y_2 of (1) are said to be principal at b if the solutions (u_1, v_1), (u_2, v_2) of (3) generated by y_1, y_2 form the columns of the principal solution (U_b, V_b) of (5). Note that the principal solution at b of (5) exists whenever equation (1) is nonoscillatory at b.

2. STATEMENT OF THE RESULTS

Theorem. *Let y_1, y_2 be the principal solutions at b of the equation*

$$(p(x)\, y'')'' = 0\,. \tag{6}$$

If there exist $c = (c_1, c_2) \in R^2$ such that

$$\lim_{t \to b^-} \int_d^t q(x)\, (c_1\, y_1(x) + c_2\, y_2(x))^2\, \mathrm{d}x = -\infty\,, \quad d \in I\,,$$

then equation (1) is oscillatory at b.

To compare the statement of this theorem with the criteria of Müller-Pfeiffer, consider the case $b = \infty$. If $\int_1^\infty p^{-1}(x)\, \mathrm{d}x = \infty$ then $y_1 = 1$, $y_2 = x$ form the principal solutions at ∞ of (6). Indeed, if $U = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$, then

$$\left[\int_d^x U^{-1}(s)\,B(s)\,U^{\mathrm{T}-1}(s)\,\mathrm{d}s\right]^{-1} = \left[\int_d^x \begin{pmatrix} s^2 & -s \\ -s & 1 \end{pmatrix} p^{-1}(s)\,\mathrm{d}s\right]^{-1} =$$

$$= \left(\int_d^x s^2\,p^{-1}(s)\,\mathrm{d}s \int_d^x p^{-1}(s)\,\mathrm{d}s\, -\right.$$

$$\left. - \left(\int_d^x sp^{-1}(s)\,\mathrm{d}s\right)^2\right)^{-1} \left(\int_d^x p^{-1}(s) \begin{pmatrix} 1 & s \\ s & s^2 \end{pmatrix}\right)\mathrm{d}s\,.$$

Denote

$$D(x) = \det\left(\int_d^x U^{-1}(s)\,B(s)\,U^{\mathrm{T}-1}(s)\,\mathrm{d}s\right) =$$

$$= \int_d^x s^2\,p^{-1}(s)\,\mathrm{d}s \int_d^x p^{-1}(s)\,\mathrm{d}s - \left(\int_d^x sp^{-1}(s)\,\mathrm{d}s\right)^2\,;$$

$$D'(x) = p^{-1}(x) \int_d^x (x - s)^2\,p^{-1}(s)\,\mathrm{d}s\,,$$

hence

$$\lim_{x\to\infty} D(x) = \lim_{x\to\infty} \int_d^x p^{-1}(t) \int_d^t (t - s)^2\,p^{-1}(s)\,\mathrm{d}s\,\mathrm{d}t = \infty\,.$$

It follows that

$$\lim_{x\to\infty} D^{-1}(x) \int_d^x s^2\,p^{-1}(s)\,\mathrm{d}s = \lim_{x\to\infty} \left(D'(x)\right)^{-1} x^2\,p^{-1}(x) =$$

$$= \lim_{x\to\infty} \left(\int_d^x (x - s)^2\,p^{-1}(s)\,\mathrm{d}s\right)^{-1} x^2 = 0\,,$$

similarly

$$\lim_{x\to\infty} D^{-1}(x) \int_d^x s\,p^{-1}(s)\,\mathrm{d}s = 0\,, \quad \lim_{x\to\infty} D^{-1}(x) \int_d^x p^{-1}(s)\,\mathrm{d}s = 0\,, \quad \text{i.e.}$$

$$\lim_{x\to\infty} \left[\int_d^x U^{-1}(s)\,B(s)\,U^{\mathrm{T}-1}(s)\,\mathrm{d}s\right]^{-1} = 0\,.$$

By the same method one can verify that $y_1 = 1$, $y_2 = x \int_d^\infty p^{-1}(s)\,\mathrm{d}s - \int_d^x (x - s)\,p^{-1}(s)\,\mathrm{d}s$ if $\int_d^\infty p^{-1}(s)\,\mathrm{d}s < \infty$, $\int_d^\infty s^2\,p^{-1}(s)\,\mathrm{d}s = \infty$ and $y_1 = \int_x^\infty (s - x)\,p^{-1}(s)\,\mathrm{d}s$, $y_2 = \int_x^\infty (s - x)\,s\,p^{-1}(s)\,\mathrm{d}s$ if $\int_d^\infty s^2\,p^{-1}(s)\,\mathrm{d}s < \infty$ form the principal solutions at ∞ of (6). Thus, we see that the criteria i) and ii) take into consideration only polynomial solutions of (6).

Now let us investigate the oscillation properties of (1) near a finite singularity, e.g. near the point $a = 0$. Then similarly as in the case of an infinite singular point one can verify that the principal solutions at $a = 0$ of (6) are of the form

$$(y_1, y_2) = \begin{cases} (1, x) & \text{if } \int_0^\delta x^2\,p^{-1}(x)\,\mathrm{d}x = \infty\,, \quad \delta > 0\,; \\ \left(x, \int_0^\delta p^{-1}(x)\,x^2\,\mathrm{d}x - \int_x^\delta (t - x)\,p^{-1}(t)\,\mathrm{d}t\right) & \text{if } \int_0^\delta x^2\,p^{-1}(x)\,\mathrm{d}x < \infty \\ & \text{and } \int_0^\delta p^{-1}(x)\,\mathrm{d}s = \infty\,, \\ \left(\int_0^x (x - t)\,p^{-1}(t)\,\mathrm{d}t\,, \int_0^x (t - x)\,t\,p^{-1}(t)\,\mathrm{d}t\right) & \text{if } \int_0^\delta p^{-1}(x)\,\mathrm{d}x < \infty\,. \end{cases}$$

Using Theorem we can now formulate the corresponding criteria for (1) to be oscillatory at 0.

3. AUXILIARY RESULTS

In this section we give several auxiliary statements which we shall use in the proof of Theorem. The main idea of this proof is described in the following statement.

Lemma 1. *If there exists a function* $y \in W_2^2(I)$, $\operatorname{supp} y \subset I_1 = [c, d] \subset I$ *such that* $I(y; c, d) < 0$ *then there exist at least two points* $x_1, x_2 \in I_1$ *which are conjugate relative to* (1).

Proof. [8]

Note that the Sobolev space $W_2^2(I)$ consists of all real-valued functions whose generalized derivatives up to order 2 belong to $L_2(I)$.

We shall also need the following statements.

Lemma 2. *Let equation* (1) *be disconjugate on an interval* $I_0 \subset I$, $x_1, x_2 \in I_0$, $d_1, d_2 \in R$. *Then there exists a unique solution* $y(x)$ *of* (1) *for which* $y(x_1) = d_1$, $y(x_2) = d_2$.

Proof. [3, Chapter I]

Lemma 3. *Let* (U, V) *be a self-conjugate solution of* (5) *such that the matrix* U *is nonsingular for* $x \in I_0 \subset I$. *Then*

$$(U_1(x), V_1(x)) = (U(x) \int_c^x U^{-1}(s)\, B(s)\, U^{\mathsf{T}-1}(s)\, \mathrm{d}s,\ V(x)\,.$$
$$.\int_c^x U^{-1}(s)\, B(s)\, U^{\mathsf{T}-1}(s)\, \mathrm{d}s + U^{\mathsf{T}-1}(x))\,, \quad c \in I_0\,,$$

is also a solution of (5) *for* $x \in I_0$.

Proof. [3, Chapter II]

Lemma 4. *Let equation* (1) *be disconjugate on* I. *If* (U_b, V_b) *is the principal solution at* b *of the corresponding LHS, then the matrix* $U_b(x)$ *is nonsingular on* I.

Proof. [11]

Lemma 5. *Consider the differential equations*

$$(7) \qquad (p_1(x)\, y')' + p_0(x)\, y = 0\,,$$

$$(8) \qquad (p_1(x)\, u')' + p_0(x)\, u = f(x)\,,$$

where $p_1(x) > 0$, $f(x) > 0$, $x \in I_0 = [t_1, t_2]$, $f(x), p_0(x) \in C(I_0)$, $p_1(x) \in C^1(I_0)$. *If there exists a solution* $u(x)$ *of* (8) *for which* $u(t_1) = 0 = u(t_2)$ *and* $u(x)$ *has a zero point on the interval* (t_1, t_2) *then there exists a solution* $y(x)$ *of* (7) *having at least two zeros on* I_0.

Proof. Let $y(x)$ be any solution of (7) and let $u(x)$ be the solution of (8) from Lemma. Multiplication of (8) by $y(x)$ and integration by parts from x_1 to x_2, $x_1, x_2 \in \in [t_1, t_2]$, gives

$$\int_{x_1}^{x_2} f(x)\, y(x)\, \mathrm{d}x = [p_1(x)(u'(x)\, y(x) - u(x)\, y'(x))]_{x_1}^{x_2} .$$

Let $y_0(x)$ be the solution of (7) satisfying the initial conditions $y_0(t_1) = 0$, $y_0'(t_1) = 1$. Then $y_0(x) > 0$ in some right deleted neighbourhood of t_1. Let $c \in (t_1, t_2)$ be such that $u(c) = 0$. First, let us consider the case $u'(c) = 0$. Then $\int_{t_1}^{c} f(x)\, y_0(x)\, \mathrm{d}x = 0$, hence $y_0(x)$ must change its sign on (t_1, c) (since $f(x) > 0$ on I_0). If $u'(c) \neq 0$, suppose that $y_0(x) > 0$ on $(t_1, t_2]$. In the case $u'(c) < 0$ we have $\int_{t_1}^{c} f(x)\, y_0(x)\, \mathrm{d}x = p_1(c) \, . \, u'(c)\, y_0(c) < 0$, a contradiction. If $u'(c) > 0$, there exists $t_3 \in (c, t_2]$ such that $u(t_3) = 0$ and $u'(t_3) \leqq 0$. Then $\int_{c}^{t_3} f(x)\, y_0(x)\, \mathrm{d}x = p(t_3)\, u'(t_3)\, y_0(t_3) - - p(c)\, u'(c)\, y_0(c) \leqq 0$ and we have again contradiction.

4. PROOF OF THEOREM

Let y_1, y_2 be principal solutions at b of (6) and let (U_b, V_b) be the principal solution at b of the corresponding LHS generated by y_1 and y_2. Denote $h(x) = c_1\, y_1(x) + + c_2\, y_2(x)$. The transformation $y = h(x)\, u$ transforms equation (6) into the equation

$$(P(x)\, u'')'' + (Q(x)\, u')' = 0 \,, \tag{9}$$

where $P = ph^2$, $Q = 2[h(ph')' + hph'' - ph'^2]$, see e.g. [1]. Since any solution of (6) has at most 3 zeros on I, there exists $d \in I$ such that $h(x) \neq 0$ on $I_1 = (d, b)$. We shall show that the second order equation

$$(P(x)\, w')' + Q(x)\, w = 0 \tag{10}$$

is disconjugate on I_1. To do it, it suffices to find a solution $w_0(x)$ of this equation which does not vanish on I_1. Let $w_0 = (y_1/h)'$ if $c_2 \neq 0$ and $w_0 = (y_2/h)'$ if $c_2 = 0$ (then, of course, $c_1 \neq 0$). Then

$$w_0 = h^{-2}(y_1'h - y_1h') = c_2h^{-2}(y_1'y_2 - y_1y_2') = c_2h^{-2} \det U_b(x) \neq 0$$

in the case $c_2 \neq 0$ and

$$w_0 = -c_1h^{-2} \det U_b(x) \neq 0 \quad \text{if} \quad c_2 = 0 \,.$$

We shall show that w_0 is a solution of (10). Let $c_2 \neq 0$, then

$$w_0' = (y_1/h)'' = h^{-1}y_1'' - 2h^{-2}h'y_1' - h^{-2}y_1h'' + 2h^{-3}y_1h'^2$$

and

$$\begin{aligned}(Pw_0')' + Qw_0 &= (phy_1'')' - 2(py_1'h')' - (ph''y_1)' + 2(ph^{-1}h'^2y_1)' + \\ &+ 2(h^{-1}y_1' - h^{-2}y_1h')(h(ph')' + hph'' - ph'^2) = \\ &= h'py_1'' + h(py_1'')' - 2y_1'(ph')' - 2h'py_1'' - y_1(ph'')' -\end{aligned}$$

$$- y_1'ph'' - 2ph^{-2}h'^3y_1 + 2h^{-1}(ph')'\,h'y_1 + 2ph^{-1}h'h''y_1 +$$
$$+ 2ph^{-1}h'^2y_1' + 2y_1'(ph')' + 2ph''y_1' - 2ph^{-1}y_1'h'^2 -$$
$$- 2h^{-1}h'(ph')'\,y_1 - 2ph^{-1}h'h''y_1 + 2ph^{-2}h'^3y_1 =$$
$$= h(py_1'')' - h'py_1'' - y_1(ph'')' + y_1'ph'' = c_2(y_2(py_1'')' - y_2'py_1'' -$$
$$- y_1(py_2'')' + y_1'py_2'') = c_2e_2^{\mathrm{T}}(U_b^{\mathrm{T}}V_b - V_b^{\mathrm{T}}U_b)\,e_1 = 0\,, \quad e_1 = (1, 0)^{\mathrm{T}}\,,$$

$e_2 = (0, 1)^{\mathrm{T}}$, since the solution (U_b, V_b) is self-conjugate. Similarly $(Pw_0')' + Qw_0 = 0$ if $c_2 = 0$. Consequently, equation (10) is disconjugate on I_1.

Let $t_1, t_2 \in I_1$, $t_1 < t_2$. As equation (6) is disconjugate on I_1, there exists a unique solution $\tilde{y}(x)$ of this equation for which

$$\tilde{y}^{(i)}(t_1) = h^{(i)}(t_1)\,, \quad \tilde{y}^{(i)}(t_2) = 0\,, \quad i = 0, 1\,. \tag{11}$$

Let $\tilde{w} = (\tilde{y}/h)'$. Then $\tilde{w}(t_1) = 0 = \tilde{w}(t_2)$ and $\tilde{w}$ is a solution of the equation

$$(Pw')' + Qw = k\,, \tag{12}$$

k being a real constant. Indeed, the function $\tilde{y}/h$ is a solution of (9), hence $\tilde{w}$ is a solution of (12). If $k = 0$, then $\tilde{w}$ is a solution of (10) with two zeros on I_1, which contradicts disconjugacy of this equation, hence $k \neq 0$. Suppose that there exists $t_3 \in (t_1, t_2)$ such that $\tilde{w}(t_3) = 0$. Then by Lemma 5 there exists a solution of (10) having two zeros on I_1, a contradiction, hence the function $\tilde{w} = (\tilde{y}/h)'$ does not vanish on (t_1, t_2).

By Lemma 3 the pair of 2-dimensional vectors

$$(u, v) = (U_b(x) \textstyle\int_x^{t_2} U_b^{-1}(s)\,B(s)\,U_b^{\mathrm{T}-1}(s)\,\mathrm{d}s(\int_{t_1}^{t_2} U_b^{-1}(s)\,B(s)\,U_b^{\mathrm{T}-1}(s)\,\mathrm{d}s)\,c\,,$$
$$(V_b(x) \textstyle\int_x^{t_2} U_b^{-1}(s)\,B(s)\,U_b^{\mathrm{T}-1}(s)\,\mathrm{d}s - U_b^{\mathrm{T}-1}(x))\,.$$
$$.\,(\textstyle\int_{t_1}^{t_2} U_b^{-1}(s)\,B(s)\,U_b^{\mathrm{T}-1}(s)\,\mathrm{d}s)\,c)$$

is a solution of the LHS corresponding to (6), hence the function $z = e_1^{\mathrm{T}}u$ is a solution of (6). One can easily verify that $z(x)$ satisfies the boundary condition (11) (with z instead of $\tilde{y}$). As the boundary value problem (6), (11) has unique solution, $z(x) = \tilde{y}(x)$. Further, we have

$$\textstyle\int_{t_1}^{t_2} p(\tilde{y}'')^2\,\mathrm{d}x = \int_{t_1}^{t_2} v^{\mathrm{T}}Bv\,\mathrm{d}x = \int_{t_1}^{t_2}(u' - Au)^{\mathrm{T}}\,v\,\mathrm{d}x = [u^{\mathrm{T}}v]_{t_1}^{t_2} -$$
$$- \textstyle\int_{t_1}^{t_2}(u^{\mathrm{T}}A^{\mathrm{T}}v + u^{\mathrm{T}}v')\,\mathrm{d}x = [u^{\mathrm{T}}v]_{t_1}^{t_2} - \int_{t_1}^{t_2} u^{\mathrm{T}}(A^{\mathrm{T}}v - A^{\mathrm{T}}v)\,\mathrm{d}x = [u^{\mathrm{T}}v]_{t_1}^{t_2} =$$
$$= -u^{\mathrm{T}}(t_1)\,v(t_1)\,.$$

Denote $\tilde{B}(x) = U_b^{-1}(x)\,B(x)\,U_b^{\mathrm{T}-1}(x)$. Then

$$u^{\mathrm{T}}(t_1)\,v(t_1) = -c^{\mathrm{T}}(\textstyle\int_{t_1}^{t_2} \tilde{B}\,\mathrm{d}x)^{-1} \int_{t_1}^{t_2} \tilde{B}\,\mathrm{d}x\, U_b^{\mathrm{T}}(t_1)\,(V_b(t_1) \int_{t_1}^{t_2} \tilde{B}\,\mathrm{d}x -$$
$$- U_b^{\mathrm{T}-1}(t_1))\,(\textstyle\int_{t_1}^{t_2} \tilde{B}\,\mathrm{d}x)^{-1}\,c = -c^{\mathrm{T}}U_b(t_1)\,V_b(t_1)\,c + c^{\mathrm{T}}(\int_{t_1}^{t_2} \tilde{B}\,\mathrm{d}x)^{-1}\,c\,.$$

Now, let $x_0 \in I$ be arbitrary, let $\delta > 0$ be such that $x_0 + \delta \in I$ and let $\varphi(x)$ be any function of the class C^2 for which $\varphi^{(i)}(x_0) = 0$, $\varphi^{(i)}(x_1) = h^{(i)}(x_1)$, $i = 0, 1$, $x_1 = x_0 + \delta$. Denote $K = I(\varphi; x_0, x_1) + [h(ph'')' - h'ph'']_{x=x_1}$. As $\lim_{t\to b-} \int_{x_1}^{t} q(x)\, h^2(x)\, dx = -\infty$, there exists $x_2 \in I$ such that $\int_{t_1}^{\tau} q(x)\, h^2(x)\, dx < -2|K|$ whenever $\tau \in (x_2, b)$. Moreover, since every solution of (6) has at most 3 zeros on I, x_2 can be chosen in such a way that $h(x) \neq 0$ on $[x_2, b)$. Further, according to the definition of the principal solution at b of (5) there exists $x_3 \in (x_2, b)$ such that $c^{\mathrm{T}}[\int_{x_2}^{\tau} U_b^{-1}(s)\, B(s)\, U_b^{\mathrm{T}-1}(s)\, ds]^{-1}\, c < |K|$ whenever $\tau \in (x_3, b)$.

Define

$$y(x) = \begin{cases} 0\,, & \text{for} \quad x \in (a, x_0)\,, \\ \varphi(x)\,, & \text{for} \quad x \in [x_0, x_1)\,, \\ h(x)\,, & \text{for} \quad x \in [x_1, x_2)\,, \\ \tilde{y}(x)\,, & \text{for} \quad x \in [x_2, x_3)\,, \\ 0\,, & \text{for} \quad x \in [x_3, b)\,, \end{cases}$$

where $\tilde{y}(x)$ is the solution of (6) satisfying (11) with $t_1 = x_2$, $t_2 = x_3$. We have

$$\begin{aligned} I(y; a, b) &= I(y; x_0, x_3) = \int_{x_0}^{x_1} (p(\varphi'')^2 + q\varphi^2)\, dx + \\ &+ \int_{x_1}^{x_2} (p(h'')^2 + qh^2)\, dx + \int_{x_2}^{x_3} (p(\tilde{y}'')^2 + q\tilde{y}^2)\, dx = \\ &= I(\varphi; x_0, x_1) + [h'ph'' - h(ph'')']_{x_1}^{x_2} + \\ &+ \int_{x_1}^{x_2} qh^2\, dx + c^{\mathrm{T}}(\int_{x_2}^{x_3} U_b^{-1}(s)\, B(s)\, U_b^{\mathrm{T}-1}(s)\, ds)^{-1}\, c - c^{\mathrm{T}} U_b(x_2)\, V_b(x_2)\, c = \\ &= K + c^{\mathrm{T}}(\int_{x_2}^{x_3} U_b^{-1}(s)\, B(s)\, U_b^{\mathrm{T}-1}(s)\, ds)^{-1}\, c + \int_{x_1}^{x_2} qh^2\, dx + \int_{x_2}^{x_3} q\tilde{y}^2\, dx\,. \end{aligned}$$

As the function $(\tilde{y}/h)$ is monotonic on (x_2, x_3) (since $(\tilde{y}/h)' \neq 0$ on (x_2, x_3)), by the second mean value theorem of integral calculus we have $\int_{x_2}^{x_3} q\tilde{y}^2\, dx = \int_{x_2}^{x_3} qh^2(\tilde{y}/h)^2\, dx = \int_{x_2}^{\xi} qh^2\, dx$, where $\xi \in (x_2, x_3)$. Consequently, $I(y; x_0, x_3) = K + c^{\mathrm{T}}(\int_{x_2}^{x_3} U_b^{-1}(s)\, B(s)\, U_b^{\mathrm{T}-1}(s)\, ds)^{-1}\, c + \int_{x_1}^{\xi} qh^2\, dx < K + |K| - 2|K| < 0$. The proof is complete.

Remark. A natural question to ask is, how to modify the above given proof to be applicable to the equation

$$(13) \qquad (-1)^n\, (p(x)\, y^{(n)})^{(n)} + q(x)\, y = 0\,.$$

The main difficulty consists in proving the monotonicity of the function (y/h) on (x_2, x_3) in order to justify the use of the second mean value theorem of integral calculus. In our Theorem we have proved this monotonicity via disconjugacy of equation (10) which implies that every solution of (12) has at most two zeros on I. Following this idea in the higher dimensional case, it is not difficult to prove that the $(2n - 2)$ order equation arising from (13) after the transformation $y = h(x)\, u$ is disconjugate on I_1 (i.e. there exists no nontrivial solution of this equation having two different different zeros of multiplicity $(n - 1)$ on I_1, but this disconjugacy is not sufficient for monotonicity of the function (y/h). To prove this monotonicity we need disconjugacy of a linear differential equation introduced by Leighton and Nehari. By their definition a linear differential equation of the n-th order is disconjugate on an interval I if no nontrivial solution of this equation has more than $(n - 1)$ zeros on I, every zero counted according to its multiplicity. However, to prove this disconjugacy is, in general, more difficult then to prove disconjugacy defined by means of conjugate points. Our method works for fourth order equations since for the second order equations the disconjugacy in the sense of Nehari coincides with the disconjugacy defined by means of conjugate points.

References

[1] *C. D. Ahlbrandt, D. B. Hinton, R. T. Lewis:* The effect of variable change on oscillation and disconjugacy criteria with applications to spectral theory and asymptotic theory. J. Math. Anal. Appl. *81* (1981), 234–277.

[2] *J. H. Barrett:* Oscillation theory of ordinary linear differential equations. Adv. in Math. *3* (1969), 415–509.

[3] *W. A. Coppel:* Disconjugacy. Lecture Notes in Math. No 220, Springer-Verlag, Berlin–Heidelberg 1971.

[4] *I. M. Glazman:* Direct Methods of Qualitative Spectral Analysis of Singular Differential Operators. Davey, Jerusalem 1965.

[5] *D. B. Hinton, R. T. Lewis:* Oscillation theory at a finite singularity. J. Diff. Equations *30* (1978), 235–247.

[6] *K. Kreith:* Oscillation Theory. Lecture Notes in Math. No 324, Springer-Verlag, Nex Yrok 1973.

[7] *R. T. Lewis:* Oscillation and nonoscillation criteria for some self-adjoint even order differential operators. Pacific J. Math. *51* (1974), 221–234.

[8] *E. Müller-Pfeiffer:* Existence of conjugate points for second and fourth order differential equations. Proc. Roy. Soc. Edinburgh *89A* (1981), 281–291.

[9] *W. T. Reid:* Sturmian Theory for Ordinary Differential Equations. Springer-Verlag, New York–Berlin–Heidelberg 1980.

[10] *C. A. Swanson:* Comparison and Oscillation Theory of Linear Differential Equations. Acad. Press, New York 1968.

[11] *E. C. Tomastik:* Singular quadratic functional of n dependent variables. Trans. Amer. Math. Soc. *124* (1966), 60–76.

Souhrn

OSCILAČNÍ KRITERIA PRO SAMOADJUNGOVANOU LINEÁRNÍ DIFERENCIÁLNÍ ROVNICI ČTVRTÉHO ŘÁDU

ONDŘEJ DOŠLÝ, JAN OSIČKA

V práci jsou odvozeny dostatečné podmínky pro koeficienty rovnice IV. řádu $(py'')'' + qy = 0$ zajišťující oscilatoričnost této rovnice v blízkosti konečného nebo nekonečného singulárního bodu. Tyto podmínky neobsahují žádná omezení týkající se znaménka funkce q.

Резюме

КРИТЕРИИ КОЛЕБАТЕЛЬНОСТИ ДЛЯ САМОСОПРЯЖЕННОГО ЛИНЕЙНОГО ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ 4-ОГО ПОРЯДКА

ONDŘEJ DOŠLÝ, JAN OSIČKA

В работе установлены достаточные условия для коеффициентов уравнения 4-ого порядка $(py'')'' + qy = 0$, которые гарантируют колебательность его решений в окресности конечной или бесконечной сингулярной точки. Эти условия не содержат никакий ограничений для знака функции q.

Authors' address: Katedra matematické analýzy Masarykovy university, Janáčkovo nám. 2a, 662 95 Brno.