Časopis pro pěstování matematiky

Pavel Pech

Inequality between sides and diagonals of a space n-gon and its integral analog

Časopis pro pěstování matematiky, Vol. 115 (1990), No. 4, 343--350

Persistent URL: `http://dml.cz/dmlcz/118412`

INEQUALITY BETWEEN SIDES AND DIAGONALS OF A SPACE n-GON AND ITS INTEGRAL ANALOG

PAVEL PECH, České Budějovice

(Received October 18, 1988)

Summary. The inequality between sides and diagonals of a closed space n-gon in E^N is established. This inequality is generalized to the integral inequality. These inequalities imply the well-known discrete and integral Wirtinger's inequalities.

Keywords: Fourier polynomials, Wirtinger's inequality.

AMS Classification: 52A40.

1. INTRODUCTION

In 1980 L. Gerber [1] proved the following statement:

Let $\mathscr{A} = A_0, A_1, \ldots, A_{n-1}$ be a closed space n-gon in a Euclidean space E^N of arbitrary dimension N. Then

$$\sum_{\nu=0}^{n-1} |A_\nu A_{\nu+2}|^2 \leqq 4\cos^2\frac{\pi}{n}\sum_{\nu=0}^{n-1}|A_\nu A_{\nu+1}|^2 .$$

The equality holds if and only if $\mathscr{A}$ is a plane affine-regular n-gon, i.e. the affine image of a regular n-gon.

In the present paper we will show that this statement is a special case of Theorem 1. Theorem 1 gives an inequality between arbitrary diagonals of a space n-gon and its sides. This inequality is generalized to the continuous case in Theorem 2. The main result is the inequality (5) and its discrete analog (1). In the latter part we deal with the well-known Wirtinger's inequality [2]. Theorems 3 and 4 state that the discrete and integral Wirtinger's inequalities follow from the inequalities (1) and (5).

The proofs are based on using the Fourier series expansion of periodic functions and its discrete analog, the finite Fourier series [3].

2. FUNDAMENTAL INEQUALITY — THE DISCRETE CASE

Theorem 1. *Let $\mathscr{A} = A_0, A_1, \ldots, A_{n-1}$ be a closed space n-gon in E^N, let $A_{n+k} = A_k$ for all $k = 0, 1, 2, \ldots$. Then for all $p = 0, 1, \ldots, n-1$,*

$$(1)\qquad \sum_{\nu=0}^{n-1} |A_\nu A_{\nu+p}|^2 \leqq \left(\frac{\sin p \dfrac{\pi}{n}}{\sin \dfrac{\pi}{n}}\right)^2 . \sum_{\nu=0}^{n-1} |A_\nu A_{\nu+1}|^2 .$$

For $p = 2, 3, \ldots, n - 2$, equality is attained if and only if $\mathscr{A}$ is a plane affine-regular n-gon or, for $N = 1$, its 1-dimensional projection. (For $p = 0, 1, n - 1$ equality occurs always).

The special case $N = 1$ is then

Theorem 1*. *Let $x_0, x_1, \ldots, x_{n-1}$ be n real numbers and let $x_{n+k} = x_k$ for $k = 0, 1, 2, \ldots$. Then for all $p = 0, 1, \ldots, n - 1$,*

$$(2)\qquad \sum_{\nu=0}^{n-1} (x_{\nu+p} - x_\nu)^2 \leqq \left(\frac{\sin p \dfrac{\pi}{n}}{\sin \dfrac{\pi}{n}}\right)^2 . \sum_{\nu=0}^{n-1} (x_{\nu+1} - x_\nu)^2 .$$

For $p = 2, 3, \ldots, n - 2$, equality is attained if and only if

$$x_\nu = A \cos \nu \frac{2\pi}{n} + B \sin \nu \frac{2\pi}{n} + C , \quad \nu = 0, 1, \ldots, n - 1$$

where A, B, C, are real constants.

First we shall prove that Theorem 1 and Theorem 1* are equivalent.

We assume Theorem 1* holds. Let $A_\nu = (x_{\nu 0}, x_{\nu 1}, \ldots, x_{\nu, N-1})$ be the cartesian coordinates of vertices of an n-gon $\mathscr{A}$. For every n-tuple $(x_{0j}, x_{1j}, \ldots, x_{n-1,j})$ the inequality (2) holds obviously. By summing these inequalities we get (1) including the case when equality holds. Conversely, let Theorem 1 hold. Putting $A_\nu = (x_\nu, 0, \ldots, 0)$ for $\nu = 0, 1, \ldots, n - 1$ in (1) we get (2).

Proof of Theorem 1. According to the previous statement it suffices to prove the theorem for $N = 2$.

Let $\mathscr{A} = (z_0, z_1, \ldots, z_{n-1})$ be a plane closed n-gon, where z_ν are complex numbers, $\nu = 0, 1, \ldots, n - 1$. We denote

$$\Pi_k = (\omega_k^0, \omega_k^1, \ldots, \omega_k^{n-1}) \quad \text{where} \quad \omega_k = e^{k.i2\pi/n} , \quad k = 0, 1, \ldots, n - 1 .$$

The system $\Pi_0, \Pi_1, \ldots, \Pi_{n-1}$ forms an orthogonal basis for the unitary vector space C^n of all n-tuples of complex numbers since

$$\Pi_r . \Pi_s = \sum_{j=0}^{n-1} \omega_r^j . \omega_s^{-j} = \begin{cases} n & \text{if} \quad r = s , \\ 0 & \text{if} \quad r \neq s . \end{cases}$$

Hence there exist numbers $\vartheta_0, \vartheta_1, \ldots, \vartheta_{n-1}$ such that $z_\nu = \sum_k \vartheta_k \omega_k^\nu$, $\nu = 0, 1, \ldots, n - 1$ or symbolically, $\mathscr{A} = \sum_k \vartheta_k \Pi_k$. A discrete analog of Parseval's relation of completeness gives an important equality

$$\sum_{\nu=0}^{n-1}|z_\nu|^2 = n\sum_{k=0}^{n-1}|\vartheta_k|^2 .$$

For all $p = 1, 2, \dots, n-1$ we obtain

$$(3)\qquad \sum_\nu |z_{\nu+p} - z_\nu|^2 - \left(\frac{\sin p\frac{\pi}{n}}{\sin\frac{\pi}{n}}\right)^2 \cdot \sum_\nu |z_{\nu+1} - z_\nu|^2 =$$

$$= n\sum_k |\vartheta_k|^2 \left[|\omega_k^p - 1|^2 - \left(\frac{\sin p\frac{\pi}{n}}{\sin\frac{\pi}{n}}\right)^2 \cdot |\omega_k - 1|^2\right] =$$

$$= n\sum_k |\vartheta_k|^2\,|\omega_k - 1|^2 \left[\left(\frac{\sin kp\frac{\pi}{n}}{\sin k\frac{\pi}{n}}\right)^2 - \left(\frac{\sin p\frac{\pi}{n}}{\sin\frac{\pi}{n}}\right)^2\right] \leqq 0$$

since for all $p = 1, 2, ., , n-1$ and $k = 1, 2, \dots, n-1$

$$(4)\qquad \left(\frac{\sin kp\frac{\pi}{n}}{\sin k\frac{\pi}{n}}\right)^2 \leqq \left(\frac{\sin p\frac{\pi}{n}}{\sin\frac{\pi}{n}}\right)^2$$

holds, with equality for $k = 1$ or $k = n - 1$. In order to prove the inequality (3) notice that the function

$$U_{p-1}(\cos x) = \frac{\sin px}{\sin x}$$

which occurs in (4) is the well-known Tchebycheff's orthogonal polynomial of the second kind. Then inequality (4) follows immediately from the properties of these polynomials [4]. Equality in (3) holds if and only if $\vartheta_k = 0$ for all $k = 2, 3, \dots, n-2$. It means the n-gon has the foım

$$\mathscr{A} = \vartheta_0\Pi_0 + \vartheta_1\Pi_1 + \vartheta_{n-1}\Pi_{n-1} .$$

It is easily shown that $\mathscr{A}$ is an affine-regular n-gon.

Remark. 1. For $p = 2$ we get the inequality from Introduction. L. Gerber proved it using the method of Lagrange multipliers. We are also able to prove Theorem 1 by this method.

2. Putting $p = 2$, $n = 4$ in (1) we get the well-known property of parallelogram (with sides a, b, c, d and diagonals e, f)

$$a^2 + b^2 + c^2 + d^2 = e^2 + f^2 .$$

3. FUNDAMENTAL INEQUALITY — THE CONTINUOUS CASE

Theorem 2. *Let $f(x)$ be a smooth function with period 2π. Then for all real t,*

$$\int_0^{2\pi} [f(x) - f(x + t)]^2 \, dx \leqq 4 \sin^2 \frac{t}{2} \int_0^{2\pi} f'(x)^2 \, dx . \tag{5}$$

Equality is attained if and only if $f(x) = A \cos x + B \sin x + C$, where A, B, C are real constants (for $t = 0$ equality holds always).

Proof. We shall prove the theorem using a Fourier series expansion of $f(x)$. We have

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) , \tag{6}$$

$$f'(x) \sim \sum_{k=1}^{\infty} (kb_k \cos kx - ka_k \sin kx) ,$$

$$f(x + t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (A_k \cos kx + B_k \sin kx) ,$$

where

$$A_k = a_k \cdot \cos kt + b_k \cdot \sin kt, \; B_k = b_k \cdot \cos kt - a_k \cdot \sin kt.$$

Parseval's relation of completeness gives

$$\int_0^{2\pi} f^2(x) \, dx = \pi \left[\frac{a_0^2}{2} + \sum_{k=1}^{\infty} (a_k^2 + b_k^2) \right]$$

so that

$$\textstyle\int_0^{2\pi} f'(x)^2 \, dx = \pi \sum_{k=1}^{\infty} k^2 (a_k^2 + b_k^2) ,$$

$$\int_0^{2\pi} [f(x) - f(x + t)]^2 \, dx = \pi \sum_{k=1}^{\infty} \left[4 \sin^2 \frac{kt}{2} (a_k^2 + b_k^2) \right] .$$

We get

$$4 \sin^2 \frac{t}{2} \int_0^{2\pi} f'(x)^2 \, dx - \int_0^{2\pi} [f(x) - f(x + t)]^2 \, dx =$$

$$= 4\pi \sum_{k=1}^{\infty} \left[\left(k^2 \sin^2 \frac{t}{2} - \sin^2 k \frac{t}{2} \right) (a_k^2 + b_k^2) \right] \geqq 0$$

since for all real x,

$$k^2 \sin^2 x - \sin^2 kx \geqq 0\,, \quad k = 0, 1, 2, \ldots$$

with equality for $k = 0$ or $k = 1$.

This inequality follows from the properties of Tchebycheff's polynomials of the second kind. It can be also proved by induction.

Equality in (5) holds if and only if $f(x)$ in (6) satisfies $a_k = b_k = 0$ for all $k = 2, 3, \ldots$, i.e.

$$f(x) = A \cos x + B \sin x + C\,, \quad A, B, C = \text{const.}$$

Now we will show that the inequality (2) is a discrete analog of the integral inequality (5).

Assume the function f is the same as in Theorem 2. Write

$$t_\nu = \nu \frac{2\pi}{n}\,, \quad \nu = 0, 1, \ldots, n - 1$$

and use Theorem 1* with $x_\nu = f(t_\nu)$.

By the Mean Value Theorem we have

$$x_{\nu+1} - x_\nu = f'(\xi_\nu) \frac{2\pi}{n}\,, \quad \xi \in (t_\nu, t_{\nu+1})\,.$$

We express (2) in the form

$$\sum_{\nu=0}^{n-1} \left[f(t_\nu) - f\left(t_\nu + p \frac{2\pi}{n} \right) \right]^2 \frac{2\pi}{n} \leqq \frac{\sin^2 p \frac{\pi}{n}}{\sin^2 \frac{\pi}{n}} \left(\frac{2\pi}{n} \right)^2 \sum_{\nu=0}^{n-1} f'(\xi_\nu)^2 \frac{2\pi}{n}\,.$$

Setting $t = p \,.\, 2\pi/n$ and passing to the limit for $n \to \infty$, we obtain the inequality (5).

4. SOME REMARKS ON WIRTINGER'S INEQUALITY

In this section we show relations holding between the inequalities (2), (5) and Wirtinger's inequality.

Let us recall the integral Wirtinger's inequality:

Let $f(x)$ be a smooth function with period 2π satisfying

$$\int_0^{2\pi} f(x)\,\mathrm{d}x = 0\,.$$

Then

$$\int_0^{2\pi} f'(x)^2\,\mathrm{d}x \geqq \int_0^{2\pi} f^2(x)\,\mathrm{d}x\,, \tag{7}$$

equality holding if and only if $f(x) = A \cos x + B \sin x$, where A, B are real constants.

This theorem was stated by W. Blaschke [5], but it is likely that it had been known before.

The discrete version of Wirtinger's inequality is as follows:

Let $x_0, x_1, \dots, x_{n-1}, x_n = x_0$ *be reals with* $\sum_{\nu=0}^{n-1} x_\nu = 0$. *Then*

$$\sum_{\nu=0}^{n-1} (x_{\nu+1} - x_\nu)^2 \geqq 4 \sin^2 \frac{\pi}{n} \sum_{\nu=0}^{n-1} x_\nu^2 . \tag{8}$$

Equality holds if and only if $x_\nu = A \cos \nu(2\pi/n) + B \sin \nu(2\pi/n)$, *where* $\nu = 0, 1, \dots, n-1$, $A, B = const.$

We can extend this theorem similarly as Theorem 1*:

Let $\mathscr{A} = A_0, A_1, \dots, A_{n-1}, A_n = A_0$ *be a closed n-gon in* E^N *with its center of gravity at the origin of the coordinate system. Then*

$$\sum_{\nu=0}^{n-1} |A_\nu A_{\nu+1}|^2 \geqq 4 \sin^2 \frac{\pi}{n} \sum_{\nu=0}^{n-1} |A_\nu|^2 . \tag{9}$$

Equality holds if and only if $\mathscr{A}$ *is a plane affine-regular n-gon.*

The case $N = 2$ of this theorem was given by B. H. Neumann [6].

K. Fan, O. Taussky, J. Todd [7] showed that the inequality (8) is the discrete analog of (7).

5. CONNECTIONS WITH WIRTINGER'S INEQUALITY

First we will prove

Lemma. *For an arbitrary closed n-gon* $\mathscr{A} = A_0, A_1, \dots, A_{n-1}$ *in* E^N *with its center of gravity at the origin of the coordinate system the equality*

$$\sum_{p,\nu=0}^{n-1} |A_\nu A_{\nu+p}|^2 = 2n \sum_{\nu=0}^{n-1} |A_\nu|^2 \tag{10}$$

holds where $A_{n+k} = A_k$ *for* $k = 1, 2, \dots$.

Proof. Let us express the squares of distances by means of a scalar product. We obtain

$$\sum_{p,\nu=0}^{n-1} |A_\nu A_{\nu+p}|^2 - 2n \sum_{\nu=0}^{n-1} |A_\nu|^2 = \sum_{p,\nu=0}^{n-1} (A_{\nu+p} - A_\nu)^2 - 2n \sum_{\nu=0}^{n-1} A_\nu^2 =$$
$$= \sum_{p,\nu=0}^{n-1} (A_{\nu+p}^2 - 2A_{\nu+p}A_\nu + A_\nu^2) - 2n \sum_{\nu=0}^{n-1} A_\nu^2 = -2n \sum_{p,\nu=0}^{n-1} A_{\nu+p}A_\nu =$$
$$= -2n \sum_{\nu=0}^{n-1} (A_\nu \sum_{p=0}^{n-1} A_{\nu+p}) = 0 .$$

Now we shall prove:

Theorem 3. *The discrete Wirtinger's inequality is a consequence of the set of inequalities* (1).

Proof. Let $\mathscr{A} = A_0, A_1, \ldots, A_{n-1}$ be a closed n-gon in E^N with its center of gravity at the origin of the coordinate system. Summing the inequalities (1) for $p = 0, 1, \ldots \ldots, n - 1$ we get

$$\sum_{p,\nu=0}^{n-1} |A_\nu A_{\nu+p}|^2 \leqq \sum_{p=0}^{n-1} \left(\frac{\sin p \dfrac{\pi}{n}}{\sin \dfrac{\pi}{n}} \right)^2 . \sum_{\nu=0}^{n-1} |A_\nu A_{\nu+1}|^2 .$$

From here, in view of the equality $\sum_{p=0}^{n-1} \sin^2 p(\pi/n) = n/2$, we obtain

$$\sum_{p,\nu=0}^{n-1} |A_\nu A_{\nu+p}|^2 \leqq \frac{n}{2 \sin^2 \dfrac{\pi}{n}} . \sum_{\nu=0}^{n-1} |A_\nu A_{\nu+1}|^2$$

and, using (10) from the previous lemma, the inequality (9).

In the next theorem we will prove an analogous relation between the integral inequality (5) and the integral Wirtinger's inequality (7).

Theorem 4. *The inequality* (5) *implies the integral Wirtinger's inequality* (7).

Proof. Let $f(x)$ be a smooth function with period 2π satisfying

$$\int_0^{2\pi} f(x)\, dx = 0 .$$

Integrating inequality (5) we get

$$(11) \qquad \int_0^{2\pi} \int_0^{2\pi} [f(x) - f(x + t)]^2 \, dx \, dt \leqq 4 \int_0^{2\pi} \sin^2 \frac{t}{2} \, dt \int_0^{2\pi} f'(x)^2 \, dx .$$

On the left-hand side of (11) we have

$$4\pi \int_0^{2\pi} f^2(x)\, dx - 2 \int_0^{2\pi} [f(x) \int_0^{2\pi} f(x + t)\, dt]\, dx = 4\pi \int_0^{2\pi} f^2(x)\, dx .$$

On the right-hand side of (11) we get

$$4 \int_0^{2\pi} \sin^2 \frac{t}{2} \, dt \int_0^{2\pi} f'(x)^2 \, dx = 4\pi \int_0^{2\pi} f'(x)^2 \, dx .$$

Dividing both sides of (11) by 4π we get the inequality (7).

References

[1] *L. Gerber:* Napoleon's theorem and the parallelogram inequality for affine-regular polygons. Amer. Math. Monthly *87* (1980), 644–648.

[2] *D. S. Mitrinovič:* Analytic inequalities. Springer, Berlin, 1970.
[3] *I. J. Schoenberg:* The finite Fourier series and elementary geometry. Amer. Math. Monthly *57* (1950), 390—404.
[4] *G. Szegö:* Orthogonal polynomials. Amer. Math. Soc. XXIII, New York, 1939.
[5] *W. Blaschke:* Kreis und Kugel. 2nd edition, Gruyter, Berlin, 1956.
[6] *B. H. Neumann:* Some remarks on polygons. J. London Math. Soc. *16* (1941), 230—245.
[7] *K. Fan, O. Taussky, J. Todd:* Discrete analogs of inequalities of Wirtinger. Monatsh. Math. *59* (1955), 73—90.

Souhrn

NEROVNOST MEZI STRANAMI A ÚHLOPŘÍČKAMI PROSTOROVÉHO n-ÚHELNÍKA A JEJÍ DISKRÉTNÍ ANALOGIE

PAVEL PECH

V článku je uvedena nerovnost mezi stranami a úhlopříčkami uzavřeného prostorového n-úhelníka v E^N. Tato nerovnost je zobecněna na integrální nerovnost. Tyto nerovnosti implikují známou diskrétní a integrální Wirtingerovu nerovnost.

Author's address: Katedra matematiky, pedagogická fakulta, Jeronýmova 10, 371 15 České Budějovice.