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TWO-SIDED SOLUTIONS OF LINEAR INTEGRODIFFERENTIAL EQUATIONS OF VOLTERRA TYPE WITH DELAY

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Summary. For the system $x = A(t)\,x + \varepsilon \int_{-\infty}^{t} R(t-s)\,x(s)\,ds + \varepsilon \int_{t-T}^{t} P(t-s)\,x(s)\,ds$, $0 < T < \infty$, where $A(t)$ is either a constant or a periodic matrix, the existence of two-sided solutions with $x(0) = x_0$ is studied in connection with the behaviour of the solutions of the unperturbed system for $\varepsilon = 0$. A Floquet type theorem for the periodic case is also proved.

Keywords: Integrodifferential equation, two-sided solution.

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Consider the integrodifferential equation

$$\frac{dx(t)}{dt} = A\,x(t) + \varepsilon \int_{-\infty}^{t} R(t-s)\,x(s)\,ds + \int_{t-T}^{t} P(t-s)\,x(s)\,ds\,, \tag{1}$$

here $x \in R^n$, $\varepsilon > 0$ is a parameter, A is a constant $n \times n$ matrix, $0 < T < \infty$, and the matrix functions R, P satisfy the conditions

(I) $R(t)$ is continuous and

$$\|R(t)\| \leqq t^{\alpha-1} e^{-\gamma t} \quad \text{for} \quad t > 0\,, \tag{2}$$

where $\alpha.\ \gamma$ are positive constants, $0 < \alpha < 1$ and $\|B\|$ is the euclidean norm of a matrix B;

(II) $P(t)$ is continuous on the interval $[0, T]$.

Definition. *A solution $x_\varepsilon(t)$ of the equation (1) is called two-sided if*

1. *x_ε is defined on the interval $(-\infty, \infty)$,*
2. *$\lim_{\varepsilon \to 0} \|x_\varepsilon - x\|_L = 0$ for any $L > 0$, where $\|x_\varepsilon - x\|_L = \max_{-L \leqq t \leqq L} \|x_\varepsilon(t) - x(t)\|$ and x is a solution of the equation*

$$\frac{dx}{dt} = Ax\,. \tag{3}$$

Remark. The above definition includes also the case of a matrix solution of (1) i.e. either $x_\varepsilon(t)$, $x(t) \in R^n$ or $x_\varepsilon(t)$, $x(t) \in M(n)$, where $M(n)$ is the set of all $n \times n$ matrices.

We study the problem of existence of two-sided solutions of the equation (1). We also study the case when the matrix A is nonconstant and periodic. Yu. A. Ryabov [2] proved a sufficient condition for the existence of a two-sided matrix solution of the equation (1) without the second integral term, i.e. when $P \equiv 0$, and has formulated a sufficient condition for the existence and uniqueness of a two-sided solution of this equation.

Theorem 1. *Let the conditions* (I), (II) *be satisfied and let the eigenvalues* $\lambda_1, \lambda_2, \ldots, \lambda_n$ *of the matrix* A *satisfy the condition*

$$\min_j \operatorname{Re} \lambda_j > -\gamma . \tag{4}$$

Then there exists an $\varepsilon^* > 0$ *such that the following assertions are valid*:

(a) *For any* $\varepsilon \in (0, \varepsilon^*]$ *there exists a two-sided matrix solution of the equation* (1) *of the form*

$$X_\varepsilon(t) = e^{Dt}, \tag{5}$$

where $D = D(\varepsilon)$ *is a matrix independent of* t *and* $\lim_{\varepsilon \to 0} D(\varepsilon) = A$, *i.e.* $\lim_{\varepsilon \to 0} \|D(\varepsilon) - A\| = 0$.

(b) *For any* $\varepsilon \in (0, \varepsilon^*]$ *and* $x_0 \in R^n$ *there exists a unique solution* $x_\varepsilon(t)$ *of the equation* (1), *satisfying the condition* $x_\varepsilon(0) = x_0$ *and* $x_\varepsilon \in U_\delta = \{z \in C^0((-\infty, \infty), R^n): z(t)\, e^{\delta t} < \infty$ *for all* $t \in (-\infty, 0]\}$, *where* δ *is a constant and* $0 < \delta < \gamma$.

The assertion (a) of this theorem concerning the case $P \equiv 0$ has been proved by Ryabov in [2], where the existence of the matrix D is proved by the method of matrix series. The proof of the assertion (b) is not given in [2]. We prove both assertions of Theorem 1 using the Banach fixed point theorem.

We need the following lemma.

Lemma 1. *Let* $0 < \bar{t} < \infty$, $u \in C^0([0, \bar{t}], R)$ *be a nonnegative function,* $a \geqq 0$, $b \geqq 0$, $k \geqq 0$, $\beta > 0$ *constants and*

$$u(t) \leqq a + k \int_0^t \int_0^s u(\tau)\, d\tau\, ds + b \int_0^t \int_0^s (s - \tau)^{\beta - 1} u(\tau)\, d\tau\, ds , \tag{6}$$

$t \in [0, \bar{t}]$. *Then*

$$u(t) \leqq a \exp\left\{\frac{k}{2} t^2 + \frac{b}{\beta(\beta + 1)} t^{\beta + 1}\right\}, \quad t \in [0, \bar{t}] . \tag{7}$$

Proof. From the Fubini theorem it follows that the inequality (6) is equivalent to

$$u(t) \leqq a + \int_0^t \left[k(t - \tau) + \frac{b}{\beta}(t - \tau)^\beta\right] u(\tau)\, d\tau$$

and applying [1, Theorem 1.4$_1$] we obtain the inequality (7).

Proof of Theorem 1. Let D be a constant $n \times n$ matrix. The matrix function $X_\varepsilon(t) = e^{Dt}$ is a matrix solution of the equation (1) if and only if

$$De^{Dt} = Ae^{Dt} + \varepsilon \int_{-\infty}^{t} R(t-s)\, e^{Ds}\, ds + \varepsilon \int_{t-T}^{t} P(t-s)\, e^{Ds}\, ds\,.$$

Let us look for the matrix solution of this equation in the form $D = A + Q$, where Q is an unknown matrix. Putting $t = 0$ in this equation we obtain the following equation for Q:

$$(8) \qquad Q = \varepsilon \int_0^\infty R(\Theta)\, e^{-(A+Q)\Theta}\, d\Theta + \varepsilon \int_0^T P(\Theta)\, e^{-(A+Q)\Theta}\, d\Theta\,.$$

Using the substitution $s = t - \Theta$ in the integrals on the right-hand side of (8) one can show that if Q is a matrix solution of (8) and $D = A + Q$ then e^{Dt} is a matrix solution of the equation (1). Therefore it suffices to solve the matrix equation (8).

The condition (4) implies that there exists μ, $-\gamma < \mu < \min_j \operatorname{Re} \lambda_j$ and a constant $k > 1$ such that

$$(9)^\sigma \qquad \|e^{-A\Theta}\| \leqq k e^{-\mu\Theta}\,, \quad \Theta \geqq 0\,.$$

Let $V_\varkappa = \{Q \in M(n)\colon \|Q\| < \varkappa\}$, where $M(n)$ is the set of all $n \times n$ matrices and $0 < \varkappa < \gamma + \mu$. Define the mapping

$$\mathscr{F}_\varepsilon\colon V_\varkappa \to M(n)\,, \quad \mathscr{F}_\varepsilon(Q) = \varepsilon \int_0^\infty R(\Theta)\, e^{-(A+Q)\Theta}\, d\Theta + $$
$$+ \varepsilon \int_0^T P(\Theta)\, e^{-(A+Q)\Theta}\, d\Theta\,.$$

Lemma 2. *There exists an $\varepsilon^* > 0$ such that the mapping $\mathscr{F}_\varepsilon$ is contractive for $\varepsilon \in (0, \varepsilon^*]$.*

Proof. If $Q_1, Q_2 \in V_\varkappa$ then using the inequalities (2), (9) we obtain

$$(10) \qquad \|\mathscr{F}_\varepsilon(Q_1) - \mathscr{F}_\varepsilon(Q_2)\| \leqq \varepsilon \Big(k \int_0^\infty \Theta^{\alpha-1} e^{-(\gamma+\mu)\Theta} \|e^{-Q_1\Theta} - e^{-Q_2\Theta}\|\, d\Theta + $$
$$+ k \int_0^T e^{-\mu\Theta} \|P(\Theta)\|\, \|e^{-Q_1\Theta} - e^{-Q_2\Theta}\|\, d\Theta \Big)\,.$$

The mean value theorem implies that

$$\|e^{-Q_1\Theta} - e^{-Q_2\Theta}\| \leqq \sup_{Q \in V_\varkappa} \|e^{Q\Theta}\|\, \|Q_1\Theta - Q_2\Theta\| \leqq \Theta e^{\varkappa\Theta} \|Q_1 - Q_2\|$$

for any Θ. Using this inequality and (10) we have

$$\|\mathscr{F}_\varepsilon(Q_1) - \mathscr{F}_\varepsilon(Q_2)\| \leqq \varepsilon k \Big(\int_0^\infty e^{-\xi\Theta} \Theta^\alpha\, d\Theta + $$
$$+ \int_0^T \Theta e^{(\varkappa-\mu)\Theta} \|P(\Theta)\|\, d\Theta \Big) \|Q_1 - Q_2\|$$

where $\zeta = \gamma + \mu - \varkappa$. If we put $s = \zeta\Theta$ in the first integral then the above inequality takes the form

$$\|\mathscr{F}_\varepsilon(Q_1) - \mathscr{F}_\varepsilon(Q_2)\| \leqq \varepsilon k[\zeta^{-\alpha-1}\,\Gamma(\alpha+1) + C]\,,$$

where $C = \int_0^T \Theta e^{(\varkappa-\mu)\Theta}\|P(\Theta)\|\,d\Theta < \infty$. Since $0 < \alpha + 1 < 2$ we have $0 < \Gamma(\alpha+1) < \infty$. Therefore if

$$0 < \varepsilon < \varepsilon_1 := \nu k^{-1}[\zeta^{\alpha+1}\,\Gamma(\alpha+1) + C]^{-1}\,, \tag{11}$$

where $0 < \nu < 1$, then

$$\|\mathscr{F}_\varepsilon(Q_1) - \mathscr{F}_\varepsilon(Q_2)\| \leqq \nu\|Q_1 - Q_2\|\,,$$

i.e. the mapping $\mathscr{F}_\varepsilon$ is contractive.

Lemma 2 implies that if $\varepsilon \in (0, \varepsilon_1)$, where ε_1 is defined by (11), then the mapping $\mathscr{F}_\varepsilon$ has a unique fixed point $Q \in V_\varkappa$. This matrix is a unique solution of (8) belonging to the set $V_\varkappa$. From (8) we have that $\lim_{\varepsilon\to 0} Q(\varepsilon) = 0$ and so $\lim_{\varepsilon\to 0} D(\varepsilon) = A$, i.e. $\lim_{\varepsilon\to 0}\|D(\varepsilon) - A\| = 0$, where $D(\varepsilon) = A + Q(\varepsilon)$. It remains to prove the assertion (b) of Theorem 1. We shall prove that for any $x_0 \in R^n$ there exists a unique solution $x(t)$ of the equation (1) satisfying the conditions $x(0) = x_0$ and $\sup_{-\infty<t\leqq 0}\|x(t)\|\,e^{\delta t} < \infty$, where

$$0 < \delta < \gamma\,, \quad \mu + \delta > 0 \tag{12}$$

and μ is the number from (9). Since $\mu + \gamma > 0$ there exists a number δ satisfying (12). Define the subspace

$$B_\delta = \{x \in C^0((-\infty, 0], R^n)\colon \sup_{-\infty<t\leqq 0}\|x(t)\|\,e^{\delta t} < \infty\}\,.$$

The set B_δ with the norm $\|x\|_\delta = \sup_{-\infty<t\leqq 0}\|x(t)\|\,e^{\delta t}$ is a Banach space. Define the mapping

$$G_\varepsilon\colon B_\delta \to C^0((-\infty, 0], R^n)\,,$$

$$(G_\varepsilon x)(t) = \varepsilon\left[\int_0^t e^{A(t-s)}\left(\int_{-\infty}^s R(s-\tau)\,x(\tau)\,d\tau + \int_{s-T}^s P(s-\tau)\,x(\tau)\,d\tau\right)ds\right], \quad -\infty < t \leqq 0\,.$$

We shall prove that $G_\varepsilon(B_\delta) \subset B_\delta$ and G_ε is contractive for $\varepsilon > 0$ sufficiently small. If $x \in B_\delta$ and $-\infty < t \leqq 0$ then

$$\|G_\varepsilon(t)\| \leqq \varepsilon k\left[\int_t^0 e^{-\mu(s-t)}\left(\int_{-\infty}^s e^{-\gamma(s-\tau)}(s-\tau)^{\alpha-1}\|x(\tau)\|\,d\tau + \int_{s-T}^s \|P(s-\tau)\|\,\|x(\tau)\|\,d\tau\right)ds\right] \leqq \varepsilon k(I_1(t) + I_2(t))\,\|x\|_\delta\,,$$

where

$$I_1(t) = \int_t^0 e^{-\mu(s-t)} \left(\int_{-\infty}^s e^{-\gamma(s-\tau)} e^{-\delta\tau} (s - \tau)^{\alpha-1} \, d\tau \right) ds ,$$

$$I_2(t) = \int_t^0 e^{-\mu(s-t)} \left(\int_{s-T}^s \|P(s - \tau)\| \, e^{-\delta\tau} \, d\tau \right) ds .$$

The function $I_1(t)$ can be written in the form

$$I_1(t) = e^{\mu t} \int_t^0 e^{-(\mu+\delta)s} \left(\int_{-\infty}^s e^{-(\gamma-\delta)(s-\tau)} (s - \tau)^{\alpha-1} \, d\tau \right) ds ,$$

and using the substitution $u = (\gamma - \delta)(s - \tau)$ we obtain

$$I_1(t) = e^{\mu t} \int_t^0 e^{-(\mu+\delta)s} \left(\int_0^\infty (\gamma - \delta)^{-\alpha} e^{-u} u^{\alpha-1} \, du \right) ds =$$

$$= \Gamma(\alpha) (\gamma - \delta)^{-\alpha} \, e^{\mu t} \int_t^0 e^{-(\mu+\delta)s} \, ds =$$

$$= \Gamma(\alpha) (\gamma - \delta)^{-\alpha} (\mu + \delta)^{-1} \, e^{\mu t} (e^{-(\mu+\delta)t} - 1) .$$

The function $I_2(t)$ can be written in the form

$$I_2(t) = \int_t^0 e^{-\mu(s-t)} \left(\int_0^T \|P(\Theta)\| \, e^{\delta\Theta} \, d\Theta) \, e^{-\delta s} \right) ds =$$

$$= e^{\mu t} \left(\int_0^T \|P(\Theta)\| \, e^{\delta\Theta} \, d\Theta \right) (\mu + \delta)^{-1} \, (e^{-(\mu+\delta)t} - 1) .$$

Therefore we have the inequality

$$\|G_\varepsilon \, x(t)\| \, e^{\delta t} \leqq \varepsilon k (K_1 + K_2) \, e^{(\mu+\delta)t} (e^{-(\mu+\delta)t} - 1) \, \|x\|_\delta ,$$

where $K_1 = \Gamma(\alpha) (\gamma - \delta)^{-\alpha} (\mu + \delta)^{-1} > 0$, $K_2 = (\int_0^T \|P(\Theta)\| \, e^{\delta\Theta} \, d\Theta) (\mu + \delta)^{-1} > > 0$. Since $\mu + \delta > 0$ we obtain

$$\sup_{-\infty < t \leqq 0} \|G_\varepsilon \, x(t)\| \, e^{\delta t} \leqq \varepsilon k (K_1 + K_2) \, \|x\|_\delta < \infty , \quad \text{i.e.} \quad G_\varepsilon x \in B_\delta$$

and therefore $G_\varepsilon B_\delta \subset B_\delta$. Since G_ε is linear we have

$$\|G_\varepsilon x_1 - G_\varepsilon x_2\|_\delta = \|G_\varepsilon (x_1 - x_2)\|_\delta \leqq \varepsilon k (K_1 + K_2) \, \|x_1 - x_2\|_\delta$$

for any $x_1, x_2 \in B_\delta$ and thus the map G_ε is contractive for any $\varepsilon \in (0, \tilde{\varepsilon})$, where $\tilde{\varepsilon} = k^{-1} (K_1 + K_2)^{-1}$. From now on we assume $\varepsilon \in (0, \tilde{\varepsilon})$. Then the map G_ε has a unique fixed point $\varphi_0 \in B_\delta$. Since this map is linear and $0 \in B_\delta$ we conclude that $\varphi_0 = 0$.

Let φ_1, φ_2 be two solutions of the equation (1) satisfying the condition $\varphi_1(0) = = \varphi_2(0) = x_0$, $\sup_{-\infty < t \leqq 0} \|\varphi_i(t)\| \, e^{\delta t} < \infty$, $i = 1, 2$ and let $\varphi(t) = \varphi_1(t) - \varphi_2(t)$. Then

$\sup_{-\infty < t \leqq 0} \|\varphi(t)\| e^{\delta t} < \infty$. The mapping $\Phi \in C^0((-\infty, 0], R^n)$, $\Phi(t) = \varphi(t)$, $-\infty < t \leqq 0$, is a fixed point of the map G_ε and therefore $\Phi(t) \equiv 0$. Thus if there is a two-sided solution of (1) belonging to B_δ then it is uniquely defined on the interval $(-\infty, 0]$. We prove that such a two-sided solution does exist and it is also uniquely defined on the interval $[0, \infty)$.

The function $\Psi(t) = e^{D(\varepsilon)t}x_0$ is a two-sided solution of the equation (1) satisfying the initial condition $\Psi(0) = x_0$. If $\varepsilon > 0$ is sufficiently small then the condition (4) and the equality $\lim_{\varepsilon \to 0} D(\varepsilon) = A$ imply that the eigenvalues $\nu_1, \nu_2, \ldots, \nu_n$ of the matrix $D(\varepsilon)$ satisfy the condition $\min_j \operatorname{Re} \nu_j > -\gamma$. Therefore there exists a constant $\tilde{\mu}$, $-\gamma < \tilde{\mu} < \min_j \operatorname{Re} \nu_j$ and a constant $\tilde{k} > 1$ such that

$$\|e^{-D(\varepsilon)\Theta}\| \leqq \tilde{k}e^{-\tilde{\mu}\Theta}, \quad \Theta \geqq 0 \quad \text{or} \quad \|e^{D(\varepsilon)t}\| \leqq \tilde{k}e^{\tilde{\mu}t}, \quad t \leqq 0 .$$

Since $\mu + \delta > 0$, where μ is the number from (9), we have $\tilde{\mu} + \delta > 0$ for ε sufficiently small. Therefore for such $\varepsilon > 0$ we obtain

$$\sup_{-\infty < t \leqq 0} \|e^{D(\varepsilon)t}x_0\| e^{\delta t} \leqq \sup_{-\infty < t \leqq 0} (\tilde{k}e^{(\tilde{\mu}+\delta)t}\|x_0\|) = \tilde{k}\|x_0\| < \infty .$$

This means that the two-sided solution $\Psi(t) = e^{D(\varepsilon)t}x_0$ belongs to the set B_δ. It suffices to prove the uniqueness of two-sided solutions of the equation (1) belonging to the set B_δ on the interval $[0, \infty)$.

Let $\varphi_1(t)$, $\varphi_2(t)$ be two-sided solutions of the equation (1) belonging to the set B_δ and satisfying the condition $\varphi_1(0) = \varphi_2(0) = x_0$. Let $\varphi = \varphi_1 - \varphi_2$. Since we have proved that $\varphi_1(t) = \varphi_2(t)$ for all $t \in (-\infty, 0]$, by (2) we obtain for $t \geqq 0$:

$$\|\varphi(t)\| \leqq \varepsilon \left\| \int_0^t e^{A(t-s)} \left(\int_0^s R(s - \tau)\, \varphi(\tau)\, d\tau + \right.\right.$$

$$\left.\left. + \int_{s-T}^s P(s - \tau)\, \varphi(\tau)\, d\tau \right) ds \right\| \leqq$$

$$\leqq \varepsilon \left[c \int_0^t e^{\nu(t-s)} \left(\int_0^s e^{-\gamma(s-\tau)}(s - \tau)^{\alpha-1} \|\varphi(\tau)\|\, d\tau + K \int_0^s \|\varphi(\tau)\|\, d\tau \right) ds \right],$$

where $\nu > \max_j \operatorname{Re} \lambda_j$, $c > 0$ ($\|e^{At}\| \leqq ce^{\nu t}$ for all $t \geqq 0$) and $K = \max_{0 \leqq t \leqq T} \|P(t)\|$.

It suffices to show that for any $0 < \bar{t} < \infty$, $\varphi(t) = 0$ for all $t \in [0, \bar{t}]$. From the above inequality we obtain

$$\|\varphi(t)\| \leqq \varepsilon \left[cM \int_0^t \int_0^s (s - \tau)^{\alpha-1} \|\varphi(\tau)\|\, d\tau\, ds + cK \int_0^t \int_0^s \|\varphi(\tau)\|\, d\tau\, ds \right].$$

Applying Lemma 1 to this inequality we obtain $\varphi(t) = 0$ for all $t \in [0, \bar{t}]$.

We have shown that for $\varepsilon > 0$ sufficiently small and any $x_0 \in R^n$ there exists a unique solution x_ε of the equation (1) satisfying the condition $x_\varepsilon(0) = x_0$ and defined on the interval $(-\infty, \infty)$. This solution has the form $x_\varepsilon(t) = e^{D(\varepsilon)t}x_0$, where

$D(\varepsilon) = \Lambda + Q(\varepsilon), \lim_{\varepsilon \to 0} Q(\varepsilon) = 0$. It remains to show that x_ε has the second property of a two-sided solution, i.e. $\lim_{\varepsilon \to 0} \|x_\varepsilon - x\|_L = 0$ for any $L > 0$, where $x(t) = e^{\Lambda t} x_0$.

If $X_\varepsilon(t) = e^{D(\varepsilon)t}$ and $X(t) = e^{\Lambda t}$ then

$$\|X_\varepsilon(t) - X(t)\| = \|e^{\Lambda t}[e^{Q(\varepsilon)t} - E]\| \leqq \|e^{\Lambda t}\| \, \|e^{Q(\varepsilon)t} - E\| \, .$$

The mean value theorem implies that for any $L > 0$

$$\max_{-L \leqq t \leqq L} \|e^{Q(\varepsilon)t} - E\| \leqq \max_{-L \leqq t \leqq L} (\|Q(\varepsilon) \, e^{Q(\varepsilon)t}\|) \, |t| \leqq LC(L) \| \, Q(\varepsilon)\| \, ,$$

where $C(L) = \max_{-L \leqq t \leqq L} \|e^{Q(\varepsilon)t}\|$. Therefore we have

$$\lim_{\varepsilon \to 0} \|x_\varepsilon - x\|_L = \lim_{\varepsilon \to 0} \max_{-L \leqq t \leqq L} \|x_\varepsilon(t) - x(t)\| \leqq LC(L) \, \|x_0\| \lim_{\varepsilon \to 0} \|Q(\varepsilon)\| = 0$$

and the proof of Theorem 1 is complete.

Let us consider the integrodifferential equation

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = A(t) \, x(t) + \varepsilon \int_{-\infty}^{t} R(t - s) \, x(s) \, \mathrm{d}s + \varepsilon \int_{t-T}^{t} P(t - s) \, x(s) \, \mathrm{d}s \, , \tag{13}$$

where R, P, T are as above and $A(t)$ is a continuous τ-periodic matrix function on $(-\infty, \infty)$, $\tau > 0$.

If $X(t)$ is the normed fundamental matrix of the linear system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = A(t) \, x \tag{14}$$

then by the Floquet theorem

$$X(t) = \Phi(t) \, e^{\Lambda \tau} \, , \tag{15}$$

where Λ is a constant matrix and $\Phi(t)$ is a continuous τ-periodic matrix function. Introducing a new variable $y = \Phi^{-1}(t) \, x$ the equation (13) becomes

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = \Lambda \, y(t) + \varepsilon \, \Phi^{-1}(t) \int_{-\infty}^{t} R(t - s) \, \Phi(s) \, y(s) \, \mathrm{d}s + \tag{16}$$
$$+ \, \varepsilon \, \Phi^{-1}(t) \int_{t-T}^{t} P(t - s) \, \Phi(s) \, y(s) \, \mathrm{d}s \, .$$

Let us look for the matrix solution e^{Dt} of the equation (16), where $D = \Lambda + Q$, Q is an unknown matrix. This is a solution of (16) if and only if

$$De^{Dt} = \Lambda e^{Dt} + \varepsilon \, \Phi^{-1}(t) \int_{-\infty}^{t} R(t - s) \, \Phi(s) \, e^{Ds} \, \mathrm{d}s + \tag{17}$$
$$+ \, \varepsilon \, \Phi^{-1}(t) \int_{t-T}^{t} P(t - s) \, \Phi(s) \, e^{Ds} \, \mathrm{d}s \, .$$

Putting $t = 0$ in this equation we obtain the equation for Q:

$$Q = \varepsilon \int_{-\infty}^{0} R(-s)\, \Phi(s)\, e^{(\Lambda+Q)s}\, ds + \varepsilon \int_{-T}^{0} P(-s)\, \Phi(s)\, e^{(\Lambda+Q)s}\, ds .$$

Introducing the substitution $-s = \sigma$ this equation becomes

$$(18) \qquad Q = \varepsilon \int_{0}^{\infty} R(\sigma)\, \Phi(-\sigma)\, e^{-(\Lambda+Q)\sigma}\, d\sigma + \varepsilon \int_{0}^{T} P(\sigma)\, \Phi(-\sigma)\, e^{-(\Lambda+Q)\sigma}\, d\sigma .$$

Let Q be a solution of (18). Then

$$De^{Dt} = \Lambda e^{Dt} + \varepsilon \int_{0}^{\infty} R(\sigma)\, \Phi(-\sigma)\, e^{-D\sigma}\, d\sigma \,.\, e^{Dt} + \\ + \varepsilon \int_{0}^{T} P(\sigma)\, \Phi(-\sigma)\, e^{-D\sigma}\, d\sigma \,.\, e^{Dt} ,$$

where $D = \Lambda + Q$. If $\sigma = t - s$ then the above equation becomes

$$(19) \qquad De^{Dt} = \Lambda e^{Dt} + \varepsilon \int_{-\infty}^{t} R(t - s)\, \Phi(s - t)\, e^{-D(t-s)}\, ds \,.\, e^{Dt} + \\ + \varepsilon \int_{t-T}^{t} P(t - s)\, \Phi(s - t)\, e^{-D(t-s)}\, ds \,.\, e^{Dt} .$$

If the conditions

$$(20) \qquad \Phi(t)\, R(t - s)\, \Phi(s - t) = R(t - s)\, \Phi(s) \quad \text{for all} \quad t, s \in R ,$$

$$(21) \qquad \Phi(t)\, P(t - s)\, \Phi(s - t) = P(t - s)\, \Phi(s) \quad \text{for all} \quad t, s \in R$$

are satisfied then the equation (19) is equivalent to the equation (17). If $A(t) = A$ is a constant matrix then these conditions are trivially satisfied.

Since the matrix functions $\Phi^{-1}(t)$, $\Phi(t)$ are continuous and periodic they must be bounded. Therefore using the same procedure as in the proof of Theorem 1 we are able to solve the equation (18) and to prove the following theorem.

Theorem 2. *Let $A(t)$ be a continuous, τ-periodic matrix function on the interval $(-\infty, \infty)$ and let the matrix functions R, P satisfy the assumptions of Theorem* 1. *Let $\Phi(t)$, Λ be the matrices defined by* (15), *let the eigenvalues $\varkappa_1, \varkappa_2, \ldots, \varkappa_n$ of Λ satisfy the condition*

$$\min_j \varkappa_j > -\gamma$$

and let the conditions (20), (21) *be satisfied. Then there exists an $\varepsilon^* > 0$ such that for any $\varepsilon \in (0, \varepsilon^*]$ the following assertions are valid*:

(a) *There exists a two-sided matrix solution of the equation* (16) *of the form*

$$Y_\varepsilon(t) = e^{Dt} ,$$

where $D = D(\varepsilon)$ is a matrix independent of t and $\lim_{\varepsilon \to 0} D(\varepsilon) = \Lambda$.

(b) *For any* $y_0 \in R^n$ *there exists a unique two-sided solution* $y_\varepsilon(t)$ *of the equation* (16) *satisfying the initial condition* $y_\varepsilon(0) = y_0$ *and* $y_\varepsilon \in U_\delta = \{z \in C^0((-\infty, \infty), R^n): \|z(t)\| e^{\delta t} < \infty$ *for all* $t \in (-\infty, 0]\}$, *where* δ *is a constant and* $0 < \delta < \gamma$.

(c) *There exists a two-sided matrix solution* X_ε *of the equation* (13) *satisfying the condition* $X_\varepsilon(0) = E$, *where E is the unit matrix. This matrix solution has the form* $X_\varepsilon(t) = \Phi(t)\, e^{D(\varepsilon)t}$, *where* $\Phi(t)$ *and* $D(\varepsilon)$ *are as above.*

(d) *For any* $x_0 \in R^n$ *there exists a unique two-sided solution* x_ε *of the equation* (13) *satisfying the initial condition* $x_\varepsilon(0) = x_0$, $x_\varepsilon \in U_\eta = \{z \in C^0((-\infty, \infty), R^n): \|z(t)\| e^{\eta t} < \infty$ *for all* $t \in (-\infty, 0]\}$, *where* η *is a constant,* $0 < \eta < \gamma$ *and* $x_\varepsilon(t) = \Phi(t)\, e^{D(\varepsilon)t} x_0$, $\Phi(t)$, $D(\varepsilon)$ *being as above.*

The assertion (c) is a generalization of the Floquet theorem.

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References

[1] *A. N. Filatov, L. V. Sharova:* Integral Inequalities and Theory of Nonlinear Oscillations. Nauka, Moscow 1976 (Russian).

[2] *Yu. A. Ryabov:* On the existence of two-sided solutions of linear integrodifferential equations of Volterra type with dealy. Časopis pěst. mat. *111* (1986), 26—33 (Russian).

Súhrn

OBOJSTRANNÉ RIEŠENIA LINEÁRNYCH INTEGRODIFERENCIÁLNYCH ROVNÍC VOLTERROVHO TYPU S ONESKORENÍM

MILAN MEDVEĎ

Pre systém $\dot{x} = A(t)\, x + \varepsilon \int_{-\infty}^{t} R(t - s)\, x(s)\, ds + \varepsilon \int_{t-T}^{t} P(t - s)\, x(s)\, ds$, $0 < T < \infty$ kde $A(t)$ je buď konštantná, alebo periodická matica, je študovaná existencia obojstranných riešení pre malé hodnoty parametra $\varepsilon > 0$. V prípade, keď je matica $A(t)$ periodická, je dokázaná veta Floquetovho typu.

Резюме

ДВУСТОРОННИЕ РЕШЕНИЯ ЛИНЕЙНЫХ ИНТЕГРО-ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ ТИПА ВОЛТЕРА С ЗАПАЗДЫВАНИЕМ

MILAN MEDVEĎ

Исследуется существование двусторонних решений с условием $x(0) = x_0$ для системы $\dot{x}(t) = A(t)\, x(t) + \varepsilon \int_{-\infty}^{t} R(t - s)\, x(s)\, ds$ $I+$ $\varepsilon \int_{t-T}^{t} P(t - s)\, x(s)\, ds$ в связи с поведением решений невозмущенной системы для $\varepsilon = 0$, где $0 < T < \infty$ и $A(t)$ — постоянная или периодическая матрица. Приведено также доказательство теоремы типа Флоке для периодического случая.

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