Časopis pro pěstování matematiky

Gary Chartrand; Ortrud R. Oellermann; Song Lin Tian; Hung Bin Zou
Steiner distance in graphs

Časopis pro pěstování matematiky, Vol. 114 (1989), No. 4, 399--410

Persistent URL: `http://dml.cz/dmlcz/118395`

STEINER DISTANCE IN GRAPHS

GARY CHARTRAND[1], ORTRUD R. OELLERMANN[1], SONGLIN TIAN, HUNG BIN ZOU, Kalamazoo

(Received December 10, 1987)

Summary. For a nonempty set S of vertices of a connected graph G, the distance $d(S)$ of S is the minimum size of a connected subgraph whose vertex set contains S. For integers n and p with $2 \leqq n \leqq p$, the minimum size of a graph G of order p is determined for which $d(S) = n - 1$ for all sets S of vertices of G having $|S| = n$. For a connected graph G of order p and integer n with $2 \leqq n \leqq p$, the n-eccentricity of a vertex v of G is the maximum value of $d(S)$ over all $S \subseteq V(G)$ with v in S and $|S| = n$. The minimum n-eccentricity $\text{rad}_n G$ is called the n-radius of G and the maximum n-eccentricity $\text{diam}_n G$ is its n-diameter. It is shown that $\text{diam}_n T \leqq \leqq [n/(n-1)]\, rad_n T$ for every tree T of order p with $2 \leqq n \leqq p$. For a graph G of order p the sequence $\text{diam}_2 G, \text{diam}_3 G, \ldots, \text{diam}_p G$ is called the diameter sequence of G. In the case of trees, the n-radius and n-diameter are investigated and the diameter sequences of trees are characterized.

1. INTRODUCTION

One of the most basic concepts associated with a graph is distance. In particular, if G is a connected graph and u and v are two vertices of G, then the distance $d(u, v)$ between u and v is the length of a shortest path connecting u and v. The goal of this paper is to introduce a generalization of distance and to investigate some of its properties. (See [1] for basic graph theory terminology.)

Let G be a connected graph of order at least 2 and let S be a nonempty set of vertices of G. Then the *Steiner distance* $d(S)$ among the vertices of S (or simply the distance of S) is the minimum size among all connected subgraphs whose vertex sets contain S. Note that if H is a connected subgraph of G such that $S \subseteq V(H)$ and $|E(H)| = d(S)$, then H is a tree. Such a tree has been referred to as a Steiner tree (see [3]). Further, if $S = \{u, v\}$, then $d(S) = d(u, v)$; while if $|S| = n$, then $d(S) \geqq n - 1$.

If G is the graph of Figure 1 and $S = \{u, v, x\}$, then $d(S) = 4$. There are several trees of size 4 containing S. One such tree T is also shown in Figure 1.

The usual distance defined on a connected graph G is a metric on its vertex set. As such, certain properties are satisfied. Among these are: (1) $d(u, v) \geqq 0$ for vertices u, v of G and $d(u, v) = 0$ if and only if $u = v$, and (2) $d(u, w) \leqq d(u, v) + d(v, w)$

[1]) Research supported in part by Office of Naval Research Contract N00014-88-K-0018.

for vertices u, v, w of G. There are extensions of these properties to the Steiner distance we have defined.

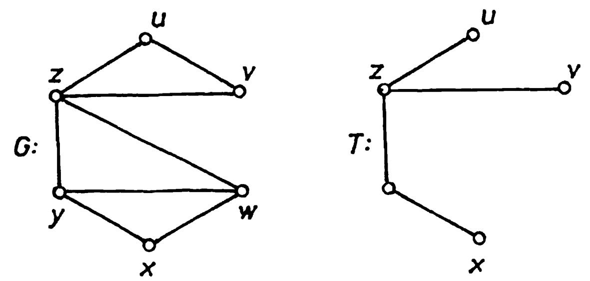

Figure 1

Let G be a connected graph and let $S \subseteq V(G)$, where $S \neq \emptyset$. Then $d(S) \geqq 0$. Further, $d(S) = 0$ if and only if $|S| = 1$. This is an extension of (1). To provide an extension of (2), let S, S_1 and S_2 be subsets of $V(G)$ such that $\emptyset \neq S \subseteq S_1 \cup S_2$ and $S_1 \cap S_2 \neq \emptyset$. Then $d(S) \leqq d(S_1) + d(S_2)$. To see this, let T_i $(i = 1, 2)$ be a tree of size $d(S_i)$ such that $S_i \subseteq V(T_i)$. Let H be the graph with vertex set $V(T_1) \cup V(T_2)$ and edge set $E(T_1) \cup E(T_2)$. Since T_1 and T_2 are connected and $V(T_1) \cap V(T_2) \neq \emptyset$, the graph H is connected. Since $S \subseteq V(H)$,

$$d(S) \leqq q(H) \leqq d(S_1) + d(S_2).$$

2. THE SIZE OF $(n; p)$ GRAPHS

Given a nonempty subset S of the vertex set of a connected graph G, the distance $d(S)$ is the minimum size of a connected graph whose vertex set contains S. Equivalently, $d(S)$ equals $|S| - 1$ plus the minimum cardinality of a subset S' of $V(G) - S$ such that $S \cup S'$ induces a connected graph. The minimum posible value for $d(S)$ is $|S| - 1$, but $d(S)$ has this value for every subset S if and only if G is complete; for otherwise, if $S^* = \{u, v\}$ consists of two nonadjacent vertices, then $d(S^*) \geqq |S^*|$. In this section, we consider the related problem of determining the minimum size of a graph G of order p having the property that $d(S) = |S| - 1$ for all subsets S of $V(G)$ with $|S| = n$ for a fixed integer $n (2 \leqq n \leqq p)$.

Let n and p be integers with $2 \leqq n \leqq p$. A graph G of order p is called an $(n; p)$ *graph* if it is of minimum size with the property that $d(S) = n - 1$ for all sets S of vertices of G with $|S| = n$. Thus our goal here is to determine the size of an $(n; p)$ graph for each pair n, p of integers with $2 \leqq n \leqq p$. For the purpose of presenting this result, we recall two basic concepts from graph theory and a theorem from the literature.

A graph G is *n-connected*, where $1 \leqq n < |V(G)|$, if the removal of fewer than n vertices from G always results in a connected graph. The kth *power* G^k of G is the graph with vertex set $V(G)$ and such that uv is an edge of G^k if and only if $d(u, v) \leqq k$ in G. We denote the cycle of order $p(\geqq 3)$ by C_p.

The following results appear within a proof of a theorem by Harary [2] and will be useful to us.

Theorem A. (Harary) (i) *If* $2 \leqq 2k = n < p$, *then* C_p^k *is n-connected.*

(ii) *Let p be an even integer satisfying* $p > n = 2k + 1 \geqq 3$. *If G is the graph obtained by joining diametrically opposite vertices of* C_p *in* C_p^k, *then G is n-connected.*

(iii) *Let p be an odd integer such that* $p > n = 2k + 1 \geqq 3$, *and let* C_p *be the cycle* $v_0, v_1, v_2, \ldots, v_{p-1}, v_0$. *If G is the graph obtained by adding* $(p + 1)/2$ *edges to* C_p^k, *namely those edges joining* v_i *and* v_j, *where* $j - i = (p - 1)/2$, *then G is n-connected.*

We precede the main result of this section by a lemma.

Lemma 1. *Let n and p be integers with* $2 \leqq n \leqq p$. *Every* $(n; p)$ *graph is* $(p - n + 1)$*-connected.*

Proof. Suppose, to the contrary, that there exists an $(n; p)$ graph G that is not $(p - n + 1)$-connected. Then there exists a vertex cutset X of cardinality $p - n$ such that $G - X$ is disconnected. Let $S = V(G) - X$. Since $|S| = n$ and $\langle S \rangle$ is disconnected, G is not an $(n; p)$ graph, producing a contradiction. □

Corollary 1. *If G is an* $(n; p)$ *graph, where* $2 \leqq n \leqq p$, *then* $\delta(G) \geqq p - n + 1$.

We are now prepared to determine the size of $(n; p)$ graphs.

Theorem 1. *Let n and p be integers with* $2 \leqq n \leqq p$. *The size of an* $(n; p)$ *graph is* $n - 1$ *if* $p = n$ *and* $[(p - n + 1)\, p/2]$ *if* $p > n$.

Proof. A graph is an $(n; n)$ graph if and only if it is a tree of order n, so that the size of such a graph is $n - 1$. Assume, then, that $p > n$. By the above corollary, if G is an $(n; p)$ graph, then $\delta(G) \geqq p - n + 1$. Therefore, if for given integers n and p, with $2 \leqq n \leqq p$, we can exhibit either a $(p - n + 1)$-regular $(n; p)$ graph or an $(n; p)$ graph all of whose vertices have degree $p - n + 1$ except one, which has degree $p - n + 2$, then the desired result follows.

Suppose first that there exists an integer $k(\geqq 2)$ such that $p = (n - 1)\, k$. Then $\overline{kK_{n-1}}$ is an appropriate $(n; p)$ graph. Hence we assume that $n - 1 \nmid p$. We may then write $p = (n - 1)\, q + r$, where $2 \leqq r \leqq n$, $r \neq n - 1$ and $q \geqq 1$. For each such integer r, we describe an $(n; n - 1 + r)$ graph H_r with the desired properties. From this, it will follow that $H_r + \overline{(q - 1) K_{n-1}}$ is an $(n; p)$ graph with the required properties and, consequently, will complete the proof.

To construct H_r, we consider two cases.

Case 1. *Assume r is even, so that* $r = 2k \geqq 2$. By Theorem A, part (i), the graph $H_r \cong C^k_{n-1+r}$ is r-connected. Let S be a set of n vertices of H_r. Since $|V(H_r) - S| = r - 1$, $\langle S \rangle$ is connected. Therefore, H_r is an $(n; n - 1 + r)$ graph with the desired properties.

Case 2. *Assume r is odd, so that* $r = 2k + 1 \geqq 3$. We consider two subcases.

Subcase 2.1. *Assume n is even.* Let H_r be the graph obtained by joining diametrically opposite vertices of C_{n-1+r} in C^k_{n-1+r}. By Theorem A, part (ii), H_r is r-connected. The proof follows as in Case 1.

Subcase 2.2. *Assume n is odd.* Let the vertices of C_{n-1+r} be labeled $v_0, v_1, \ldots, v_{n-2+r}, v_0$, and let H_r be the graph obtained by adding $(n + r)/2$ edges to C^k_{n-1+r}, namely those edges joining v_i and v_j, where $j - i = (n + r)/2$. By Theorem A, part (iii), H_r is r-connected and, again, the proof follows as in Case 1. □

3. ON THE n-RADIUS AND n-DIAMETER OF A TREE

If v is a vertex of a connected graph G, then the *eccentricity* $e(v)$ of v is defined by

$$e(v) = \max \{d(u, v) \mid u \in V(G)\} .$$

The *radius* rad G and *diameter* diam G of G are defined by

$$\text{rad } G = \min \{e(v) \mid v \in V(G)\} \quad \text{and} \quad \text{diam } G = \max \{e(v) \mid v \in V(G)\} .$$

These last two concepts are related by the inequalities rad $G \leqq$ diam $G \leqq 2$ rad G (see [1, p. 9], for example). In this section, we generalize eccentricity, radius and diameter.

Let G be a connected graph of order $p \geqq 2$ and let n be an integer with $2 \leqq n \leqq p$. The *n-eccentricity* $e_n(v)$ of a vertex v of G is defined by

$$e_n(v) = \max \{d(S) \mid S \subseteq V(G), \ |S| = n, \text{ and } v \in S\} .$$

The *n-radius* of G is

$$\text{rad}_n G = \min \{e_n(v) \mid v \in V(G)\} ,$$

while the *n-diameter* of G is

$$\text{diam}_n G = \max \{e_n(v) \mid v \in V(G)\} .$$

Note for every connected graph G that $e_2(v) = e(v)$ for all vertices v of G and that $\text{rad}_2\, G = \text{rad}\, G$ and $\text{diam}_2\, G = \text{diam}\, G$.

Each vertex of the graph G of Figure 2 is labeled with its 3-eccentricity, so that $\text{rad}_3\, G = 4$ and $\text{diam}_3\, G = 6$.

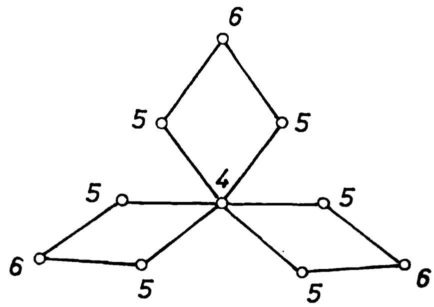

Figure 2

We now turn our attention to trees. It is useful to observe that if T is a nontrivial tree and $S \subseteq V(T)$, where $|S| \geqq 2$, then there is a unique subtree T_S of size $d(S)$ containing the vertices of S. We refer to such a tree as the *tree generated by S*. If S and S' are sets of vertices of a tree T with $S \subset S'$, then $T_S \subset T_{S'}$; otherwise, T_S contains an edge e, say, that does not belong to $T_{S'}$. Since T_S is a tree of minimum size that contains S, there exists a pair u, v of vertices in S such that the $u - v$ path in T_S contains the edge e. However, since $T_{S'}$ contains a $u - v$ path that does not contain e, there are at least two distinct $u - v$ paths in T, which is not possible since T is a tree. Hence if S is a set of vertices of a tree T and v is a vertex in $V(T) - S$, then the tree generated by $S \cup \{v\}$ contains the tree generated by S. Let w be the (necessarily unique) vertex of T_S whose distance from v is a minimum. Then $T_{S \cup \{v\}}$ contains the unique $v - w$ path and $d(S \cup \{v\}) = d(S) + d(v, w)$. If H is a subgraph of a graph G and v is a vertex of G, then $d(v, H)$ denotes the minimum distance from v to a vertex of H. Therefore, $d(S \cup \{v\}) = d(S) + d(v, T_S)$.

For a tree T, we denote by $V_1(T)$ the set of end-vertices of T and $p_1 = |V_1(T)|$. If $S = V_1(T)$, then $T_S = T$ so that $d(S) = q(T)$ and $d(S \cup \{v\}) = q(T)$ for all $v \in V(T)$. Hence if T is a tree and $n \geqq 2$ and integer with $p_1 < n$, then $e_n(v) = q(T)$ for all $v \in V(T)$. The next result considers n-eccentricities of vertices in trees T with at least n end-vertices.

Proposition 1. *Let $n \geqq 2$ be an integer and suppose that T is a tree of order p with $p_1 \geqq n$. Let $v \in V(T)$. If $S \subseteq V(T)$ such that $v \notin S$, $|S| = n - 1$ and $d(S \cup \{v\}) = e_n(v)$, then $S \subseteq V_1(T)$.*

Proof. Suppose, to the contrary, that there exists a set S of vertices of T satisfying the hypothesis of the proposition such that $S \nsubseteq V_1(T)$. Then there exists $w \in S$

such that $\deg_T w \geqq 2$. Let T_0 denote the subtree of T generated by $S_0 = S \cup \{v\}$, and let T_0' be the branch of T at w that contains v. If there exists an end-vertex x of T in a branch of T at w different from T_0' such that $x \notin S$, then

$$d((S_0 \cup \{x\}) - \{w\}) > d(S_0),$$

which produces a contradiction. Hence, there is no such end-vertex x. But then T_0 is also the tree generated by $S_1 = S_0 - \{w\}$. Let $y \in V_1(T)$ such that $y \notin S$. Then

$$d(S_1 \cup \{y\}) > d(S_0),$$

again a contradiction. □

Corollary 2. *Let $n \geqq 2$ be an integer and T a tree with $p_1 \geqq n$. Then* $\text{diam}_n T = d(S)$, *where S is a set of n end-vertices of T.*

Proof. If $n = 2$, then the result follows immediately. Assume thus that $n \geqq 3$. Suppose that $v \in V(T)$ with $e_n(v) = \text{diam}_n T$. Let S' be a set of $n - 1$ vertices such that $d(S' \cup \{v\}) = e_n(v)$. By Proposition 1, $S' \subseteq V_1(T)$. If $u \in S'$, then $e_n(u) \geqq d(S' \cup \{v\}) \geqq e_n(v)$, implying that $e_n(u) = \text{diam}_n T$. However, then it follows by Proposition 1 that $S' \cup \{v\} - \{u\}$ is a set of $n - 1$ end-vertices of T, so that $S = S' \cup \{v\}$ is a set of n end-vertices of T with $d(S) = \text{diam}_n T$. □

We now state, without proof, a basic lemma that will prove to be useful.

Lemma 2. *Let S be a set of $n \geqq 3$ end-vertices of a tree T and suppose that $v \in S$. Then $T_{S-\{v\}}$ can be obtained from T_S by deleting v and every vertex of degree 2 on a shortest path from v to a vertex of degree at least 3 in T_S.*

The next result will serve as a useful tool when proving the main theorem of this section.

Proposition 2. *Let $n \geqq 3$ be an integer and suppose that T is a tree with $p_1(\geqq n)$ end-vertices. If v is a vertex of T with $e_n(v) = \text{rad}_n T$, then there exists a set S of $n - 1$ end-vertices of T such that $d(S \cup \{v\}) = e_n(v)$ and $v \in V(T_S)$.*

Proof. Assume that the proposition is false. Then there exists a tree T that is a counterexample to the proposition and a vertex v of T for which the conclusion fails. By Proposition 1, there exists a set S of $n - 1$ end-vertices of T such that $e_n(v) = d(S \cup \{v\})$. From our assumption, it follows that S belongs to a component T_1 of $T - v$. Let u be the unique vertex of T_1 that is adjacent to v in T, and let T_2 be the component of $T - u$ containing v. Then T is decomposed into T_1, T_2 and a complete graph of order 2 whose edge is uv.

Suppose that there exists a set S' of $n - 1$ end-vertices of T_2 such that $e_n(v) = d(S' \cup \{v\})$. Let $x_1 \in S$ and $x_2 \in S'$; and define

$$S_1 = (S - \{x_1\}) \cup \{x_2\} \quad \text{and} \quad S_2 = (S' - \{x_2\}) \cup \{x_1\}.$$

Then the tree $T_{S_1 \cup S_2} = T_{S \cup S'}$ contains one more edge than the forest $T_{S \cup \{u\}} \cup T_{S' \cup \{v\}}$, namely uv. Hence

$$|E(T_{S_1 \cup S_2})| = |E(T_{S \cup \{u\}})| + |E(T_{S' \cup \{v\}})| + 1 .$$

Since each of $T_{S \cup \{u\}}$ and $T_{S' \cup \{v\}}$ has size $e_n(v) - 1$, it follows that

$$|E(T_{S_1 \cup S_2})| = |E(T_{S_1})| + |E(T_{S_2})| - |E(T_{S_1}) \cap E(T_{S_2})| = \\ = |E(T_{S \cup \{u\}})| + |E(T_{S' \cup \{v\}})| + 1 = 2e_n(v) - 1 .$$

Therefore, either $|E(T_{S_1})| \geqq e_n(v)$ or $|E(T_{S_2})| \geqq e_n(v)$, which implies that $|E(T_{S_1})| = e_n(v)$ or $|E(T_{S_2})| = e_n(v)$. However, since T_{S_1} and T_{S_2} both contain v, this contradicts our assumption about T and v. Of course, if S' is a set of $n - 1$ end-vertices such that $e_n(v) = d(S' \cup \{v\})$, then it follows from our assumption that S' cannot contain vertices from different branches of T at v. Therefore, every set S' of $n - 1$ end-vertices for which $e_n(v) = d(S' \cup \{v\})$ is contained in T_1.

Let R be a set of $n - 1$ end-vertices such that $e_n(u) = d(R \cup \{u\})$. If R is contained in T_1, then $d(R \cup \{v\}) > d(R \cup \{u\})$, contradicting the fact that v has minimum n-eccentricity. If R contains vertices from both T_1 and T_2, then

$$d(R \cup \{v\}) = d(R) = d(R \cup \{u\}) = e_n(u) \geqq e_n(v) \geqq d(R \cup \{v\}) ,$$

contrary to our assumption about T and v. Thus, R is contained in T_2.

Now,

$$e_n(u) = d(R \cup \{u\}) = d(R \cup \{v\}) + 1 .$$

As we have seen, since R is a set of $n - 1$ end-vertices contained in T_2. then $d(R \cup \{v\}) < e_n(v)$. Thus, $e_n(u) \leqq e_n(v)$ so that $e_n(u) = e_n(v)$, and u also has minimum n-eccentricity.

Let $x \in R$ and $y \in S$, and define

$$X = (R - \{x\}) \cup \{y\} \quad \text{and} \quad Y = (S - \{y\}) \cup \{x\} .$$

Then

$$\begin{aligned} 2e_n(v) &= 2d(S \cup \{v\}) = d(R \cup \{u\}) + d(S \cup \{v\}) \\ &= d((R \cup \{u\}) \cup (S \cup \{v\})) + 1 \\ &= d(R \cup S) + 1 \\ &= d(X \cup Y) + 1 \leqq d(X) + d(Y) \leqq 2e_n(v) \end{aligned}$$

since uv belongs to both T_X and T_Y. However, then, $d(X) \geqq e_n(v)$ or $d(Y) \geqq e_n(v)$, which implies that $d(X) = e_n(v)$ and $d(Y) = e_n(v)$. However, X is a set of $n - 1$ end-vertices such that T_X contains v. This again contradicts our choice of T and v. □

Corollary 3. Let $n \geqq 3$ be an integer and suppose that T is a tree with at least n end-vertices. If v is a vertex of T with $e_n(v) = \text{rad}_n\, T$, then v is not an end-vertex of T.

We now establish a relationship between the n-diameter and $(n-1)$-diameter of a tree, where $n \geqq 3$ is an integer.

Proposition 3. *Let $n \geqq 3$ be an integer and T a tree of order $p \geqq n$. Then*

$$\operatorname{diam}_{n-1} T \leqq \operatorname{diam}_n T \leqq \left(\frac{n}{n-1}\right) \operatorname{diam}_{n-1} T.$$

Proof. If S is a set of $n-1$ vertices such that $d(S) = \operatorname{diam}_{n-1} T$, then for every set S' of n vertices of T with $S \subseteq S'$, we have $\operatorname{diam}_{n-1} T = d(S) \leqq d(S') \leqq \operatorname{diam}_n T$. Hence the left inequality of the proposition follows.

To verify that $\operatorname{diam}_n T \leqq (n/(n-1)) \operatorname{diam}_{n-1} T$, we observe first that if T has at most $n-1$ end-vertices, then

$$\operatorname{diam}_{n-1} T = \operatorname{diam}_n T = p - 1,$$

so that $\operatorname{diam}_n T < (n/(n-1)) \operatorname{diam}_{n-1} T$ in this case.

Assume now that T has at least n end-vertices. By Corollary 2, there is a set S of n end-vertices such that $\operatorname{diam}_n T = d(S)$. Let $S = \{v_1, v_2, \ldots, v_n\}$ and let l_i $(1 \leqq i \leqq n)$ denote the shortest distance from v_i to a vertex of degree at least 3 in T_S.

We show now that there exists at least one i $(1 \leqq i \leqq n)$ such that $l_i \leqq (1/(n-1)) \operatorname{diam}_{n-1} T$. Suppose that $l_i > (1/(n-1)) \operatorname{diam}_{n-1} T$ for all i $(1 \leqq i \leqq n)$. Since by Lemma 2, $T_{S-\{v_n\}}$ can be obtained from T_S by deleting v_n and every vertex of degree 2 on a shortest path from v_n to a vertex having degree at least 3 in T_S, it follows that

$$q(T_{S-\{v_n\}}) \geqq \sum_{i=1}^{n-1} l_i > (n-1) \frac{1}{n-1} \operatorname{diam}_{n-1} T = \operatorname{diam}_{n-1} T.$$

This is not possible because

$$\operatorname{diam}_{n-1} T \geqq d(S - \{v_n\}) = q(T_{S-\{v_n\}}).$$

We may therefore assume that $l_n \leqq (1/(n-1)) \operatorname{diam}_{n-1} T$. Then

$$\operatorname{diam}_n T = d(S) = d(S - \{v_n\}) + d(v_n, T_{S-\{v_n\}}) \leqq$$

$$\leqq \operatorname{diam}_{n-1} T + \frac{1}{n-1} \operatorname{diam}_{n-1} T = \frac{n}{n-1} \operatorname{diam}_{n-1} T. \quad \square$$

The following proposition will aid us in deriving a relationship between the n-diameter and n-radius of a tree.

Proposition 4. *Let $n \geqq 3$ be an integer and T a tree of order $p \geqq n$. Then*

$$\operatorname{diam}_{n-1} T = \operatorname{rad}_n T.$$

Proof. If $p_1 \leqq n-1$, then $\text{rad}_n T = \text{diam}_{n-1} T = p - 1$. Assume then that $p_1 \geqq n$. We show first that $\text{rad}_n T \geqq \text{diam}_{n-1} T$. Let v be any vertex of T and S a set of $n-1$ end-vertices of T such that $d(S) = \text{diam}_{n-1} T$. Then

$$e_n(v) \geqq d(S \cup \{v\}) \geqq d(S) = \text{diam}_{n-1} T.$$

Hence $\text{rad}_n T = \min_{v \in V(T)} e_n(v) \geqq \text{diam}_{n-1} T$.

We now verify that $\text{diam}_{n-1} T \geqq \text{rad}_n T$. Let v be a vertex of T such that $e_n(v) = \text{rad}_n T$. By Proposition 2, there exists a set S of $n-1$ end-vertices of T such that $d(S) = e_n(v) = \text{rad}_n T$ and $v \in V(T_S)$. Therefore,

$$\text{diam}_{n-1} T = \max\{d(S') : |S'| = n-1,\ S' \subseteq V_1(T)\} \geqq d(S) = \text{rad}_n T.$$

Hence $\text{diam}_{n-1} T = \text{rad}_n T$. □

Corollary 4. *If $n \geqq 2$ is an integer and T a tree of order $p \geqq n$, then*

$$\text{rad}_n T \leqq \text{diam}_n T \leqq \frac{n}{n-1} \text{rad}_n T.$$

Proof. The result is well-known for $n = 2$. If $n \geqq 3$, then Propositions 3 and 4 provide the desired inequalities. □

We conjecture that Corollary 3 can be extended to any connected graph.

Conjecture. *If $n \geqq 2$ is an integer and G is a connected graph of order $p \geqq n$, then*

$$\text{rad}_n G \leqq \text{diam}_n G \leqq \frac{n}{n-1} \text{rad}_n G.$$

For a graph G of order $p \geqq 2$, the *diameter sequence* of G is defined as the sequence

$$\text{diam}_2 G, \text{diam}_3 G, \ldots, \text{diam}_p G,$$

while the *radius sequence* is the sequence

$$\text{rad}_2 G, \text{rad}_3 G, \ldots, \text{rad}_p G.$$

In order to characterize diameter sequences of trees, we first introduce an additional term and state a useful result.

Let G be a connected graph of order p. For $2 \leqq n \leqq p$, a set S consisting of n vertices of G is called an *n-diameter set* of G if $d(S) = \text{diam}_n (G)$. The following result appears in [4].

Theorem B. *Let T be a nontrivial tree with $k(\geqq 2)$ end-vertices. For every integer n with $2 \leqq n \leqq k$, there exists an n-diameter set S_n of T (consisting of n end-vertices of T) such that $S_2 \subset S_3 \subset \ldots \subset S_k$.*

We are now prepared to present the desired characterization of diameter sequences of trees.

Theorem 2. *A sequence $a_2, a_3, \ldots, a_p$ of positive integers is the diameter sequence of a tree of order p having k end-vertices if and only if*

(1) $a_{n-1} < a_n \leqq (n/(n-1))\, a_{n-1}$ *for* $3 \leqq n \leqq k$,

(2) $a_n = p - 1$ *for* $k \leqq n \leqq p$, *and*

(3) $a_{n+1} - a_n \leqq a_n - a_{n-1}$ *for* $3 \leqq n \leqq p - 1$.

Proof. Let T be a tree of order p with $k(\geqq 2)$ end-vertices and having diameter sequence $a_2, a_3, \ldots, a_p$. By Proposition 3,

$$a_{n-1} \leqq a_n \leqq \left(\frac{n}{n-1}\right) a_{n-1}$$

for $3 \leqq n \leqq k$. By Theorem B, there exists an n-diameter set S_n and an $(n-1)$-diameter set S_{n-1}, each consisting only of end-vertices of T, such that $S_{n-1} \subset S_n$; so $S_n = S_{n-1} \cup \{v\}$ for some end-vertex $v \in V(T) - S_{n-1}$. Thus,

$$a_n = \operatorname{diam}_n T = d(S_n) = d(S_{n-1} \cup \{v\}) \geqq$$
$$\geqq d(S_{n-1}) + 1 > d(S_{n-1}) = \operatorname{diam}_{n-1} T = a_{n-1},$$

which verifies (1).

If $n \geqq k$, then $\operatorname{diam}_n T = p - 1$, so that $a_k = a_{k+1} = \ldots = a_p = p - 1$ and (2) is established.

To verify (3), we again employ Theorem B. Let $a_{n-1} = d(S_{n-1})$, $a_n = d(S_n)$ and $a_{n+1} = d(S_{n+1})$, where

$$S_n = S_{n-1} \cup \{v\} \quad \text{and} \quad S_{n+1} = S_n \cup \{u\}.$$

Let T_{n-1} be the tree generated by S_{n-1} and T_n the tree generated by S_n. By the remark preceding Proposition 1,

$$d(S_n) = d(S_{n-1}) + d(v, T_{n-1}),$$

so that

$$a_n = d(S_n) = d(S_{n-1} \cup \{v\}) =$$
$$= d(S_{n-1}) + d(v, T_{n-1}) = a_{n-1} + d(v, T_{n-1}).$$

Similarly, $a_{n+1} = a_n + d(u, T_n)$. Therefore,

$$a_{n+1} - a_n = d(u, T_n) \leqq d(u, T_{n-1}) \leqq d(v, T_{n-1}) = a_n - a_{n-1},$$

which verifies (3).

For the converse, suppose that $a_2, a_3, \ldots, a_p$ is a sequence of positive integers satisfying properties (1)–(3). Let H_2 be a path of length a_2 and suppose H_2: $v_0, v_1, \ldots$

..., v_{a_2}. For $3 \leqq i \leqq k$, let H_i: $v_{i,0}, v_{i,1}, \ldots, v_{i,a_i - a_{i-1}}$ be a path of length $a_i - a_{i-1}$. Define T to be the tree obtained by identifying $v_{i,0}$ $(3 \leqq i \leqq k)$ with v_r, where $r = \lceil a_2/2 \rceil$. Then T has size

$$a_2 + (a_3 - a_2) + (a_4 - a_3) + \ldots + (a_k - a_{k-1}) = a_k = p - 1 ,$$

and therefore has order p. Further, T has diameter sequence $a_2, a_3, \ldots, a_p$. □

Corollary 5. *A sequence $a_2, a_3, \ldots, a_p$ of positive integers is the radius sequence of a tree of order $p \geqq 2$ having k end-vertices if and only if* (1) $a_{n-1} < a_n \leqq (n/(n-1))\, a_{n-1}$ *for* $3 \leqq n \leqq k + 1$, (2) $a_n = p - 1$ *for* $k + 1 \leqq n \leqq p$ *and* (3) $a_{n+1} - a_n \leqq a_n - a_{n-1}$ *for* $4 \leqq n \leqq p$.

Further work on this subject has been done in [4].

References

[1] *G. Chartrand, L. Lesniak:* Graphs & Digraphs, 2nd Edition. Monterey, CA (1986).
[2] *F. Harary:* The maximum connectivity of a graph. Proc. Acad. Sci. USA *48* (1962) 1142—1146.
[3] *R. M. Karp:* Reducibility among combinatorial problems. Complexity of Computer Computations (eds., R. E. Miller and J. W. Thatcher). Plenum Press, New York (1972) 85—103.
[4] *O. R. Oellermann, S. Tian:* Steiner centers in graphs. J. Graph Theory (to appear).

Souhrn

STEINEROVA VZDÁLENOST V GRAFECH

GARY CHARTRAND, ORTRUD R. OELLERMANN, SONGLIN TIAN, HUNG GING ZOU, Michigan

Je-li S neprázdná množina uzlů souvislého grafu G, pak vzdálenost $d(S)$ množiny S je minimální velikost souvislého podgrafu jehož množina uzlů obsahuje S. Pro celá čísla n, p, $2 \leqq n \leqq p$, je určena nejmenší velikost grafu G řádu p pro který platí $d(S) = n - 1$ pro každou množinu S uzlů grafu G, pro niž $|S| = n$. Pro souvislý graf G řádu p a celé číslo n, $2 \leqq n \leqq p$, definujeme n-excentricitu uzlu v grafu G jako maximální hodnotu $d(S)$ přes všechny $S \subseteq V(G)$, kde v leží v S a $|S| = n$. Minimální n-excentricita $\mathrm{rad}_n\, G$ se nazývá n-poloměr G, maximální n-excentricita $\mathrm{diam}_n\, G$ se nazývá jeho n-průměr. Je dokázáno, že platí $\mathrm{diam}_n\, T \leqq [n/(n-1)]\, \mathrm{rad}_n\, T$ pro každý strom řádu p, $2 \leqq n \leqq p$. Je-li G graf řádu p, pak posloupnost $\mathrm{diam}_2\, G, \mathrm{diam}_3\, G, \ldots, \mathrm{diam}_p\, G$ se nazývá posloupnost průměrů grafu G. V případě stromů jsou vyšetřovány pojmy n-poloměr, n-průměr a jsou charakterizovány posloupnosti průměrů stromů.

Резюме

РАССТОЯНИЕ ШТЕЙНЕРА В ГРАФАХ

GARY CHARTRAND, ORTRUD R. OELLERMANN, SONGLIN TIAN, HUNG BIN ZOU

Расстояние $d(S)$ непустого множества S вершин связного графа G — это минимальное число ребер связаного подграфа, множество вершин которого содержит S. В статье для

любых целых чисел $p \geqq n \geqq 2$ определено наименьшее число ребер граф G, для которого $d(S) = n - 1$ для каждого множества S его вершин мощности $|S| = n$. Для связного графа G порядка p и целого числа n, $2 \leqq n \leqq p$, определен n-эксцентриситет вершины v графа G как максимум чисел $d(S)$ для всех $S \subset V(G)$ мощности $|S| = u$, содержащих v. Минимальный n-эксцентриситет $\text{rad}_n\, G$ называется n-радиусом графа G и максимальный n-эксцентриситет $\text{diam}_n\, G$ называется n-диаметром графа G. Доказано неравенство $\text{diam}_n\, T \leqq [n(n-1)]\, \text{rad}_n\, T$ для каждого дерева порядка p и для $2 \leqq n \leqq p$. Для графа порядка p последовательность $\text{diam}_2\, G$, $\text{diam}_3\, G, \ldots, \text{diam}_p\, G$ называется последовательнсстью диаметров графа G. В случае деревьев исследуются понятия n-радиуса и n-диаметра и характеризуются последовательность диаметров деревьев.

Authors' adresses: G. Chartrand, O. R. Oellermann, S. Tian, Department of Mathematics and Statistics Western Michigan University, Kalamazoo, Michigan 49008; *H. B. Zou*, Intermagnetics General Corporation.