Kouei Sekigawa; Lieven Vanhecke

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HARMONIC MAPS AND s-REGULAR MANIFOLDS

K. SEKIGAWA, Niigata, L. VANHECKE, Leuven

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1. INTRODUCTION

The following result has been proved in [2]:
A Riemannian manifold is locally symmetric if and only if all local geodesic symmetries are harmonic.

The locally symmetric Riemannian manifolds form a subset of the class *of locally s-regular manifolds* and *generalized symmetric spaces*. These manifolds may be defined by using local diffeomorphisms which are generalizations of the geodesic symmetries. The main purpose of this note is to derive a characterization of these spaces which extends the above one for locally symmetric spaces.

2. LOCALLY s-REGULAR MANIFOLDS

Let (M, g) be a smooth connected manifold of dimension n with Riemannian metric g. Further, let T_q^p denote the algebra of all smooth tensor fields on M with contravariant and covariant orders p and q, respectively. In particular, we put $T_q^0 = T_q$ and $T_0^p = T^p$. Let ∇ denote the Riemannian connection and R the curvature tensor field on M where we define the curvature operator R_{XY} by

$$R_{XY} = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$$

for all $X, Y \in T^1$.

Any $S \in T_1^1$ may be considered as a field of endomorphisms of tangent spaces and a tensor field $P \in T_q^p$ is then called *S-invariant* if for all $\omega_1, \ldots, \omega_p \in T_1$ and $X_1, \ldots$ $\ldots, X_q \in T^1$

$$P(\omega_1 S, \ldots, \omega_p S, X_1, \ldots, X_q) = P(\omega_1, \ldots, \omega_p, SX_1, \ldots, SX_q)$$

where $(\omega S) X = \omega(SX)$ for $\omega \in T_1$ and $X \in T^1$.

Next, S is called a *symmetry tensor field* if $I - S$ is non-singular and g is S-invariant. In particular, if ∇S and $\nabla^2 S$ are S-invariant, then we say that S is *regular*.

Write $B(m, r)$ for the geodesic ball on M with center m and sufficiently small radius r. For any symmetry tensor field S on M we define a *local symmetry* s_m on $B(m, r)$ by

$$s_m = \exp_m \circ S_m \circ \exp_m^{-1} .$$

Clearly s_m is a diffeomorphism of $B(m, r)$. If $\{x^1, \ldots, x^n\}$ is a normal coordinate system on $B(m, r)$ such that $\{\partial/\partial x^1, \ldots, \partial/\partial x^n\}$ is an orthonormal basis at m, then

(1) $$x^i \circ s_m = S^i_j(m)\, x^j$$

where $S^i_j(m)$ are the natural components of S at m and form an orthogonal matrix. We denote by s the map $m \mapsto s_m$ so defined on M and note that for each $m \in M$

$$s_{m*} \mid M_m = S_m .$$

Hence S is determined uniquely by s.

We recall from [5] (see also [8]) that (M, g) together with s is called a *(Riemannian) locally s-regular manifold* if each s_m is also a *local isometry* which preserves S, that is

$$s_{m*}(SX) = S(s_{m*}X)$$

for each $m \in M$ and each vector field X defined on some neighborhood of m. Then s is called a *local regular s-structure* on (M, g). We refer to [8] for an extensive study and many nice and non-trivial examples.

For later use we state the following lemmas from [5]:

Lemma 1. *If S is regular and R and ∇R are S-invariant, then (M, g) is a locally s-regular manifold with symmetry tensor field S.*

Lemma 2. *If S is regular and the tensor field P and ∇P are S-invariant, then $\nabla^2 P$ and all higher order covariant differentials of P are S-invariant.*

3. HARMONIC MAPS

Let (M, g) and (N, h) be two Riemannian manifolds with metrics g and h and let $f: (M, g) \to (N, h)$ be a smooth map. The pullback f^*h is a semidefinite symmetric covariant tensor of order two, called the *first fundamental form*. Further, the covariant differential $\nabla(df)$ is a symmetric tensor of order two with values in $f^{-1}(TN)$, called the *second fundamental form* of f (see [3], [4]). A map with vanishing second fundamental form is said to be *totally geodesic*.

The trace of $\nabla(df)$ is denoted by $\tau(f)$ and it is called the *tension field* of f. A map with vanishing tension field is called a *harmonic* map.

If $\mathcal{U} \subset M$ is a domain with coordinates $(x^1, \ldots, x^m)$ and $\mathcal{V} \subset N$ a domain with coordinates $(y^1, \ldots, y^n)$ such $f(\mathcal{U}) \subset \mathcal{V}$, then f can be locally represented by $y^\alpha = f^\alpha(x^1, \ldots, x^m)$, $\alpha = 1, \ldots, n$. The metric tensor g is represented by $g(x) = g_{ij}(x)\, dx^i\, dx^j$, $i, j = 1, \ldots, m$, and similarly we have $h(y) = h_{\alpha\beta}(y)\, dy^\alpha\, dy^\beta$, $\alpha, \beta = 1, \ldots, n$. $df(x)$ is represented by the matrix $(\partial f^\alpha/\partial x^i)$. In this case we have

$$(fh^*)_{ij} = \frac{\partial f^\alpha}{\partial x^i} \frac{\partial f^\beta}{\partial x^j} h_{\alpha\beta} ,$$

(2) $$(\nabla(\mathrm{d}f))^{\gamma}_{ij} = \frac{\partial^2 f^{\gamma}}{\partial x^i \, \partial x^j} - {}^{M}\Gamma^{k}_{ij} \frac{\partial f^{\gamma}}{\partial x^k} + {}^{N}\Gamma^{\gamma}_{\alpha\beta} \frac{\partial f^{\alpha}}{\partial x^i} \frac{\partial f^{\beta}}{\partial x^j},$$

where ${}^{M}\Gamma^{k}_{ij}$ and ${}^{N}\Gamma^{\gamma}_{\alpha\beta}$ are the Christoffel symbols for (M, g) and (N, h), respectively.

It follows that f is harmonic if and only if

(3) $$\tau(f) = g^{ij}(\nabla(\mathrm{d}f))^{\gamma}_{ij} = 0 .$$

Now we return to the local diffeomorphisms s_m defined in Section 2. Using normal coordinates $\{x^1, \ldots, x^n\}$, (1), (2) and (3) we get at once

Theorem 3. *The local symmetries s_m, $m \in (M, g)$, are harmonic if and only if for each $p \in B(m, r)$ we have*

(4) $$\tau(s_m)^k (p) = g^{ij}(-\Gamma^{l}_{ij}(p)\, S^{k}_{l}(m) + \Gamma^{k}_{ab}(s_m(p))\, S^{a}_{i}(m)\, S^{b}_{j}(m) = 0$$

for $k = 1, 2, \ldots, n$.

We note that the Christoffel symbols Γ^{k}_{ij} are given by

(5) $$\Gamma^{k}_{ij} = \frac{1}{2} g^{kl} \left\{ \frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l} \right\}.$$

4. POWER SERIES EXPANSIONS OF g AND g^{-1} IN NORMAL COORDINATES

To prove our main result we will write down a power series expansion for the tension field $\tau(s_m)$. Therefore we need power series expansions for g and g^{-1} with respect to normal coordinates. This can be done by using the general method given in [6] (see also [1], [7]) or by using the technique of Jacobi vector fields (see [1], [9] for details and further references). We now write down the result but we delete the long but straightforward computations.

Put

$$p = \exp_m (ru)$$

where

$$ru = \sum x^i \frac{\partial}{\partial x^i}(m), \quad \|u\| = 1 .$$

Then we have

Theorem 4. *With respect to normal coordinates the metric g is given by*

(6) $$\begin{aligned} g_{ij}(p) = {} & \delta_{ij} - \tfrac{1}{3} x^a x^b R_{aibj}(m) - \tfrac{1}{6} x^a x^b x^c (\nabla_a R_{bicj})(m) + \\ & + \tfrac{1}{60} x^a x^b x^c x^d \big(-3\nabla^2_{ab} R_{cidj} + \tfrac{8}{3} \sum_s R_{aibs} R_{cjds}\big)(m) + \\ & + \tfrac{1}{90} x^a x^b x^c x^d x^k \big[-\nabla^3_{abc} R_{dikj} + 2 \sum_s (\nabla_a R_{bics} R_{djks} + \nabla_a R_{bjcs} R_{diks})\big](m) + \end{aligned}$$

$$+ \frac{1}{7!} x^a x^b x^c x^d x^k x^l [-10\nabla^4_{abcd} R_{kilj} + 34 \sum_s (\nabla^2_{ab} R_{cids} R_{kjls} + \nabla^2_{ab} R_{cjds} R_{kils}) +$$
$$+ 55 \sum_s \nabla_a R_{bics} \nabla_d R_{kjls} - 16 \sum_{s,t} R_{aibs} R_{cjdt} R_{kslt}] (m) + O(r^7) .$$

From this we get

Theorem 5. *With respect to normal coordinates g^{-1} is given by*

$$(7) \qquad g^{ij}(p) = \delta_{ij} + \tfrac{1}{3} x^a x^b R_{aibj}(m) + \tfrac{1}{6} x^a x^b x^c (\nabla_a R_{bicj})(m) +$$
$$+ \tfrac{1}{60} x^a x^b x^c x^d (3\nabla^2_{ab} R_{cidj} + 4 \sum_s R_{aibs} R_{cjds})(m) + O(r^5) .$$

5. MAIN THEOREM

We are now ready to prove the

Main Theorem. *Let (M, g) be a Riemannian manifold with a regular tensor field S. If the local symmetry s_m at each point $p \in M$ is a harmonic map, then (M, g) is a locally s-regular manifold with symmetry tensor field S, and conversely.*

Proof. First we note that the converse is trivial since any isometric map is harmonic.

To prove the direct result we put

$$(8) \qquad -\Gamma^l_{ij}(p) S^k_l(m) + \Gamma^k_{ab}(s_m(p)) S^a_i(m) S^b_j(m) = \sum_{t=1}^{5} \alpha^k_{tij}(m, u) r^t + O(r^6)$$

where $p = \exp_m(ru)$, $\|u\| = 1$. Then, using this and (7), the condition (4) gives the following necessary conditions:

A. $\sum_i \alpha^k_{1ii}(m, u) = 0;$

B. $\sum_i \alpha^k_{2ii}(m, u) = 0;$

C. $\sum_i \alpha^k_{3ii}(m, u) + \frac{1}{3} \sum_{i,j} \alpha^k_{1ij}(m, u) R_{uiuj}(m) = 0;$

D. $\sum_i \alpha^k_{4ii}(m, u) + \frac{1}{3} \sum_{i,j} \alpha^k_{2ij}(m, u) R_{uiuj}(m) + \frac{1}{6} \sum_{i,j} \alpha^k_{1ij}(m, u) (\nabla_u R_{uiuj})(m) = 0;$

E. $\sum_i \alpha^k_{5ii}(m, u) + \frac{1}{3} \sum_{i,j} \alpha^k_{3ij}(m, u) R_{uiuj}(m) + \frac{1}{6} \sum_{i,j} \alpha^k_{2ij}(m, u) (\nabla_u R_{uiuj})(m) +$
$+ \frac{1}{60} \sum_{i,j} \alpha^k_{1ij}(m, u) (3\nabla^2_{uu} R_{uiuj} + 4 \sum_s R_{uius} R_{ujus})(m) = 0 .$

Hence we need to compute α^k_{tij} for $t = 1, \dots, 5$. These computations are straightforward using (5) and (6) but they are very long. Therefore we delete the detailed expressions and consider at once the consequences of the conditions A, ..., E.

The conditions A. Here we obtain for the Ricci tensor ϱ at m

$$\varrho_{uS^{-1}k} - \varrho_{Suk} = 0$$

or equivalently,

$$\varrho_{SuSk} - \varrho_{uk} = 0\,,$$

where Sk and $S^{-1}k$, denote the vectors $S_m(\partial/\partial x^k)(m)$ and $S_m^{-1}(\partial/\partial x^k)(m)$, respectively $k = 1, \ldots, n$. Since $(\partial/\partial x^k)(m)$ and u are are arbitrary, this implies that the *Ricci tensor ϱ is S-invariant.*

The conditions B. These conditions turn out to be equivalent to

(9) $$-6\nabla_u\varrho_{S^{-1}ku} + \nabla_{S^{-1}k}\varrho_{uu} + 6\nabla_{Su}\varrho_{kSu} - \nabla_k\varrho_{SuSu} = 0\,.$$

First, put $S^{-1}k = u$. Then we get

(10) $$\nabla_u\varrho_{uu} = \nabla_{Su}\varrho_{SuSu}\,.$$

Next, put $S^{-1}k = v$ in (9). Then we have

(11) $$6\nabla_u\varrho_{uv} - \nabla_v\varrho_{uu} - 6\nabla_{Su}\varrho_{SuSv} + \nabla_{Sv}\varrho_{SuSu} = 0\,,$$

and by interchanging u and v,

(12) $$6\nabla_v\varrho_{uv} - \nabla_u\varrho_{vv} - 6\nabla_{Sv}\varrho_{SuSv} + \nabla_{Su}\varrho_{SvSv} = 0\,.$$

Further, replace u by $u + v$ in (10) and develop. This gives

(13) $$2\nabla_u\varrho_{uv} + 2\nabla_v\varrho_{uv} + \nabla_u\varrho_{vv} + \nabla_v\varrho_{uu} = $$
$$= 2\nabla_{Su}\varrho_{SuSv} + 2\nabla_{Sv}\varrho_{SuSv} + \nabla_{Su}\varrho_{SvSv} + \nabla_{Sv}\varrho_{SuSu}\,.$$

Now we put

$$A = \nabla_u\varrho_{uv} - \nabla_{Su}\varrho_{SuSv}\,, \quad A' = \nabla_v\varrho_{uv} - \nabla_{Sv}\varrho_{SuSv}\,,$$
$$B = \nabla_v\varrho_{uu} - \nabla_{Sv}\varrho_{SuSu}\,, \quad B' = \nabla_u\varrho_{vv} - \nabla_{Su}\varrho_{SvSv}\,.$$

Then we obtain from (11), (12) and (13)

$$6A - B = 0\,,$$
$$6A' - B' = 0\,,$$
$$2(A + A') + B + B' = 0$$

and so

$$A + A' = 0 = B + B'\,.$$

The last expression gives

(14) $$\nabla_v\varrho_{uu} - \nabla_{Sv}\varrho_{SuSu} = \nabla_u\varrho_{vv} - \nabla_{Su}\varrho_{SvSv}\,.$$

Finally we put $v = x + y$ in (14) and develop. This yields

$$\nabla_u \varrho_{xy} = \nabla_{Su} \varrho_{SxSy} ,$$

which implies that $\nabla\varrho$ is *S-invariant.*

Note that, since S is regular, it follows from the conditions A and B and Lemma 2 that *all the covariant derivatives $\nabla^k \varrho$ are S-invariant.*

The conditions C. Taking into account the results obtained above, these conditions become, with $S^{-1}k = u$,

$$\sum_{i,j} R_{uiuj}(R_{uiuj} - R_{SuSiSuSj}) = 0 . \tag{15}$$

To derive a useful result from (15) we proceed as follows. We identify $T_m M$ with an n-dimensional Euclidean space $\mathbb{R}^n$ via the orthonormal basis $\{(\partial/\partial x^i)(m), i = 1, \ldots, n\}$. Then the left hand side of (15), after multiplication with r^4, may be regarded as a homogeneous polynomial of order 4. Further, let D denote the Laplacian of $\mathbb{R}^n$. Then, taking twice the Laplacian in (15) we easily obtain

$$\sum_{a,b,i,j} R_{aibj}(R_{aibj} - R_{SaSiSbSj}) = 0 \tag{16}$$

and hence also

$$\sum_{a,b,i,j} (R_{aibj} - R_{SaSiSbj})^2 = 0 . \tag{17}$$

So we have from (17)

$$R_{aibj} = R_{SaSiSbSj} ,$$

which yields that *R is S-invariant.*

Remark. Instead of using the Laplacian we may also use integration over $S^{n-1}(1)$ in $T_m M$ (see [1], [7] for details).

The conditions D. Putting again $S^{-1}k = u$ and taking into account the results from conditions A, B and C we obtain

$$\sum_{i,j} R_{uiuj}(\nabla_u R_{uiuj} - \nabla_{Su} R_{SuSiSuSj}) = 0 . \tag{18}$$

(We will not need this conditions to prove the theorem.)

The conditions E. Proceeding in the same way, a long computation leads to

$$\begin{aligned} &4 \sum_{i,j} R_{uiuj}(\nabla^2_{uu} R_{uiuj} - \nabla^2_{SuSu} R_{SuSiSuSj}) + \\ &+ 22 \sum_{i,j} [(\nabla_u R_{uiuj})^2 - (\nabla_{Su} R_{SuSiSuSj})^2] - \\ &- 70 \sum_{i,j} \nabla_u R_{uiuj}(\nabla_u R_{uiuj} - \nabla_{Su} R_{SuSiSuSj}) = 0 , \end{aligned} \tag{19}$$

where we used the fact that R is S-invariant. Using again the method of the Laplacian or integration over the unit sphere, we obtain

$$\sum_{a,b,c,i,j} \nabla_a R_{bicj}(\nabla_a R_{bicj} - \nabla_{Sa} R_{SbSiScSj}) = 0$$

and hence also

$$\sum_{a,b,c,i,j} (\nabla_a R_{bicj} - \nabla_{Sa} R_{SbSiScSj})^2 = 0 .$$

This yields

$$\nabla_a R_{bicj} = \nabla_{Sa} R_{SbSiScSj} ,$$

which means that ∇R *is S-invariant.*

The required result follows now from Lemma 1.

Remark. The result about locally symmetric manifolds mentioned above follows at once from our Main Theorem by putting $S = -I$.

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Souhrn

HARMONICKÉ ZOBRAZENÍ A *s*-REGULÁRNÍ VARIETY

K. SEKIGAWA, L. VANHECKE

V [2] bylo dokázáno: Riemannovská varieta je lokálně symetrická právě když všechny lokálně geodetické symetrie jsou harmonické. Lokálně symetrické Riemannovské variety tvoří podmnožinu třídy lokálně *s*-regulárních variet a zobecněných symetrických prostorů. Tyto variety lze definovat pomocí lokálních difeomorfismů, jež jsou zobecněním geodetických symetrií. Hlavním cílem práce je odvodit charakterizaci těchto prostorů, která rozšiřuje výše uvedenou charakterizaci lokálně symetrických prostorů.

Резюме

ГАРМОНИЧЕСКИЕ ОТОБРАЖЕНИЯ И s-РЕГУЛЯРНЫЕ МНОГООБРАЗИЯ

K. Sekigawa, L. Vanhecke

В работе [2] доказано, что риманово многообразие является локально симметрическим тогда и только тогда, когда все локальные геодезические симметрии являются гармоническими отображениями. Локально симметрические римановы многообразия образуют подмножество класса всех локально s-регулярных многообразий и обобщенных симметрических пространств. Эти многообразия можно определить при помощи локальных диффереоморфизмов, обобщающих геодезические симметрии. Цель статьи — вывести характеризацию этих пространств, аналогичную приведенной выше характеризации локально симметрических пространств.

Authors' addresses: K. Sekigawa, Niigata University, Department of Mathematics, Niigata, 950-21 Japan; *L. Vanhecke,* Katholieke Universiteit Leuven, Department of Mathematics, Celestijnenlaan 200B, B-3030 Leuven, Belgium.