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THE JOIN OF GRAPHS AND THE BINDING MINIMALITY

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Summary. The important notion of the binding number was introduced by D. R. Woodall in [5]. The main theorem of Woodall's paper is a sufficient condition for the existence of a Hamiltonian circuit given in terms of the binding number. Later, other authors considered the binding number of some well-known classes of graphs and their products, see for example [3], [4]. In this paper we establish some general properties of the join of hallian graphs. Further, we study the binding minimal graphs, restricting our investigations to the join of some special graphs.

Keywords: binding number, join of graphs.

I. DEFINITIONS, NOTATION AND PRELIMINARIES

We consider only finite, undirected graphs without loops or multiple edges. Most of the concepts used in this paper can be found in [2].

For a graph $G = (V(G), E(G))$ and a vertex $x \in V(G)$ we denote by $\Gamma_G(x)$ the set of all vertices of G adjacent to x. If $X \subseteq V(G)$, then we writte $\Gamma_G(X) = \bigcup_{x \in X} \Gamma_G(x)$ or shortly $\Gamma(X)$.

Let $\mathscr{F}_G = \{X : X \subseteq V(G),\ X \neq \emptyset \text{ and } \Gamma_G(X) \neq V(G)\}$. *The binding number of* G, denoted by $\operatorname{bind}(G)$, is defined as

$$\operatorname{bind}(G) = \min_{X \in \mathscr{F}_G} \frac{|\Gamma(X)|}{|X|}.$$

Woodall calculated the binding number of graphs of some well-known classes. Let us recall some propositions of [5].

Proposition 1. *bind*$(K_n) = n - 1$, *for* $n \geqq 1$.

Proposition 2. *If* $n \geqq 3$, *then*

$$\operatorname{bind}(C_n) = \begin{cases} 1 & \text{if } n \text{ is even} \\ \dfrac{n-1}{n-2} & \text{if } n \text{ is odd}. \end{cases}$$

Proposition 3. *If* $n \geqq 1$, *then*

$$\operatorname{bind}(P_n) = \begin{cases} 1 & \text{if } n \text{ is even} \\ \dfrac{n-1}{n+1} & \text{if } n \text{ is odd} . \end{cases}$$

Proposition 4. $\operatorname{bind}(K_{m,n}) = m/n,\ m \leqq n,\ m \geqq 1,\ n \geqq 1.$

A graph G is said to be *binding minimal* if for each edge $e \in E(G)$, $\operatorname{bind}(G - e) < \operatorname{bind}(G)$.

It is not difficult to show that the following types of graphs are binding minimal: K_n for $n \geqq 2$, C_n if n is odd, P_n if n is odd, $K_{1,n}$ if $n \geqq 1$.

The join of graphs $G_1, \ldots, G_s$, $s \geqq 2$, $V(G_i) \cap V(G_j) = \emptyset$ $i \neq j$, $1 \leqq i, j \leqq s$ is a graph $G_1 + \ldots + G_s = (V(G_1) \cup \ldots \cup V(G_s),\ E(G_1) \cup \ldots \cup E(G_s) \cup \mathscr{E})$ where $\mathscr{E} = \{\{u, v\} : u \in V(G_i) \text{ and } v \in V(G_j),\ i \neq j,\ 1 \leqq i, j \leqq s\}$.

A graph G is *hallian* if $|\Gamma(X)| \geqq |X|$ holds for any set $X \subseteq V(G)$ or equivalently, if G has a (1,2)-factor. By a (1,2)-factor of G we mean a set of independent edges or vertex disjoint cycles which cover all vertices of G.

A graph G is *k-hallian* if for any set A of vertices of order at most k the subgraph of G induced by the set $V(G) - A$ is hallian. The largest k such that G is k-hallian is called *the hallian index* of G and is denoted by $h(G)$.

The vertex connectivity $\kappa(G)$ of a graph G is the minimum number of vertices whose removal results in a disconnected or trivial graph. Clearly $h(G) \leqq \delta(G) - 1$ and $\kappa(G) \leqq \delta(G)$ where $\delta(G)$ denotes the minimum degree of a vertex of G.

In our investigations we will use the following propositions of [1].

Proposition 5. [1]. *Let G be an l-connected and k-hallian graph on n vertices. Then $|\Gamma(X)| \geqq |X| + r$ for any set $X \in \mathscr{F}_G$ where $r = \min\{l, k\}$.*

Proposition 6. [1]. *Let G be a graph on n vertices. If any set $X \in \mathscr{F}_G$ satisfies $|\Gamma(X)| \geqq |X| + k$, then G is k-hallian and k-connected.*

Proposition 7. [1]. *If a graph on n vertices has $h(G) = \delta(G) - 1$ and $\kappa(G) \geqq h(G)$, then*

$$\operatorname{bind}(G) = \frac{n-1}{n - \delta(G)} .$$

Since $h(G) \leqq \delta(G) - 1$, thus the conditions $h(G) = \delta(G) - 1$ and $\kappa(G) \geqq h(G)$ are equivalent to the condition $\min\{h(G), \kappa(G)\} = \delta(G) - 1$.

Proposition 8. [1]. *Let G be a graph on n vertices. If $X \in \mathscr{F}_G$, then $|X| \leqq n - \delta(G)$.*

The main purpose of this section is to calculate the binding number of the join of graphs $G_1, \ldots, G_s$. First we show some general properties of the join of hallian graphs. The following convention will be useful in the subsequent considerations. If $G_1, \ldots, G_s$ are graphs, then $|V(G_i)| = n_i$, $\sum_{i=j}^{s} n_i = N$ and for any $1 \leqq i \leqq j \leqq s$, $n_i \leqq n_j$. The following lemma will be useful in the subsequent investigations.

Lemma 1. *Let $H = G_1 + \ldots + G_s$. Then $X \in \mathcal{F}_H$ if and only if $X \in \mathcal{F}_{G_i}$, for some i, $1 \leqq i \leqq s$.*

Proof. If $X \in \mathcal{F}_{G_i}$, $1 \leqq i \leqq s$, then the lemma is obvious. Suppose $X \in \mathcal{F}_H$ and $X \notin \mathcal{F}_{G_i}$ for any i, $1 \leqq i \leqq s$. Thus $X = \emptyset$ or $\Gamma_{G_i}(X) = V(G_i)$ or X contains vertices of at least two graphs G_i, G_j, $i \neq j$, $1 \leqq i, j \leqq s$. It is clear that $X \notin \mathcal{F}_H$ in the above cases and the lemma is proved.

Theorem 2. *Let $H = G_1 + \ldots + G_s$. For any i, $1 \leqq i \leqq s$ let $h(G_i) = k$, $\kappa(G_i) = l$. Then* $\min\{h(H), \kappa(H)\} = N - n_s + r$, *where* $r = \min\{k, l\}$.

Proof. Suppose $r = k$. Putting $A = V(G_1) \cup \ldots \cup V(G_{s-1}) \cup B$ where $B \subseteq V(G_s)$ is such that $\langle V(G_s) \setminus B \rangle$ is non hallian and $|B| = r + 1$, we obtain that $\langle V(H) \setminus A \rangle$ is non hallian. Hence $h(H) \leqq N - n_s + r$. Now, suppose $r = l$. Putting $C = V(G_1) \cup \ldots \cup V(G_{s-1}) \cup D$ where $D \subseteq V(G_s)$ is such that $\langle V(G_s) \setminus D \rangle$ is not a connected graph and $|D| = r$, we obtain that $\langle V(H) \setminus C \rangle$ is not connected. Hence $\kappa(H) \leqq N - n_s + r$.

Consequently

$$\min\{h(H), \kappa(H)\} = N - n_s + r . \tag{1}$$

Let $X \in \mathcal{F}_H$. By Lemma 1, $X \in \mathcal{F}_{G_i}$ for an arbitrary i, $1 \leqq i \leqq s$ and according to Proposition 5, $|\Gamma_{G_i}(X)| \geqq |X| + r$.

This implies $|\Gamma_H(X)| = N - n_i + |\Gamma_{G_i}(X)| \geqq N - n_i + |X| + r \geqq |X| + N - n_s + r$. Applying Proposition 6 we obtain that H is $(N - n_s + r)$-hallian and $(N - n_s + r)$-connected. This implies

$$\min\{h(H), \kappa(H)\} \geqq N - n_s + r . \tag{2}$$

By (1) and (2) the theorem is proved. □

Theorem 2 and Proposition 7 imply

Theorem 3. *Let $H = G_1 + \ldots + G_s$ where $h(G_i) = k$, $\kappa(G_i) = l$ for any i, $1 \leqq i \leqq s$. Denote* $r = \min\{k, l\}$. *If $\delta(H) = N - n_s + r + 1$, then* $\operatorname{bind}(H) = (N - 1)/(N - \delta(H))$.

Corollary 3.1. *If G_i, $1 \leqq i \leqq s$ is an 1-regular graph on n_i vertices, then* $\operatorname{bind}(H) = (N - 1)/(n_s - 1)$.

Proof. Obviously that $h(G_i) = 0$ and $\kappa(G_i) \geqq 0$, so $r = \min\{h(G_i), \kappa(G_i)\} = 0$. It is easy to observe that $\delta(H) = N - n_s + 1$, hence applying Theorem 3, we have $\operatorname{bind}(H) = (N - 1)/(n_s - 1)$. □

Corollary 3.2. *If G_i, $1 \leqq i \leqq s$ is an elementary circuit C_{n_i} where n_i is odd, $n_i \geqq 3$, then* $\operatorname{bind}(H) = (H - 1)/(n_s - 2)$.

Proof. If n_i is odd, then $h(C_{n_i}) = 1$ and $\kappa(C_{n_i}) = 2$. Observing that $\delta(H) = N - n_s + 2$, we conclude from Theorem 3 that $\operatorname{bind}(H) = (N - 1)/(n_s - 2)$. □

Similarly we show the following:

Corollary 3.3. *If G_i is an elementary path P_{n_i}, n_i is even, $n_i \geqq 2$, $1 \leqq i \leqq s$, then* $\operatorname{bind}(H) = (N - 1)/(n_s - 1)$.

Theorem 4. *Let $H = G_1 + \ldots + G_s$. If $n_i = n$, G_i is hallian with $h(G_i) = h_i$ and $\kappa(G_i) = k_i$ for each i, $1 \leqq i \leqq s$, then* $\min\{h(H), \kappa(H)\} = n(s - 1) + r$, *where* $r = \min\{h_1, \ldots, h_s, k_1, \ldots, k_s\}$.

Proof. Suppose $r = h_i$ for some i, $1 \leqq i \leqq s$. Putting $A = V(G_1) \cup \ldots \cup V(G_{i-1}) \cup V(G_{i+1}) \cup \ldots \cup V(G_s) \cup B$ where $B \subseteq V(G_i)$, $|B| = r + 1$ and $\langle V(G_i) \setminus B\rangle$ is non hallian, we get that $\langle V(H) \setminus A\rangle$ is a non hallian graph. Hence $h(H) \leqq n(s - 1) + r$. Now suppose $r = k_i$ for some i, $1 \leqq i \leqq s$. Putting $C = V(G_1) \cup \ldots \cup V(G_{i-1}) \cup V(G_{i+1}) \cup \ldots \cup V(G_s) \cup D$ where $D \subseteq V(G_i)$, $|D| = r$ and $\langle V(G_i) \setminus D\rangle$ is not connected we have that $\langle V(H) \setminus C\rangle$ is not connected. Hence $\kappa(H) \leqq n(s - 1) + r$. Thus

$$\min\{h(H), \kappa(H)\} \leqq n(s - 1) + r\,. \tag{3}$$

Let $X \in \mathscr{F}_H$. By Lemma 1, $X \in \mathscr{F}_{G_i}$ for some i, $1 \leqq i \leqq s$. Proposition 5 implies $|\Gamma_{G_i}(X)| \geqq |X| + \min\{h_i, k_i\}$. Since $|\Gamma_H(X)| = n(s - 1) + |\Gamma_{G_i}(X)|$, thus $|\Gamma_H(X)| \geqq n(s - 1) + |X| + \min\{h_i, k_i\} \geqq |X| + n(s - 1) + r$. According to Proposition 6 we obtain that H is $[n(s - 1) + r]$-hallian an $[n(s - 1) + r]$-connected. This implies

$$\min\{h(H), \kappa(H)\} \geqq n(s - 1) + r\,. \tag{4}$$

The inequalities (3) and (4) show the theorem. □

Using Theorem 4 and Proposition 7, we get the following theorem.

Theorem 5. *Let $H = G_1 + \ldots + G_s$. For any i, $1 \leqq i \leqq s$, let $n_i = n$, $h(G_i) = h_i$, $\kappa(G_i) = k_i$. If $\delta(H) = n(s - 1) + r + 1$, then* $\operatorname{bind}(H) = (ns - 1)/(n - r - 1)$, *where* $r = \min\{h_1, \ldots, h_s, k_1, \ldots, k_s\}$.

Corollary 5.1. *Let $H = G_1 + \ldots + G_s$. If $G_i = P_n$, n even, $n \geqq 4$ for $1 \leqq i \leqq \leqq t < s$ and $G_i = C_n$ for $t + 1 \leqq i \leqq s$, then* $\operatorname{bind}(H) = (ns - 1)/(n - 1)$.

Proof. Since n is even, thus $h(P_n) = 0$, $h(C_n) = 0$, $\kappa(P_n) = 1$, $\kappa(C_n) = 2$. Applying Theorem 5 with $\delta(H) = n(s - 1) + 1$, and $r = 0$ we show the corollary. □

Now we shall calculate the binding number of join graphs for which the above method cannot be used. Namely, let $H = G_1 + \ldots + G_s$ and $G_i = C_{n_i}$ where n_i is even, $n_i \geqq 4$ for any i, $1 \leqq i \leqq s$. Since $h(G_i) = 0$ and $\kappa(G_i) = 2$, we have $\min\{h(H), \kappa(H)\} = N - n_s$, by Theorem 2. Note, that $\delta(H) = N - n_s + 2$. Hence $\operatorname{bind}(H) \geqq (N - 2)/(n_s - 2)$ (see [1]).

Theorem 6. *If $H = G_1 + \ldots + G_s$, where $G_i = C_{n_i}$, n_i is even, $n_i \geqq 4$ for each i, $1 \leqq i \leqq s$, then*

$$\operatorname{bind}(H) = \begin{cases} 2s - 1 & \text{if} \quad n_i = 4 \quad \text{for each } i, \quad 1 \leqq i \leqq s \\ \dfrac{N - 1}{n_s - 2} & \text{otherwise.} \end{cases}$$

Proof. Since G_i is a O-hallian graph, thus $|\Gamma_{G_i}(X)| \geqq |X|$ for any $X \in \mathscr{F}_{G_i}$. Considering all sets $X \in \mathscr{F}_{G_i}$, we distinguish the following possibilities:

a) X is the largest stable set of vertices of G_i, thus $|\Gamma_{G_i}(X)| = |X|$, $|X| = n_i/2$,

b) X is any other set of $\mathscr{F}_{G_i}$, thus $|\Gamma_{G_i}(X)| \geqq |X| + 1$.

Now we estimate $|\Gamma_H(X)|$ for $X \in \mathscr{F}_H$, i.e., $X \in \mathscr{F}_{G_i}$ for any i, $1 \leqq i \leqq s$. In case a),

$$|\Gamma_H(X)| = N - n_i + \frac{n_i}{2} = N - \tfrac{1}{2}n_i \geqq N - \tfrac{1}{2}n_s .$$

Hence

$$\frac{|\Gamma_H(X)|}{|X|} \geqq \frac{N - \frac{1}{2}n_s}{|X|} \geqq \frac{N - \frac{1}{2}n_s}{\frac{1}{2}n_s} .$$

It is evident that the equality holds for the largest stable set of G_s.

In case b), $|\Gamma_H(X)| = N - n_i + |X| + 1 \geqq N - n_s + |X| + 1$. Hence

$$\frac{|\Gamma_H(X)|}{|X|} \geqq 1 + \frac{N - n_s + 1}{|X|} \geqq 1 + \frac{N - n_s + 1}{n_i - 2} \geqq 1 + \frac{N - n_s + 1}{n_s - 2} = \frac{N - 1}{n_s - 2} .$$

Moreover, putting $X = V(G_s) - \Gamma_{G_s}(v)$ for some $v \in V(G_s)$, we obtain the equality. The definition of the binding number implies that

$$\operatorname{bind}(H) = \min\left\{\frac{2(N - \frac{1}{2}n_s)}{n_s}, \frac{N - 1}{n_s - 2}\right\} .$$

Suppose that

$$(5) \qquad \frac{2(N - \frac{1}{2}n_s)}{n_s} \geqq \frac{N-1}{n_s - 2}$$

or equivalently

$$N\left(\frac{2}{n_s} - \frac{1}{n_s - 2}\right) \geqq 1 - \frac{1}{n_s - 2}.$$

Since $N \geqq 4 + n_s$, thus

$$(6) \qquad N\left(\frac{2}{n_s} - \frac{1}{n_s - 2}\right) \geqq (4 + n_s)\left(\frac{2}{n_s} - \frac{1}{n_s - 2}\right).$$

We easily verify that $(4 + n_s)(2/n_s - 1/(n_s - 2)) \geqq 1 - 1/(n_s - 2)$ is equivalent to $n_s \geqq 6$, and by (6) the inequality (5) is true for $n_s \geqq 6$. If $n_s = 4$, then $G_i = C_4$ for any i, $1 \leqq i \leqq s$. So, (5) can also be written as $2s - 1 \geqq 2s - \frac{1}{2}$, but this is false which completes the proof. □

Now we shall investigate the cases when the graphs G_i are non-hallian and $E(G_i) = \emptyset$ for each i, $1 \leqq i \leqq s$. In fact, $H = G_1 + \ldots + G_s$ is a complete s-partite graph.

Theorem 7. *Let be a complete s-partite graph, then*

$$\operatorname{bind}(H) = \frac{N - n_s}{n_s}.$$

Proof. Let $X \in \mathscr{F}_H$. This implies by Lemma 1 that $X \in \mathscr{F}_{G_i}$ for some i, $1 \leqq i \leqq s$. We observe that $\Gamma_{G_i}(X) = \emptyset$, hence $|\Gamma_H(X)| = N - n_i \geqq N - n_s$.

Since $|X| \leqq n_s$, we have $|\Gamma_H(X)|/|X| = (N - n_s)/|X| \geqq (N - n_s)/n_s$. Moreover putting $X = V(G)$ we have $\Gamma_H(X)/|X| = (N - n_s)/n_s$ and the theorem is proved. □

Now we shall consider the case when G_i is the graph P_{n_i}, n_i is odd, $n_i \geqq 3$ for any i, $1 \leqq i \leqq s$.

Theorem 8. *Let* $H = G_1 + \ldots + G_s$ *and* $G_i = P_{n_i}$, n_i *odd*, $n_i \geqq 3$, $1 \leqq i \leqq s$. *Then*

$$\operatorname{bind}(H) = \begin{cases} \dfrac{2N - n_s - 1}{n_s + 1} & \text{if} \quad n_s = 3 \quad \text{or} \quad n_s = 5 \quad \text{and} \quad N = 8 \\[2ex] \dfrac{N - 1}{n_s - 1} & \text{otherwise.} \end{cases}$$

Proof. Let $X \in \mathscr{F}_H$, hence $X \in \mathscr{F}_{G_i}$ for any i, $1 \leqq i \leqq s$, we have the following possibilities to consider:

1° X is the largest stable set of vertices of G_i. Thus $|X| = \frac{1}{2}(n_i + 1)$, $|\Gamma_{G_i}(X)| = \frac{1}{2}(n_i - 1)$.

$$\frac{|\Gamma_H(X)|}{|X|} = \frac{N - n_i + \frac{1}{2}(n_i - 1)}{\frac{1}{2}(n_i + 1)} = \frac{2N - n_i - 1}{n_i + 1} \geqq \frac{2N - n_s - 1}{n_s + 1}.$$

The equality holds for the largest stable set of G_s.

2° For any other set $X \in \mathcal{F}_{G_i}$, $|\Gamma_{G_i}(X)| \geqq |X|$ holds. Thus

$$\frac{|\Gamma_H(X)|}{|X|} = \frac{N - n_i + |\Gamma_{G_i}(X)|}{|X|} \geqq \frac{N - n_i}{|X|} + 1 \geqq \frac{N-1}{n_i - 1} \geqq \frac{N-1}{n_s - 1}$$

and the equality holds if $X \subseteq V(G_s)$ and $X = V(G_s) - \Gamma_{G_s}(v)$ where v is a vertex of degree 1 in G_s.

From 1° and 2° we have

$$\operatorname{bind}(H) = \min\left\{\frac{2N - n_s - 1}{n_s + 1}, \frac{N-1}{n_s - 1}\right\}.$$

Suppose that

$$\frac{2N - n_s - 1}{n_s + 1} \geqq \frac{N-1}{n_s - 1} \tag{7}$$

or simply $N(n_s - 3) \geqq n_s^2 - n_s - 2$. Since $N \geqq n_s + 3$, then $N(n_s - 3) \geqq \geqq (n_s + 3)(n_s - 3)$. The inequality $n_s^2 - 9 \geqq n_s^2 - n_s - 2$ is true for $n_s \geqq 7$. Now we consider (7) for $n_s = 3$ and $n_s = 5$. We obtain that (7) is false for $n_s = 3$ or $n_s = 5$ and $N = 8$ and (7) is true for $n_s = 5, N = 9$.

This completes the proof. □

III. THE JOIN OF GRAPHS AND THE BINDING MINIMALITY

In this section we will prove that some graphs which are join of graphs are binding minimal while some are not. In all proofs we use the following lemma which is an immediate consequence of the definition of the binding number.

Lemma 9. *Let G be a graph and* $\operatorname{bind}(G) = c$. *If $e \in E(G)$, then for any set $X \in \mathcal{F}_{G-e}$ such that $X \cap e = \emptyset$ we have*

$$\frac{|\Gamma_{G-e}(X)|}{|X|} \geqq c .$$

Theorem 10. *Let $H = G_1 + \ldots + G_s$ where G_i, $1 \leqq i \leqq s$ are 1-regular graphs. H is binding minimal if and only if $n_i = n_s$ for each i, $1 \leqq i \leqq s$.*

Proof. By Corollary 3.1, $\operatorname{bind}(H) = (N - 1)/(n_s - 1)$. Assume there exists a graph G_i, $1 \leqq i \leqq s$ such that $n_i \neq n_s$. This implies $n_i \leqq n_s - 2$. Let $e \in E(G_i)$ and $e = \{x, y\}$. According to Lemma 9 we have to consider all sets $X \in \mathcal{F}_{H-e}$ such that $X \cap e \neq \emptyset$. By Lemma 1 $X \in \mathcal{F}_{G_i - e}$, $X \cap e \neq \emptyset$ and we distinguish two possibilities:

a) $x \in X$ and $y \in X$, $|X| \leqq n_i$, or

b) either $x \in X$ or $y \in X$, hence $|X| \leqq n_i - 1$.

In case a) we obtain

$$\frac{|\Gamma_{H-e}(X)|}{|X|} = \frac{N - n_i + |\Gamma_{G_i-e}(X)|}{|X|} = \frac{N - n_i + |\Gamma_{G_i}(X)| - 2}{|X|} \geqq$$

$$\geqq 1 + \frac{N - n_i - 2}{|X|} \geqq \frac{N-2}{n_i} \geqq \frac{N-2}{n_s - 2} \geqq \frac{N-1}{n_s - 1} = \operatorname{bind}(H).$$

In case b) we have

$$\frac{|\Gamma_{H-e}(X)|}{|X|} = \frac{N - n_i + |\Gamma_{G_i-e}(X)|}{|X|} = \frac{N - n_i + |\Gamma_{G_i}(X)| - 1}{|X|} \geqq$$

$$\geqq \frac{N - n_i + |X| - 1}{|X|} \geqq 1 + \frac{N - n_i - 1}{n_i - 1} = \frac{N-2}{n_i - 1} \geqq \frac{N-2}{n_i} > \operatorname{bind}(H).$$

Finally, H is not binding minimal.

Conversely, assume that $n_i = n$ for each i, $1 \leqq i \leqq s$. In this case $\operatorname{bind}(H) = (ns - 1)/(n - 1)$.

Let $e \in E(G_i)$ for an arbitrary i, $1 \leqq i \leqq s$. Putting $X = V(G_i)$ we obtain

$$\frac{|\Gamma_{H-e}(X)|}{|X|} = \frac{ns - 2}{n} < \frac{ns - 1}{n - 1} = \operatorname{bind}(H).$$

Now let $e \in E(H)$, $e = \{x, y\}$ and $x \in V(G_i)$, $y \in V(G_j)$, $i \neq j$. If we put $X = (V(G_i) - \{x\}) \cup \{y\}$, then

$$\frac{|\Gamma_{H-e}(X)|}{|X|} = \frac{ns - 1}{n} < \operatorname{bind}(H).$$

Thus $\operatorname{bind}(H - e) < \operatorname{bind}(H)$ for each edge $e \in E(H)$, i.e., H is binding minimal and this completes the proof. □

Theorem 11. *Let H be an s-partite graph, $H = G_1 + \ldots + G_s$ where $E(G_i) = \emptyset$, $1 \leqq i \leqq s$. H is binding minimal if and only*

if $s = 2$, then $H = K_{1,N-1}$,

if $s = 3$, then $n_i = n_s$ and $n_s < s$ for each i, $2 \leqq i \leqq n_s$.

Proof. By Theorem 7, $\operatorname{bind}(H) = (N - n_s)/n_s$. Suppose that $s = 2$ and $H \neq K_{1,N-1}$. It is not difficult to see that H is not binding minimal. Further, suppose that $s \geqq 3$ and either $n_i \neq n_s$ for some i, $2 \leqq i < s$ or $n_i = n_s$ for each i, $2 \leqq i \leqq s$ and $n_s \geqq s$.

First, let $n_i < n_s$, $2 \leqq i < s$ and $e = \{x, y\}$, $x \in V(G_1)$, $y \in V(G_i)$. Considering all sets $X \in \mathscr{F}_{H-e}$ such that $X \cap e \neq \emptyset$ we have the following possibilities.

1. If $X \subseteq V(G_1)$ or $X \subseteq V(G_i)$ and $|X| \geqq 2$, then $\Gamma_{H-e}(X) = \Gamma_H(X)$, i.e., $|\Gamma_{H-e}(X)|/|X| \geqq \text{bind}(H)$.
2. If $X = \{x\}$, then $|\Gamma_{H-e}(X)|/|X| = N - n_1 - 1 \geqq N - n_s \geqq (N - n_s)/n_s = \text{bind}(H)$. If $X = \{y\}$, then $|\Gamma_{H-e}(X)|/|X| = N - n_i - 1 \geqq \text{bind}(H)$.
3. If $X = \{x, y\}$, then $|\Gamma_{H-e}(X)|/|X| = (N - 2)/2 \geqq \text{bind}(H)$.
4. If $X = X' \cup \{x\}$, $X' \subseteq V(G_i)$, $2 \leqq |X'| \leqq n_i$ or $X = X' \cup \{y\}$, $X' \subseteq V(G_1)$, $2 \leqq |X'| \leqq n_1$, then

$$\frac{|\Gamma_{H-e}(X)|}{|X|} = \frac{N-1}{|X|} \geqq \frac{N-1}{n_1+1} \geqq \frac{N-1}{n_i+1} \geqq \frac{N-1}{n_s} \geqq \frac{N-n_s}{n_s}.$$

In these cases H is not binding minimal.

Now let $3 \leqq s \leqq n_s$ and $n_i = n_s$ for each i, $2 \leqq i \leqq s$. Let $e = \{x, y\}$ and $x \in V(G_i)$, $y \in V(G_j)$ where $i \neq j$, $i \geqq 2$, $j \geqq 2$. Taking all sets $X \in \mathscr{F}_{H-e}$ such that $X \cap e \neq \emptyset$, we have the following possibilities to consider.

a) If $X \subseteq V(G_k)$ where $k = i$ or $k = j$ and $|X| \geqq 2$, then $\Gamma_{H-e}(X) = \Gamma_H(X)$, i.e., $|\Gamma_{H-e}(X)|/|X| \geqq \text{bind}(H)$.
b) If $X \subseteq V(G_k)$ where $k = i$ or $k = j$ and $|X| = 1$, then $|\Gamma_{H-e}(X)|/|X| = N - n_s - 1$. Since $N \geqq 2n_s + 1$, hence $N - n_s + 1 \geqq (N - n_s)/n_s = \text{bind}(H)$.
c) If $X = X' \cup \{x\}$ or $X = X' \cup \{y\}$ where $|X| \geqq 2$ and $X' \subseteq V(G_j)$ or $X' \subseteq V(G_i)$, respectively, then $|\Gamma_{H-e}(X)|/|X| = (N-1)/|X| \geqq (N-1)/(n_s+1)$.

Suppose that

$$\frac{N-1}{n_s+1} \geqq \frac{N-n_s}{n_s} \tag{8}$$

or equivalently $N/n_s \leqq n_s$. It is clear that $N/n_s \leqq s$. From this and the assumption that $s \leqq n_s$ we have $N/n_s \leqq s \leqq n_s$. This implies that (8) is true.

Finally, in every case there exists $e \in E(H)$ such that $|\Gamma_{H-e}(X)|/|X| = \text{bind}(H)$, i.e., H is not binding minimal.

Conversely, let $s = 2$ and $H = K_{1,N-1}$. It is obvious that H is binding minimal.

Now, let $s \geqq 3$, $H = G_1 + \ldots + G_s$, and for each i, $2 \leqq i \leqq s$ let $n_i = n$ and $n < s$. In this case $\text{bind}(H) = (N - n)/n$. It is obvious that $n_1 \leqq n$ and we distinguish two cases $n_1 < n$ or $n_1 = n$. In the first case we have to calculate $\text{bind}(H - e)$ where $e = \{x, y\}$, $x \in V(G_1)$, $y \in V(G_i)$, $2 \leqq i \leqq s$ or $e = \{x', y'\}$, $x' \in V(G_i)$ and $y' \in V(G_j)$, $i \neq j$, $2 \leqq i, j \leqq s$. Putting $X = V(G_i) \cup \{x\}$ or $X = V(G_i) \cup \{y'\}$, respectively, we obtain $|\Gamma_{H-e}(X)|/|X| = (N-1)/(n+1)$.

Suppose that

$$\frac{N-1}{n+1} < \text{bind}(H) = \frac{N-n}{n} \tag{9}$$

or simply $n^2 < N$.

By the assumption $n < s$, we obtain $n^2 < ns$. It is clear that $ns < N$. Thus $n^2 < N$ is true and (9) is true, too. If $n_1 = n_s = n$, then for any edge $e \in E(H)$, $e = \{x, y\}$,

$x \in V(G_i)$, $y \in V(G_j)$, $i \neq j$, $1 \leqq i$, $j \leqq s$, we can prove that $|\Gamma_{H-e}(X)|/X) = (N-1)/(n+1) < \text{bind}(H)$. Finally for each edge of H there exists $X \in \mathscr{F}_{H-e}$ such that $|\Gamma_{H-e}(X)|/|X| < \text{bind}(H)$, i.e., H is binding minimal. The theorem is proved. □

Theorem 12. *Let $H = G_1 + \ldots + G_s$, $s \geqq 2$ and $G_i = C_n$, where n_i is odd, $n_i \geqq 3$ for any i, $1 \leqq i \leqq s$. H is binding minimal if and only if $n_i = n_s$ for each i, $1 \leqq i \leqq s$.*

Proof. By Corollary 3.2, $\text{bind}(H) = (N-1)/(n_s - 2)$. Suppose that there exists a graph G_i, $1 \leqq i \leqq s$ such that $n_i < n_s$. This implies $n_i \leqq n_s - 2$. We choose an edge $\{x, y\} \in E(G_i)$. According to Lemma 9 we have to consider only the sets $X \in \mathscr{F}_{H-e}$ such that $X \cap e \neq \emptyset$. We observe that $G_i - e$ is a path on an odd number of vertices. By Lemma 1 it suffices to consider all $X \in \mathscr{F}_{G_i - e}$.

Two possibilities can occur:

a) X is the largest stable set of $G_i - e$. Then $|X| = \frac{1}{2}(n_i + 1)$, and $|\Gamma_{G_i-e}(X)| = \frac{1}{2}(n_i - 1)$.

b) For any other set $X \in \mathscr{F}_{G_i - e}$ we have $|\Gamma_{G_i-e}(X)| \geqq |X|$.

In a) we have

$$\frac{|\Gamma_{H-e}(X)|}{|X|} = \frac{N - n_i + \frac{1}{2}(n_i - 1)}{\frac{1}{2}(n_i + 1)} = \frac{N - \frac{1}{2}n_i - \frac{1}{2}}{\frac{1}{2}n_i + \frac{1}{2}}.$$

We have to consider the following inequality

$$\text{(10)} \qquad \frac{N - \frac{1}{2}n_i - \frac{1}{2}}{\frac{1}{2}n_i + \frac{1}{2}} \geqq \frac{N-1}{n_s - 2}.$$

Using the assumption $n_i \leqq n_s - 2$ we obtain $(N - \frac{1}{2}n_i - \frac{1}{2})(n_s - 2) \geqq (N - \frac{1}{2}n_i - 1\frac{1}{2})\, n_i$ and because the inequality $(N - \frac{1}{2}n_i - \frac{1}{2})\, n_i \geqq (N-1) \cdot (\frac{1}{2}n_i + \frac{1}{2})$ is equivalent to $N \geqq n_i + 1$, thus (10) is true, too. In case b) we have

$$\frac{|\Gamma_{H-e}(X)|}{|X|} = \frac{|\Gamma_{Gi-e}(X)| + N - n_i}{|X|} \geqq 1 + \frac{N - n_i}{|X|} \geqq 1 + \frac{N - n_i}{n_i - 1} = $$
$$= \frac{N-1}{n_i - 1} > \frac{N-1}{n_s - 2} = \text{bind}(H).$$

Consequently, H is not binding minimal.

Conversely, suppose $n_i = n_s = n$ for each i, $1 \leqq i \leqq s$. In this case $\text{bind}(H) = (ns - 1)/(n - 2)$. If $e \in E(G_i)$, $1 \leqq i \leqq s$, $e = \{x, y\}$, then putting $X = V(G_i) - \Gamma_{G_i-e}(x)$ we obtain

$$\frac{|\Gamma_{H-e}(X)|}{|X|} = \frac{n(s-1) + n - 1}{n-1} = \frac{ns-1}{n-1} < \frac{ns-1}{n-2} = \text{bind}(H).$$

If $e \in E(H)$, $e = \{x, y\}$, $x \in V(G_i)$ and $y \in V(G_j)$, $i \neq j$, then putting $X = (V(G_i) - \{x\}) \cup \{y\}$ we obtain

$$\frac{|\Gamma_{H-e}(X)|}{|X|} = \frac{ns - 1}{n - 1} < \text{bind}(H).$$

Hence H is binding minimal and the theorem is proved. □

Theorem 13. *Let $H = G_1 + \ldots + G_s$, $s \geqq 2$ and for any i, $1 \leqq i \leqq s$. Let $G_i = C_{n_i}$ where $n_i \geqq 4$, n_i even. The graph H is binding minimal if and only if $n_i = n_s$ for any i, $1 \leqq i \leqq s$.*

Proof. According to Theorem 6, $\text{bind}(H) = (N - 1)/(n_s - 2)$. Suppose that there exists a graph G_i for some $1 \leqq i < s$ such that $n_i \neq n_s$ i.e., $n_i \leqq n_s - 2$. Let $e \in E(G_i)$. Notice that $G_i - e$ is a path P_{n_i}, n_i even, hence for any set $X \in \mathscr{F}_{G_i}$, $|\Gamma_{G_i-e}(X)| \geqq |X|$ (see Proposition 3).

We have to consider all sets $X \in \mathscr{F}_{H-e}$ such that $X \cap e \neq \emptyset$. It is clear that it suffices to consider all sets $X \in \mathscr{F}_{G_i-e}$.

Hence

$$\frac{|\Gamma_{H-e}(X)|}{|X|} = \frac{N - n_i + |\Gamma_{G_i-e}(X)|}{|X|} \geqq \frac{N - n_i}{|X|} + 1.$$

Since $|X| \leqq n_i - 1$ we have

$$\frac{|\Gamma_{H-e}(X)|}{|X|} \geqq \frac{N - 1}{n_i - 1} > \frac{N - 1}{n_s - 2} = \text{bind}(H).$$

Thus H is not binding minimal.

Conversely, let $n_i = n$ for each i, $1 \leqq i \leqq s$. By Theorem 6 we have

$$\text{bind}(H) = \begin{cases} 2s - 1 & \text{if } n = 4 \\ \dfrac{N - 1}{n - 2} & \text{if } n \geqq 6. \end{cases}$$

Let $\{x, y\} = e \in E(G_i)$ for some i, $1 \leqq i \leqq s$. Observe that $G_i - e = P_n$. Putting $X = V(G_i) - \Gamma_{G_i}(x)$, we obtain

$$\frac{|\Gamma_{H-e}(X)|}{|X|} = \frac{4s - 1}{3} < 2s - 1 = \text{bind}(H) \quad \text{if} \quad n = 4$$

and

$$\frac{|\Gamma_{H-e}(X)|}{|X|} = \frac{ns - 1}{n - 1} < \frac{ns - 1}{n - 2} = \text{bind}(H) \quad \text{if} \quad n \geqq 6.$$

Let $e = \{x, y\}$ where $x \in V(G_i)$, $y \in V(G_j)$, $i \neq j$. Then $X = (V(G_i) - \{x\}) \cup \{y\}$

satisfies

$$\frac{|\Gamma_{H-e}(X)|}{|X|} = \frac{4s-1}{4} \leqq 2s - 1 = \mathrm{bind}(H) \quad \text{if} \quad n = 4$$

and

$$\frac{|\Gamma_{H-e}(X)|}{|X|} = \frac{ns-1}{n} < \frac{ns-1}{n-2} = \mathrm{bind}(H) \quad \text{if} \quad n \geqq 6\,.$$

This implies that H is binding minimal and the theorem is proved. □

Theorem 14. *Let $H = G_1 + \ldots + G_s$, $s \geqq 2$ and for any i, $1 \leqq i \leqq s$ let $G_i = P_{n_i}$ where n_i is even, $n_i \geqq 2$. The graph H is binding minimal if and only if $n_i = 2$ for any i, $1 \leqq i \leqq s$, i.e., H is the complete graph on $2s$ vertices.*

Proof. If H is the complete graph, then H is binding minimal. Conversely, by Corollary 3.3, $\mathrm{bind}(H) = (N-1)/(n_s - 1)$. Suppose there exists a graph G_i such that $n_i > 2$, so $n_s \geqq 4$.

Let $x_1, x_2, \ldots, x_{2k}$ be vertices of G_s and $\{x_i, x_{i+1}\} \in E(G_s)$ for any i, $1 \leqq i \leqq 2k - - 1$. If $e = \{x_2, x_3\}$, then $G_s - e$ is a hallian graph, hence $|\Gamma_{G_s-e}(X)| \geqq |X|$ for any $X \in \mathscr{F}_{G_s-e}$. Moreover $|X| \leqq n_s - 1$ (Proposition 8) and we obtain

$$\frac{|\Gamma_{H-e}(X)|}{|X|} = \frac{N - n_s + |\Gamma_{G_s-e}(X)|}{|X|} \geqq \frac{N-n_s}{|X|} + 1 \geqq \frac{N-1}{n_s-1} = \mathrm{bind}(H)\,.$$

It is clear that for any other set $X \in \mathscr{F}_{H-e}$ we have $\Gamma_{H-e}(X)|/|X| \geqq \mathrm{bind}(H)$ (see Lemma 9).

Thus H is not binding minimal and the proof is complete. □

Theorem 15. *Let $H = G_1 + \ldots + G_s$, $s = 2$ where $G_i = P_{n_i}$, n_i is odd, $n_i \geqq 3$, $1 \leqq i \leqq s$. Then H is not binding minimal.*

Proof. By Theorem 8,

$$\mathrm{bind}(H) = \begin{cases} \dfrac{2N - n_s - 1}{n_s + 1} & \text{if} \quad n_s = 3 \quad \text{or} \quad n_s = 5 \quad \text{and} \quad N = 8 \\[2ex] \dfrac{N-1}{n_s - 1} & \text{otherwise}\,. \end{cases}$$

Let $n_s = 3$. Then $\mathrm{bind}(H) = \frac{3}{2}s - 2$. Further, let $e \in E(H)$, $e = \{x, y\}$, $x \in V(G_i)$, $y \in V(G_j)$, $i \neq j$, and $\deg_{G_i} x = \deg_{G_j} y = 2$. Considering all $X \in \mathscr{F}_{H-e}$ such that $X \cap e \neq \emptyset$ we have the following possibilties: If $X = \{x\}$ (or $X = \{y\}$), then $|\Gamma_{H-e}(X)|/|X| = 3s - 2 > \frac{3}{2}s - 2 = \mathrm{bind}(H)$. If $X = \{x, y\}$, then $|\Gamma_{H-e}(X)|/|X| = = \frac{3}{2}s - 1 > \frac{3}{2}s - 2 = \mathrm{bind}(H)$. Thus H is not binding minimal.

Let $n_s = 5$ and $N = 8$, i.e., $n_1 = 3$, $n_2 = 5$ and $s = 2$. If $e \in E(G_1)$, then for all $X \in \mathscr{F}_{H-e}$ such that $X \cap e \neq \emptyset$ we obtain

$$\frac{|\Gamma_{H-e}(X)|}{|X|} \geqq \frac{5}{3} = \text{bind}(H).$$

Let $n_s \geqq 5$ and $N \geqq 9$. Let $e = \{x, y\} \in E(H)$ where $x \in V(G_i)$, $y \in V(G_s)$, $1 \leqq i < s$ and $\deg_{G_i} x = \deg_{G_s} y = 2$. We consider all $X \in \mathscr{F}_{H-e}$ where $X \cap e \neq \emptyset$, and we have the following possibilities:

a) If $X = \{y\}$, then $|\Gamma_{H-e}(X)| = N - n_s + 1 \geqq (N-1)/(n_s - 1) = \text{bind}(H)$.
 If $X = \{x\}$, then $|\Gamma_{H-e}(X)| = N - n_i + 1 > \text{bind}(H)$.

b) If $X = \{x, y\}$, then $|\Gamma_{H-e}(X)|/X = \frac{1}{2}(N-2)$.

Now suppose that

$$\frac{N-2}{2} \geqq \frac{N-1}{n_s - 1} = \text{bind}(H). \tag{11}$$

The inequality (11) is equivalent to $N(n_s - 3) \geqq 2n_s - 4$. By the assumption $N \geqq 9$, which yields $N(n_s - 3) \geqq 9(n_s - 3)$. Solving the inequality $9(n_s - 3) \geqq 2n_s - 4$, we obtain it is true for $n_s \geqq 5$. Hence (11) is also true for $n_s \geqq 5$, and $N \geqq 9$.

c) If $X = X' \cup \{x\}$, $X' \subseteq V(G_s)$, $2 \leqq |X'| \leqq n_s - 2$ and $y \notin \Gamma_{G_s}(X)$ or
 $X = X' \cup \{y\}$, $X' \subseteq V(G_i)$, $2 \leqq |X'| \leqq n_i - 2$ and $x \notin \Gamma_{G_i}(X)$,
 then estimating $|\Gamma_{H-e}(X)|/|X|$ we obtain $(N-1)/(n_s - 1)$ or $(N-1)/(n_i - 1)$, respectively. Clearly $|\Gamma_{H-e}(X)|/|X| = \text{bind}(H)$.

Thus H is not binding minimal. □

Theorem 16. *If $H = G_1 + \ldots + G_s$ and $G_i = P_n$ $1 \leqq i \leqq t < s$ and n is even, $n \geqq 4$, and $G_i = C_n$ for $t + 1 \leqq i \leqq s$, then H is not binding minimal.*

We omitt the proof of this theorem. Let us only notice that for an edge $e \in E(P_n)$ such that $P_n - e$ is hallian, it is not difficult to show that $|\Gamma_{H-e}(X)|/|X| \geqq \text{bind}(H)$ for any set $X \in \mathscr{F}_{H-e}$.

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Souhrn

SPOJENÍ GRAFŮ A VAZEBNÁ MINIMALITA

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Autorky vyšetřují Zykovovu sumu grafů a vazebnou minimalitu související s vazebným číslem zavedeným Woodallem. S použitím některých vlastností Hallových grafů vypočítávají toto číslo pro Zykovovu sumu s grafů dobře známých tříd ($s \geqq 2$). Dále formulující podmínky, za kterých Zykovova suma je vazebně minimální.

Резюме

СОЕДИНЕНИЕ ГРАФОВ И „МИНИМАЛЬНОСТЬ" ОТНОСИТЕЛЬНО СВЯЗЫВАЮЩЕГО ЧИСЛА

MARIA KWAŚNIK, DANUTA MICHALAK

В статье изучаются соединение графов, введенное Зыковым, и свойство „быть минимальным графом относительно связывающего числа", введённого Вудаллом. Это число определяется при помощи свойств графов Холла для соединения s ($s \geqq 2$) графов разных известных видов и приводятся условия, при которых соединение минимально.

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