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CONTINUOUS DEPENDENCE ON A PARAMETER OF SOLUTIONS OF GENERALIZED DIFFERENTIAL EQUATIONS

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Summary. In the theory of generalized differential equations an interesting convergence effect occurs which was described by J. Kurzweil as the R-emphatic convergence. Using the notion of a generalized differential equation with a substitution, so called convergence under substitution will be defined and will appear to be very similar to the R-emphatic convergence. A sequence of equations which is convergent under substitution can be transformed to another sequence of equations which converges to its limit equation in a classical way, i.e. with the uniform convergence of solutions and of right-hand sides of these equations.

Keywords: generalized differential equation, generalized differential equation with a substitution, continuous dependence on a parameter, R-emphatic convergence, convergence under substitution.

AMS classification: 34A10, 34C20.

INTRODUCTION

If we study the behavior of solutions of a sequence of ordinary differential equations

$$\frac{dx}{dt} = f(x, t) + g(x)\,\varphi_k(t)$$

where the functions φ_k "tend to the Dirac function", we find that the classical continuous dependence theorems cannot be used. J. Kurzweil investigated this problem in 1958 in his paper [K2] and introduced the so-called R-emphatic convergence of the right-hand sides of generalized differential equations

$$\frac{dx}{d\tau} = DF_k(x, t),$$

which ensures the pointwise convergence of solutions of these equations.

In this paper an auxiliary notion of the generalized differential equation with a substitution

$$x(t) = y(v(t)), \quad \frac{dy}{d\tau'} = DH(y, t')$$

is defined; it enables us to give an effective approach to the concept of the R-emphatic convergence.

1. THE GENERALIZED DIFFERENTIAL EQUATION WITH A SUBSTITUTION

1.1. Let $\mathbb{N}$ denote the set of all positive integers, let $\mathbb{R}^N$ ($N \in \mathbb{N}$) be the N-dimensional Euclidean space with the norm $|\cdot|$, $\mathbb{R}^1 = \mathbb{R}$. The symbol $(a_n)_{n=n_0}^{\infty}$ denotes a sequence.

1.2. If a function $g\colon [a, b] \to \mathbb{R}^N$, $-\infty < a < b < +\infty$, is of bounded variation, it can be written as a sum of its continuous and jump parts; these will be denoted by g^C, g^J, respectively. We assume that $g^C(a) = g(a)$, $g^J(a) = 0$.

We will write $g(t-) = \lim_{\tau \to t-} g(\tau)$, $g(t+) = \lim_{\tau \to t+} g(\tau)$, if the limits exist. The symbol $g(v(t+)+)$ denotes the same as $g(s+)$ where $s = v(t+)$. If v is an increasing function, then evidently $g(v(t+)+) = \lim_{\tau \to t+} g(v(\tau))$ provided the left-hand side has sense.

1.3. A function $x\colon [a, b] \to \mathbb{R}^N$ is called *regulated* if the onesided limits $x(t-)$ and $x(t+)$ exist and are finite for all $t \in (a, b]$ and $t \in [a, b)$, respectively. Since every regulated function $x\colon [a, b] \to \mathbb{R}^N$ is bounded, we may denote $\|x\| = \sup\{|x(t)|;\ t \in [a, b]\}$.

Let us denote by $\mathscr{R}_N[a, b]$ the normed linear space of all regulated functions from $[a, b]$ to $\mathbb{R}^N$ with the norm $\|\cdot\|$.

Then $\mathscr{R}_N[a, b]$ is a Banach space. For information about regulated functions see [F2].

1.4. A set $\mathscr{A} \subset \mathscr{R}_N[a, b]$ is called *equiregulated* if it has the following property: For every $\varepsilon > 0$ and $t_0 \in [a, b]$ there is $\delta > 0$ such that

(i) if $x \in \mathscr{A}$, $t' \in [a, b]$ and $t_0 - \delta < t' < t_0$, then $|x(t_0-) - x(t')| < \varepsilon$,

(ii) if $x \in \mathscr{A}$, $t'' \in [a, b]$ and $t_0 < t'' < t_0 + \delta$, then $|x(t'') - x(t_0+)| < \varepsilon$.

In [F2], Th. 2.18 it is proved that for a set $\mathscr{A} \subset \mathscr{R}_N[a, b]$ the following conditions are equivalent:

(i) $\mathscr{A}$ is relatively compact in $\mathscr{R}_N[a, b]$;

(ii) $\mathscr{A}$ is equiregulated and for every $t \in [a, b]$ the set $\{x(t);\ x \in \mathscr{A}\}$ is bounded;

(iii) the set $\{x(a);\ x \in \mathscr{A}\}$ is bounded and there is an increasing continuous function $\eta\colon [0, \infty) \to [0, \infty)$, $\eta(0) = 0$ and an increasing function $K\colon [a, b] \to \mathbb{R}$ such that

$$|x(t_2) - x(t_1)| \leqq \eta(K(t_2) - K(t_1)) \quad \text{for every} \quad x \in \mathscr{A}\,, \quad a \leqq t_1 < t_2 \leqq b\,.$$

1.5. In this paper we will use the generalized Perron integral, which was introduced by J. Kurzweil in [K1]. A treatise of this integral which is sufficient for our purposes can be found in [S1]. We will use the notation from [S1].

A finite sequence of numbers $A = \{\alpha_0, \tau_1, \alpha_1, \ldots, \alpha_{k-1}, \tau_k, \alpha_k\}$ is called *a partition* of the interval $[a, b]$ if

$$a = \alpha_0 < \alpha_1 < \ldots < \alpha_{k-1} < \alpha_k = b \quad \text{and} \quad \alpha_{i-1} \leqq \tau_i \leqq \alpha_i\,, \quad i = 1, 2, \ldots, k\,.$$

Given a function $\delta\colon [a, b] \to (0, \infty)$, we denote by $\mathscr{A}(\delta)$ the set of all partitions A such that

$$[\alpha_{i-1}, \alpha_i] \subset [\tau_i - \delta(\tau_i), \tau_i + \delta(\tau_i)] \quad \text{for} \quad i = 1, 2, \ldots, k\,.$$

The symbol $\mathscr{S}[a, b]$ denotes the system of all sets $S \subset [a, b] \times [a, b]$ satisfying the following condition: For every $\tau \in [a, b]$ there is $\delta(\tau) > 0$ such that $(\tau, t) \in S$ for every $t \in [a, b] \cap [\tau - \delta(\tau), \tau + \delta(\tau)]$.

Let $S \in \mathscr{S}[a, b]$, assume that a function $U\colon S \to R^N$ is given. If δ is a function on $[a, b]$ which corresponds to S then for every partition $A \in \mathscr{A}(\delta)$, $A = \{\alpha_0, \tau_1, \alpha_1, \ldots$ $\ldots, \alpha_{k-1}, \tau_k, \alpha_k\}$ the finite sum $s(U, A) = \sum_{i=1}^{k} [U(\tau_i, \alpha_i) - U(\tau_i, \alpha_{i-1})]$ is defined; $s(U, A)$ is the integral sum corresponding to the function U and the partition A.

A function $U\colon S \to R^N$, $S \in \mathscr{S}[a, b]$ is called *integrable over* $[a, b]$ if there exists $\gamma \in R^N$ such that for every $\varepsilon > 0$ there exists $\delta\colon [a, b] \to (0, \infty)$ such that for every $A \in \mathscr{A}(\delta)$ the inequality $|s(U, A) - \gamma| < \varepsilon$ holds. The element $\gamma \in R^N$ is called the generalized Perron integral of U over the interval $[a, b]$ and will be denoted by $\int_a^b DU(\tau, t)$. If $\int_a^b DU(\tau, t)$ exists then we define $\int_b^a DU(\tau, t) = -\int_a^b DU(\tau, t)$. We set $\int_a^a DU(\tau, t) = 0$ if $a = b$.

In [K1], Def. 1,1,1 and Def. 1,1,4 an equivalent definition of the generalized Perron integral is given (the equivalence is proved in Th. 1,2,1 in [K1]). This definition can be formulated as follows:

The function $U\colon S \to R$, $S \in \mathscr{S}[a, b]$ is integrable over $[a, b]$ and has the integral $\gamma \in R$ if for every $\varepsilon > 0$ there is $\delta\colon [a, b] \to (0, \infty)$ and functions $m, M\colon [a, b] \to R$ such that $\gamma - \varepsilon < m(b) - m(a) \leqq M(b) - M(a) < \gamma + \varepsilon$ and $(t - \tau)[m(t)\, m(\tau)] \leqq$ $\leqq (t - \tau)[U(\tau, t) - U(\tau, \tau)] \leqq (t - \tau)[M(t) - M(\tau)]$ for every $(\tau, t) \in S$ such that $|\tau - t| < \delta(\tau)$.

This definition will be convenient for proving the following lemma:

1.6. Lemma. *Let a function $U\colon S \to R$, $S \in \mathscr{S}[a, b]$ be given, assume that there is a nondecreasing function $h^*\colon [a, b] \to R$ which has zero continuous part, is left-continuous on $(a, b]$ and such that $|U(\tau, t) - U(\tau, \tau)| \leqq |h^*(t) - h^*(\tau)|$ for every $(\tau, t) \in S$. Then the function U is integrable over $[a, b]$ and*

$$\int_a^b DU(\tau, t) = \sum_{a \leqq t < b} [U(t, t+) - U(t, t)]\,.$$

Proof. For every $t \in [a, b)$ the limit $U(t, t+)$ exists because $|U(t, s'') - U(t, s')| \leqq$ $\leqq h^*(s'') - h^*(s')$ if $(t, s'), (t, s'') \in S$, $t < s' < s''$. Denote $\alpha_t = U(t, t+) - U(t, t)$. Owing to the estimate $|\alpha_t| \leqq h^*(t+) - h^*(t)$ the series $\sum_{a \leqq t < b} \alpha_t$ is absolutely convergent; let its sum be denoted by γ.

Let $\varepsilon > 0$ be given. There are points $a = t_1 < t_2 < \ldots < t_{k+1} = b$ such that

$$\sum_{t\in[a,b)\setminus\{t_1,\ldots,t_k\}} [h^*(t+) - h^*(t)] = \sum_{i=1}^{k} [h^*(t_{i+1}) - h^*(t_i+)] < \frac{\varepsilon}{2}.$$

Define $\delta(\tau) = \min\{|\tau - t_i|, i = 1, 2, \ldots, k+1\}$ for $\tau \in [a, b] \setminus \{t_1, t_2, \ldots, t_k\}$;

$$\delta(t_j) = \min\{|t_i - t_j|;\ i = 1, 2, \ldots, k+1,\ i \neq j\} \quad \text{for} \quad j = 1, 2, \ldots, k.$$

If we define $\chi(t) = \sum_{s\in[a,t)\setminus\{t_1,\ldots,t_k\}} [h^*(s+) - h^*(s)]$, $t \in [a, b]$ then the function χ is nondecreasing and $\chi(b) - \chi(a) < \varepsilon/2$. Let us define functions

$$m(t) = \sum_{a\leqq s<t} \alpha_s - 2\,\chi(t), \quad M(t) = \sum_{a\leqq s<t} \alpha_s + 2\,\chi(t), \quad t \in [a, b].$$

Then

$$m(b) - m(a) = \gamma - 2[\chi(b) - \chi(a)] > \gamma - \varepsilon;$$
$$M(b) - M(a) = \gamma + 2[\chi(b) - \chi(a)] < \gamma + \varepsilon.$$

If the pair (τ, t) belongs to S and $\tau < t < \tau + \delta(\tau)$ then none of the points $t_1, t_2, \ldots, t_k$ belongs to the interval (τ, t). Hence $\chi(t) - \chi(\tau) = h^*(t) - h^*(\tau)$ provided $\tau \notin \{t_1, t_2, \ldots, t_k\}$ and $\chi(t) - \chi(\tau) = h^*(t) - h^*(\tau+)$ provided $\tau = t_j$ for some $j \in \{1, 2, \ldots, k\}$. We have the inequality

$$U(\tau, t) - U(\tau, \tau) = [U(\tau, \tau+) - U(\tau, \tau)] + [U(\tau, t) - U(\tau, \tau+)] \leqq$$
$$\leqq \alpha_\tau + [h^*(t) - h^*(\tau+)] \leqq \alpha_\tau + \sum_{\tau<s<t} \alpha_s + 2[h^*(t) - h^*(\tau+)] \leqq$$
$$\leqq \sum_{\tau\leqq s<t} \alpha_s + 2[\chi(t) - \chi(\tau)] = M(t) - M(\tau).$$

Similarly it can be proved that if $(\tau, t) \in S$ and $\tau - \delta(\tau) < t < \tau$ then

$$U(\tau, \tau) - U(\tau, t) \leqq h^*(\tau) - h^*(t) \leqq M(\tau) - M(t).$$

The inequality

$$(t - \tau)[m(t) - m(\tau)] \leqq (t - \tau)[U(\tau, t) - U(\tau, \tau)],\ (\tau, t) \in S,\ |\tau - t| < \delta(\tau)$$

can be verified analogously.

1.7. In [S1], [S2] we can find basic results concerning the generalized differential equation

$$\text{(1.1)} \qquad \frac{\mathrm{d}x}{\mathrm{d}\tau} = \mathrm{D}F(x, t).$$

The function F on the right-hand side of (1.1) is a vector-valued function from G to $\mathbb{R}^N$, where G is a subset of $\mathbb{R}^{N+1}$.

An N-vector valued function x is a solution of the equation (1.1) on an interval $I \subset \mathbb{R}$, if $(x(t), t) \in G$ for all $t \in I$ and if for every $s_1, s_2 \in I$ the identity

$$x(s_2) - x(s_1) = \int_{s_1}^{s_2} DF(x(\tau), t) \tag{1.2}$$

holds. The integral used on the right-hand side of (1.2) is the generalized Perron integral of the function $U(\tau, t) = F(x(\tau), t)$.

Assume that I, I' are intervals of the form $[t_0, t_0 + \sigma]$, $[t_0, t_0 + \sigma']$ or $[t_0, t_0 + \sigma)$, $[t_0, t_0 + \sigma')$. Let x, y be solutions of the equation (1.1) on the intervals I, I', respectively. The solution y is called a continuation of x if $I \subset I'$ and if $x(t) = y(t)$ for every $t \in I$. If $I \neq I'$ then the solution y is called a proper continuation of the solution x.

Solutions to which there is no proper continuation are called maximal solutions of (1.1).

1.8. Throughout this paper let $T > 0$ be a fixed number and $\Omega \subset \mathbb{R}^N$ a fixed open set. Denote $G = \Omega \times (-T, T)$.

Assume that $h, k, l: [-T, T] \to \mathbb{R}$ are nondecreasing functions which are continuous from the left on $(-T, T]$ and continuous from the right at the point $-T$, and let $\omega: [0, \infty) \to [0, \infty)$ be a continuous increasing function such that $\omega(0) = 0$.

We will be concerned with the class $\Phi(G, k, l, \omega)$ of functions F occurring on the right-hand side of (1.1).

Definition. A function $F: G \to \mathbb{R}^N$ belongs to the class $\Phi(G, k, l, \omega)$ if

$$|F(x, t_2) - F(x, t_1)| \leqq |k(t_2) - k(t_1)| \quad \text{for all} \quad (x, t_1), (x, t_2) \in G, \tag{1.3}$$

$$|F(x, t_2) - F(x, t_1) - F(y, t_2) + F(y, t_1)| \leqq \omega(|x - y|)\,|l(t_2) - l(t_1)| \tag{1.4}$$
$$\text{for all} \quad (x, t_1), (x, t_2), (y, t_1), (y, t_2) \in G.$$

We denote $\mathscr{F}(G, h, \omega) = \Phi(G, h, h, \omega)$.

Whenever the symbol $\Phi(G, k, l, \omega)$ or $\mathscr{F}(G, h, \omega)$ is used in this paper, it will be assumed that the set G and the functions h, k, l, ω have the properties described above.

1.9. Remark. (i) In [K1], [K2] and [S1], [S2] the set $\mathscr{F}(G, h, \omega)$ is used except Chap. 5 in [S2]. For one function F this is not important since if $F \in \Phi(G, k, l, \omega)$ and we denote $h(t) = k(t) + l(t)$ then $F \in \mathscr{F}(G, h, \omega)$. Nevertheless, to distinguish the two functions k and l is of importance when one is concerned with an infinite set of such functions F.

(ii) The continuity at the endpoints of the interval $[-T, T]$ of functions h, k, l is assumed only for technical purposes. Since we will work on the set $G = \Omega \times (-T, T)$, nothing changes if e.g. a function h is only left-continuous on $(-T, T)$; it can be re-defined by the value $h((-T)+)$ at the point $-T$ and by $h(T-)$ at T.

1.10. Remark. It follows from (1.3) that for every $x \in \Omega$ the function $F(x, \cdot)$ has bounded variation on $(-T, T)$. If a function x is a solution of (1.1) on $[a, b]$ then

$$|x(t_2) - x(t_1)| \leqq k(t_2) - k(t_1), \quad a \leqq t_1 < t_2 \leqq b$$

according to Lemma 2.6 in [S1]; hence the function x has bounded variation.

1.11. Let us denote by V^- the set of all increasing functions $v: [-T, T] \to [-T, T]$ which are continuous from the left on $(-T, T]$ and continuous from the right at the point $-T$, $v(-T) = -T$, $v(T) = T$.

By Λ let us denote the set of all functions $\lambda: [-T, T] \to [-T, T]$ which are continuous and increasing on $[-T, T]$, $\Lambda(-T) = -T$, $\Lambda(T) = T$.

1.12. Definition. Assume that functions $H \in \Phi(G, k, l, \omega)$ and $v \in V^-$ are given, let $I \subset (-T, T)$ be an interval.

(i) An N-vector valued function x is a *solution* of the *generalized differential equation with a substitution*

$$x(t) = y(v(t)), \quad \frac{dy}{d\tau'} = DH(y, t') \tag{1.5}$$

on the interval I, if there exists an interval $J \subset (-T, T)$ and a solution y of the generalized differential equation

$$\frac{dy}{d\tau'} = DH(y, t') \tag{1.6}$$

on the interval J such that the equality $x(t) = y(v(t))$ holds for every $t \in I$.

(ii) We say that the solution x is a *maximal solution* of (1.5) if it has no proper continuation (defined as in 1.7).

(iii) Let x be a solution of (1.5) on $[t_0, c]$ and let $v(c) < v(c+)$. We say that x *disappears* at the point c if $x(t) = y(v(t))$ holds on $[t_0, c]$ for some maximal solution y of (1.6) which is defined on an interval J such that its right endpoint belongs to $[v(c), v(c+)]$ and $v(c) \in J$.

1.13. Remark. It is possible that a solution x disappears at a point c but it can be continued to the right. This situation occurs when there are two solutions y_1, y_2 of (1.6) on intervals J_1, J_2, respectively, such that $J_1 \supsetneqq [v(t_0), v(c+)]$, $J_2 = [v(t_0), d]$ or $[v(t_0), d)$ where $d \in (v(c), v(c+)]$, y_2 is a maximal solution of (1.6) on J_2 and $x(t) = y_1(v(t)) = y_2(v(t))$ for every $t \in [t_0, c]$.

1.14. Example. Assume that $T = 2$, $H(y, t) = yt$; $v(t) = t$ for $t \in [-2, 0]$ and $v(t) = 1 + t/2$ for $t \in (0, 2]$. By [S1], Chap. 4A the equation with substitution (1.5) can be written in the form

$$x(t) = y(v(t)), \quad \frac{dy}{ds} = y \tag{1.7}$$

and its solutions are the functions $x(t) = x_0 e^t$, $t \in [-2, 0]$, $x(t) = x_0 e^{1+t/2}$, $t \in (0, 2]$.

1.15. Definition. Assume that $F \in \Phi(G, k, l, \omega)$, $v \in A$ are given. A function $H: G \to \mathbb{R}^N$ is called *the prolongation of the function F along v*, if

$$H(x, v(t)) = F(x, t) \quad \text{for every} \quad (x, t) \in G . \tag{1.7}$$

1.16. Proposition. *Assume that $F \in \Phi(G, k, l, \omega)$ and $v \in A$ are given, let the function $H: G \to \mathbb{R}^N$ be the prolongation of F along v. Then $H \in \Phi(G, k \circ v^{-1}, l \circ v^{-1}, \omega)$.*

The proof is evident.

1.17. Theorem. *Assume that functions $F \in \Phi(G, k, l, \omega)$ and $v \in A$ are given, let $H \in \Phi(G, k', l', \omega)$ be the prolongation of F along the function v. Then the equations (1.1) and (1.5) have the same solutions.*

Proof. First assume that x is a solution of (1.1) on I and define $J = \{v(t);\ v \in I\}$, $y(t') = x(v^{-1}(t'))$ for every $t' \in J$. For every $\sigma_1, \sigma_2 \in J$ we have

$$y(\sigma_2) - y(\sigma_1) = x(v^{-1}(\sigma_2)) - x(v^{-1}(\sigma_1)) = \int_{v^{-1}(\sigma_1)}^{v^{-1}(\sigma_2)} \mathrm{D}F(x(\tau), t) .$$

By Th. 1.24 in [S1] we conclude that

$$\int_{v^{-1}(\sigma_1)}^{v^{-1}(\sigma_2)} \mathrm{D}F(x(\tau), t) = \int_{\sigma_1}^{\sigma_2} \mathrm{D}F(x(v^{-1}(\tau)), v^{-1}(t)) = \int_{\sigma_1}^{\sigma_2} \mathrm{D}H(y(\tau), t) .$$

This means that the function y is a solution of (1.6) on J and consequently the function x is a solution of (1.5) on I.

On the other hand, if the function x is a solution of (1.5) on I then there is a solution y of the equation (1.6) on J such that $x(t) = y(v(t))$ for $t \in I$. Th. 1.24 in [S1] implies that

$$x(t_2) - x(t_1) = y(v(t_2)) - y(v(t_1)) = \int_{v(t_1)}^{v(t_2)} \mathrm{D}H(y(\tau), t) =$$

$$= \int_{t_1}^{t_2} \mathrm{D}H(y(v(\tau)), v(t)) = \int_{t_1}^{t_2} \mathrm{D}F(x(\tau), t) \quad \text{for every} \quad t_1, t_2 \in I .$$

1.18. Let functions $H \in \Phi(G, k, l, \omega)$ and $v \in V^-$ be given. By $R_{(H,v)}$ we denote the set of all pairs $(x, t) \in G$ with the following properties:

(i) If v is continuous at t then $x + H(x, v(t)+) - H(x, v(t)) \in \Omega$; let us denote $p(x, t) = 0$.

(ii) if $v(t) < v(t+)$ then there exist $\delta > 0$ and a unique solution y of the initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}\tau'} = \mathrm{D}H(y, t') , \quad y(v(t)) = x \tag{1.8}$$

on the interval $[v(t), v(t+\delta)]$. Moreover,

(1.9) there exists $\varrho > 0$ such that $z \in \Omega$ for any $z \in \mathbb{R}^N$ satisfying

$$|z - y(s)| < \varrho \quad \text{for some} \quad s \in [v(t), v(t+\delta)].$$

Denote $p(x, t) = y(v(t+)+) - x$.

1.19. Proposition. *Assume that functions $H \in \Phi(G, k, l, \omega)$ and $v \in V^-$ are given. Then*

(i) $|p(x, t)| \leqq k(v(t+)+) - k(v(t))$ for every $(x, t) \in R_{(H,v)}$;

(ii) *for every $(x, t) \in G$ the series*

$$\sum_{\substack{-T<s<t \\ (x,s)\in R_{(H,v)}}} [p(x, s) - H(x, v(s+)+) + H(x, v(s))]$$

is absolutely convergent.

Proof. (i) Using Lemma 2.5 in [S1] we get the estimate

$$|p(x, t)| = |y(v(t+)+) - x| = \lim_{s\to t+} |y(v(s)) - x| = \lim_{s\to t+} \left|\int_{v(t)}^{v(s)} \mathrm{D}H(y(\tau'), t')\right| \leqq$$

$$\leqq \lim_{s\to t+} [k(v(s)) - k(v(t))] = k(v(t+)+) - k(v(t)).$$

(ii) Since the composition $k \circ v$ is a nondecreasing function, the set of all its points of discontinuity is at most countable. Hence there is a sequence $(s_j)_{j=1}^{\infty}$ of pairwise different points from $(-T, T)$ such that $k(v(t+)+) = k(v(t))$ for every $t \in (-T, T) \setminus \{s_1, s_2, \ldots\}$. For every $(x, t) \in R_{(H,v)}$ we have

$$|p(x, t) - H(x, v(t+)+) + H(x, v(t))| \leqq$$

$$\leqq |p(x, t)| + |H(x, v(t+)+) - H(x, v(t))| \leqq 2[k(v(t+)+) - k(v(t))];$$

hence

$$\sum_{(x,s)\in R_{(H,v)}} |p(x, s) - H(x, v(s+)+) + H(x, v(s))| \leqq$$

$$\leqq 2 \sum_{-T<s<T} [k(v(s+)+) - k(v(s))] = 2\sum_{j=1}^{\infty} [k(v(s_j+)+) - k(v(s_j))] \leqq$$

$$\leqq 2[k(v(T)) - k(v(-T))].$$

1.20. Definition. Assume that functions $H \in \Phi(G, k, l, \omega)$, $v \in V^-$ are given. The function

(1.10)
$$F(x, t) = H(x, v(t)) +$$

$$+ \sum_{\substack{-T<s<t \\ (x,s)\in R_{(H,v)}}} [p(x, s) - H(x, v(s+)+) + H(x, v(s))], \quad (x, t) \in G$$

is called *the reduction of the function H by the function v.*

1.21. Proposition. *Assume that the function $F: G \to \mathbb{R}^N$ is the reduction of a function $H \in \Phi(G, k, l, \omega)$ by a function $v \in V^-$. Define*

$$h(t) = 2 \sum_{\substack{-T<s<t \\ v(s)<v(s+)}} [k(v(s+)+) - k(v(s))], \quad t \in [-T, T].$$

Then

$$(1.11) \qquad |F(x, t_2) - F(x, t_1) - H(x, v(t_2)) + H(x, v(t_1))| \leq$$
$$\leq |h(t_2) - h(t_1)| \quad \text{for every} \quad (x, t_1), (x, t_2) \in G.$$

Proof. The proposition follows immediately from the proof of Prop. 1.19.

1.22. Example. Let us return to Example 1.14. In this case the reduction of the function H by the function v will have the form $F(x, t) = xt$ for $t \in [-2, 0]$, $F(x, t) = x(e - 1 + t/2)$ for $t \in (0, 2]$.

1.23. Lemma. *Let functions $H \in \Phi(G, k, l, \omega)$ and $v \in V^-$ be given, assume that F is the reduction of the function H by v. Assume that the function $y: [v(a), v(b)] \to \mathbb{R}^N$ $(-T < a < b < T)$ satisfies the following conditions:*

(i) $(y(v(t)), t) \in R_{(H,v)}$ *for every* $t \in [a, b)$;

(ii) *the function* $y \circ v$ *is regulated*;

(iii) *the integral* $\int_{v(s)}^{v(s+)} DH(y(\tau'), t')$ *exists for every* $s \in [a, b)$.

Then the integrals $\int_a^b DF(y(v(\tau)), t)$ *and* $\int_{v(a)}^{v(b)} DH(y(\tau'), t')$ *exist and the equality*

$$(1.12) \quad \int_a^b DF(y(v(\tau)), t) - \int_{v(a)}^{v(b)} DH(y(\tau'), t') = \sum_{a \leq s < b} \Big[F(y(v(s)), s+) - F(y(v(s)), s) -$$
$$- \int_{v(s)}^{v(s+)} DH(y(\tau'), t') - H(y(v(s+)), v(s+)+) + H(y(v(s+)), v(s+))\Big]$$

holds.

Proof. Since the function $H(x, v(t))$ obviously belongs to $\mathcal{F}(G, h, \omega)$ with $h(t) = k(v(t)) + l(v(t))$, the existence of the integral $\int_a^b DH(y(v(\tau)), v(t))$ follows from Corollary 2.11 in [S1].

All assumptions of the Theorem in [F1] being satisfied, the existence of $\int_a^b DH(y(v(\tau)), v(t))$ implies that the integral $\int_{v(a)}^{v(b)} DH(y(\tau'), t')$ exists and the equality

$$(1.13) \quad \int_a^b DH(y(v(\tau)), v(t)) - \int_{v(a)}^{v(b)} DH(y(\tau'), t') = \sum_{a \leq s < b} \Big[H(y(v(s)), v(s+)+) -$$
$$- H(y(v(s)), v(s)) - \int_{v(s)}^{v(s+)} DH(y(\tau'), t') - H(y(v(s+)), v(s+)+) +$$
$$+ H(y(v(s+)), v(s+))\Big]$$

holds. Let us denote $F^*(x, t) = F(x, t) - H(x, v(t))$ for $(x, t) \in G$. By Proposition 1.21 the assumptions of Lemma 1.6 are fulfilled for every component $[F^*(y(v(\tau)), t)]_j = U(\tau, t)$, $j = 1, 2, \ldots, N$ of the vector-valued function $F^*(y(v(\tau)), t)$. Consequently, the integral $\int_a^b \mathrm{D}F^*(y(v(\tau)), t)$ exists and we have

$$(1.14) \qquad \int_a^b \mathrm{D}[F(y(v(\tau)), t) - H(y(v(\tau)), v(t))] = \int_a^b \mathrm{D}F^*(y(v(\tau)), t) =$$

$$= \sum_{a \leqq s < b} [F^*(y(v(s)), s+) - F^*(y(v(s)), s)] = \sum_{a \leqq s < b} [F(y(v(s)), s+) -$$

$$- F(y(v(s)), s) - H(y(v(s)), v(s+) +) + H(y(v(s)), v(s))] .$$

From the existence of the integrals $\int_a^b \mathrm{D}F^*(y(v(\tau)), t)$ and $\int_a^b \mathrm{D}H(y(v(\tau)), v(t))$ we conclude that the integral $\int_a^b \mathrm{D}F(y(v(\tau)), t)$ exists. Combining (1.13) and (1.14) we get the equality (1.12).

1.24. Theorem. *Let functions $H \in \Phi(G, k, l, \omega)$ and $v \in V^-$ be given, assume that the function $F: G \to \mathbb{R}^N$ is the reduction of H by v. Assume that a function $x: [\alpha, \beta] \to \mathbb{R}^N$ $(-T < \alpha < \beta < T)$ is given such that $(x(t), t) \in R_{(H,v)}$ for every $t \in [\alpha, \beta)$.*

Then the function x is a solution of the equation (1.1) on $[\alpha, \beta]$ if and only if it is a solution of the equation (1.5) on $[\alpha, \beta]$.

Proof. (i) Let x be a solution of (1.1) on $[\alpha, \beta]$. By Lemma 2.6 in [S1] the function x is of bounded variation. Let us define a function $y: [v(\alpha), v(\beta)] \to \mathbb{R}^N$ in the following way:

For every σ such that $\sigma = v(t)$ for some $t \in [\alpha, \beta]$ let us define $y(\sigma) = x(t)$.

If $t \in [\alpha, \beta)$ is such that $v(t) < v(t+)$, then $(x(t), t) \in R_{(H,v)}$ by the assumption of this proposition, and therefore by 1.18 (ii) there exist $\delta_t > 0$ and an N-vector valued function y_t which is a solution of the initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}\tau'} = \mathrm{D}H(y, t'), \quad y(v(t)) = x(t)$$

on the interval $[v(t), v(t+\delta_t)]$. By 1.18 (ii) we have $p(x, t) = y_t(v(t+) +) - x$. It follows from (1.10) that $F(x(t), t+) - F(x(t), t) = p(x(t), t)$. Consequently,

$$(1.15) \quad F(x(t), t+) - F(x(t), t) = y_t(v(t+) +) - x(t) = \lim_{s \to t+} \int_{v(t)}^{v(s)} \mathrm{D}H(y_t(\tau'), t') =$$

$$= \int_{v(t)}^{v(t+)} \mathrm{D}H(y_t(\tau'), t') + H(y_t(v(t+)), v(t+) +) - H(y_t(v(t+)), v(t+))$$

(here Th. 1.15 from [S1] was used).

Now let us define $y(\sigma) = y_t(\sigma)$ for every $\sigma \in [v(t), v(t+)]$.

Lemma 1.23 implies that for every $s_1, s_2 \in [\alpha, \beta]$ the integral $\int_{v(s_1)}^{v(s_2)} \mathrm{D}H(y(\tau'), t')$ exists. By (1.15) the sum on the right-hand side of the relation (1.12) is zero if a, b are replaced by s_1, s_2. Hence

$$(1.16)\qquad \int_{v(s_1)}^{v(s_2)} \mathrm{D}H(y(\tau'), t') = \int_{s_1}^{s_2} \mathrm{D}F(y(v(\tau)), t) = \int_{s_1}^{s_2} \mathrm{D}F(x(\tau), t) =$$
$$= x(s_2) - x(s_1) = y(v(s_2)) - y(v(s_1)).$$

Assume that $v(\alpha) \leqq \sigma_1 < \sigma_2 \leqq v(\beta)$. Let us find $s_1, s_2 \in [\alpha, \beta]$ such that $v(s_i) \leqq \leqq \sigma_i \leqq v(s_i+)$, $i = 1, 2$. If $s_1 = s_2$ then we have the equality

$$(1.17)\quad y(\sigma_2) - y(\sigma_1) = y_{s_1}(\sigma_2) - y_{s_1}(\sigma_1) = \int_{\sigma_2}^{\sigma_1} \mathrm{D}H(y_{s_1}(\tau'), t') = \int_{\sigma_1}^{\sigma_2} \mathrm{D}H(y(\tau'), t').$$

If $s_1 < s_2$ then

$$(1.18)\qquad y(\sigma_2) - y(\sigma_1) = [y(\sigma_2) - y(v(s_2))] + [y(v(s_2)) - y(v(s_1))] +$$
$$+ [y(v(s_1)) - y(\sigma_1)] = [y_{s_2}(\sigma_2) - y_{s_2}(v(s_2))] +$$
$$+ [y(v(s_2)) - y(v(s_1))] + [y_{s_1}(v(s_1)) - y_{s_1}(\sigma_1)] =$$
$$= \int_{v(s_2)}^{\sigma_2} \mathrm{D}H(y_{s_2}(\tau'), t') + \int_{v(s_1)}^{v(s_2)} \mathrm{D}H(y(\tau'), t') - \int_{v(s_1)}^{\sigma_1} \mathrm{D}H(y_{s_1}(\tau'), t') =$$
$$= \int_{\sigma_1}^{\sigma_2} \mathrm{D}H(y(\tau'), t').$$

From (1.16), (1.17) and (1.18) it follows that the function y is a solution of (1.6) on the interval $[v(\alpha), v(\beta)]$, and consequently the function x is a solution of (1.5) on $[\alpha, \beta]$.

(ii) If the function x is a solution of the equation (1.5) then by Definition 1.12 there is a solution y of the equation (1.6) on $[v(\alpha), v(\beta)]$ such that $x(t) = y(v(t))$ for every $t \in [\alpha, \beta]$. Analogously as in part (i) we conclude from Lemma 1.23 that

$$x(s_2) - x(s_1) = y(v(s_2)) - y(v(s_1)) = \int_{v(s_1)}^{v(s_2)} \mathrm{D}H(y(\tau'), t') = \int_{s_1}^{s_2} \mathrm{D}F(x(\tau), t)$$

for every $s_1, s_2 \in [\alpha, \beta]$, which implies that the function x is a solution of (1.1) on $[\alpha, \beta]$.

2. CLASSICAL CONTINUOUS DEPENDENCE THEOREMS

2.1. Lemma. *Assume that a function $F \in \Phi(G, k, l, \omega)$ is given. Then for every two regulated functions $x, y\colon [\alpha, \beta] \to \Omega$ $(-T < \alpha < \beta < T)$ the inequality*

$$(2.1)\qquad \left\|\int_\alpha^\beta \mathrm{D}[F(x(\tau), t) - F(y(\tau), t)]\right\| \leqq \omega(\|x - y\|)\,(l(\beta) - l(\alpha))$$

holds.

Proof. The integral in (2.1) exists owing to Corollary 2.11 in [S1]. By (1.4) we have

$$|t - \tau| \, |F(x(\tau), t) - F(y(\tau), t) - F(x(\tau), \tau) + F(y(\tau), \tau)| \leqq$$
$$\leqq (t - \tau)\, \omega(|x(\tau) - y(\tau)|)\,(l(t) - l(\tau)) \leqq (t - \tau)\, \omega(\|x - y\|)\,(l(t) - l(\tau))$$

for every $\tau, t \in [\alpha, \beta]$. Corollary 1.18 in [S1] implies that

$$\left\|\int_\alpha^\beta \mathrm{D}[F(x(\tau), t) - F(y(\tau), t)]\right\| \leqq \int_\alpha^b \omega(\|x - y\|)\, \mathrm{d}l = \omega(\|x - y\|)\,(l(\beta) - l(\alpha)).$$

2.2. Lemma. *Let a sequence of functions* $F_n \in \Phi(G, k_n, l_n, \omega)$, $n = 0, 1, 2, \ldots$ *be given*; *assume that*

(2.2) *there is* $c > 0$ *such that*

$$l_n(T) - l_n(-T) \leqq c \quad \textit{for every} \quad n = 0, 1, 2, \ldots;$$

(2.3) $F_n(x, t) \to F_0(x, t)$ *and* $F_n(x, t+) \to F_0(x, t+)$ *for every* $(x, t) \in G$.

If $[a, b] \subset (-T, T)$ *and if a function* $\varphi\colon [a, b] \to \Omega$ *is constant on the open interval* (a, b), *then*

$$\lim \int_a^b \mathrm{D}F_n(\varphi(\tau), t) = \int_a^b \mathrm{D}F_0(\varphi(\tau), t). \tag{2.4}$$

Proof. Assume that φ has a value d on (a, b). From Th. 1.15 in [S1] we conclude that

$$\int_a^b \mathrm{D}F_n(\varphi(\tau), t) = F_n(d, b) - F_n(d, a+) + F_n(\varphi(a), a+) - F_n(\varphi(a), a) \tag{2.5}$$

for $n = 0, 1, 2, \ldots$. From (2.3) we then obtain (2.4).

2.3. Lemma. *Assume that functions* $F_n \in \Phi(G, k_n, l_n, \omega)$, $n = 0, 1, 2, \ldots$ *satisfy* (2.2), (2.3). *Then* (2.4) *holds for every finite step function* $\varphi\colon [a, b] \to \Omega$ $(-T < a < b < T)$.

Proof. Assume that φ has the form $\varphi(\tau) = d_i$ for $\tau \in (t_{i-1}, t_i)$, where $a = t_0 < t_1 < \ldots < t_m = b$. Since

$$\int_a^b \mathrm{D}F_n(\varphi(\tau), t) = \sum_{i=1}^m \int_{t_{i-1}}^{t_i} \mathrm{D}F_n(\varphi(\tau), t) \quad \text{for every} \quad n = 0, 1, 2, \ldots,$$

the relation (2.4) follows from Lemma 2.3.

2.4. Theorem. *Assume that a sequence of functions* $F_n \in \Phi(G, k_n, l_n, \omega)$, $n = 0, 1, 2, \ldots$ *satisfies* (2.2), (2.3).

Let $[\alpha, \beta] \subset (-T, T)$. *For any* $n \in \mathbb{N}$, *let* x_n *be a solution to the equation*

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \mathrm{D}F_n(x, t) \tag{2.6$_n$}$$

on $[\alpha, \beta]$. Furthermore, let us assume that x_n tend uniformly on $[\alpha, \beta]$ to such a function x_0 that $x_0(t) \in \Omega$ for any $t \in [\alpha, \beta]$.

Then the function x_0 is a solution of the equation

$$\frac{dx}{d\tau} = DF_0(x, t) \tag{2.7}$$

on the interval $[\alpha, \beta]$.

Proof. Since the functions x_n have bounded variations by [S1], Corollary 2.7, the function x_0 is regulated on $[\alpha, \beta]$; hence the integral $\int_{t_1}^{t_2} DF_0(x_0(\tau), t)$ exists for every $t_1, t_2 \in [\alpha, \beta]$. By definition of solutions of $(2.6)_n$ we have

$$x_n(t_2) - x_n(t_1) = \int_{t_1}^{t_2} DF_n(x_n(\tau), t) \quad \text{for every} \quad n \in \mathbb{N}\,.$$

If we prove that

$$\lim \int_{t_1}^{t_2} DF_n(x_n(\tau), t) = \int_{t_1}^{t_2} DF_0(x_0(\tau), t)\,,$$

we obtain the equality

$$x_0(t_2) - x_0(t_1) = \int_{t_1}^{t_2} DF_0(x_0(\tau), t)$$

for every $t_1, t_2 \in [\alpha, \beta]$, which implies that x_0 is a solution of (2.7).

Let $\varepsilon > 0$ be given; let us find $\lambda > 0$ such that $\omega(\lambda) < \varepsilon$. Since the function x_0 is regulated, there is a finite step function $\varphi\colon [\alpha, \beta] \to \Omega$ such that $\|x_0 - \varphi\| < \lambda/2$. Let n_1 be such an integer that $\|x_n - x_0\| < \lambda/2$ for every $n \geqq n_1$. By Lemma 2.3 there is $n_2 \in \mathbb{N}$ such that

$$\left|\int_{t_1}^{t_2} DF_n(\varphi(\tau), t) - \int_{t_1}^{t_2} DF_0(\varphi(\tau), t)\right| < \varepsilon$$

for every $n \geqq n_2$. Denote $n_0 = \max(n_1, n_2)$.

For arbitrary $n \geqq n_0$ the inequality $\|x_n - \varphi\| < \lambda$ holds; using Lemma 2.1 we get

$$\begin{aligned}\left|\int_{t_1}^{t_2} DF_n(x_n(\tau), t) - \int_{t_1}^{t_2} DF_0(x_0(\tau), t)\right| &\leqq \left|\int_{t_1}^{t_2} D[F_n(x_n(\tau), t) - F_n(\varphi(\tau), t)]\right| + \\ + \left|\int_{t_1}^{t_2} DF_n(\varphi(\tau), t) - \int_{t_1}^{t_2} DF_0(\varphi(\tau), t)\right| &+ \left|\int_{t_1}^{t_2} D[F_0(\varphi(\tau), t) - F_0(x_0(\tau), t)]\right| \leqq \\ \leqq \omega(\|x_n - \varphi\|)(l_n(t_2) - l_n(t_1)) &+ \varepsilon + \omega(\|\varphi - x_0\|)(l_0(t_2) - l_0(t_1)) \leqq \\ &\leqq 2\omega(\eta)\, c + \varepsilon < \varepsilon(2c + 1)\,.\end{aligned}$$

2.5. Theorem. *Assume that a sequence of functions $F_n \in \Phi(G, k_n, l_n, \omega)$, $n = 0, 1, 2, \ldots$ satisfies (2.2);*

(2.8) *there is a continuous incresing function* $\eta: [0, \infty) \to [0, \infty)$, $\eta(0) = 0$ *and an increasing function* $K: [-T, T] \to \mathbb{R}$ *which is left-continuous on* $(-T, T]$, $K(-T) = K((-T)+)$ *and such that*

$$k_n(t_2) - k_n(t_1) \leqq \eta(K(t_2) - K(t_1)) \quad \text{for every} \quad n \in \mathbb{N}, \quad -T \leqq t_1 < t_2 \leqq T;$$

$$(2.9) \qquad F_n(x, t) \to F_0(x, t) \quad \text{for every} \quad (x, t) \in G.$$

(i) *If for any* $n \in \mathbb{N}$ x_n *is a solution of* $(2.6)_n$ *on* $[\alpha, \beta]$ *and the set* $\{x_n(\alpha), n \in \mathbb{N}\}$ *is bounded, then the sequence* $(x_n)_{n=1}^{\infty}$ *contains a subsequence which is convergent uniformly on* $[\alpha, \beta]$ *to a function* $x_0 \in \mathscr{R}_N[\alpha, \beta]$.

(ii) *If* $x_0(t) \in \Omega$ *for every* $t \in [\alpha, \beta]$, *then the function* x_0 *is a solution of* (2.7) *on* $[\alpha, \beta]$.

Proof. (i) By [S1], Lemma 2.6 we get from the assumption (2.8) that $|x_n(t_2) - x_n(t_1)| \leqq k_n(t_2) - k_n(t_1) \leqq \eta(K(t_2) - K(t_1))$ for every $n \in \mathbb{N}$, $\alpha \leqq t_1 < t_2 \leqq \beta$. According to Theorem 1.4 about equiregulated sets the functions $x_1, x_2, \ldots$ are contained in a compact subset of $\mathscr{R}_N[\alpha, \beta]$; hence there is a subsequence $(x_{n_k})_{k=1}^{\infty}$ which converges uniformly on $[\alpha, \beta]$ to a function $x_0 \in \mathscr{R}_N[\alpha, \beta]$.

(ii) Since $|F_n(x, t_2) - F_n(x, t_1)| \leqq k_n(t_2) - k_n(t_1) \leqq \eta(K(t_2) - K(t_1))$, we get by 1.4 that for every $x \in \Omega$ the functions $F_n(x, \cdot)$ uniformly converge to $F_0(x, \cdot)$. This implies that (2.3) holds; now Theorem 2.4 can be used.

2.6. Theorem. *Assume that functions* $F_n \in \Phi(G, k_n, l_n, \omega)$, $n = 0, 1, 2, \ldots$ *satisfy* (2.2), (2.8) *and* (2.9).

Let an N-vector valued function x_0 *be a solution of* (2.7) *on* $[\alpha, \beta] \subset (-T, T)$ *which has the following uniqueness property*:

(2.10) *If* x *is a solution of* (2.7) *on* $[\alpha, \gamma] \subset [\alpha, \beta]$ *such that* $x(\alpha) = x_0(\alpha)$, *then* $x(t) = x_0(t)$ *for every* $t \in [\alpha, \gamma]$.

Assume further that

(2.11) *there is* $\varrho > 0$ *such that if* $y \in \mathbb{R}^N$, $s \in [\alpha, \beta]$ *and* $|y - x_0(s)| < \varrho$ *then* $y \in \Omega$.

Assume that a sequence $(y_n)_{n=1}^{\infty} \subset \mathbb{R}^N$ *is given such that* $\lim_{n \to \infty} y_n = x_0(\alpha)$.

Then there is an integer n_0 *such that for every* $n \geqq n_0$ *there exists a solution* x_n *of* $(2.6)_n$ *on* $[\alpha, \beta]$, $x_n(\alpha) = y_n$, *and* $\lim x_n(t) = x_0(t)$ *uniformly on* $[\alpha, \beta]$.

The proof is in fact the same as the proof of Theorem 2.4 in [S2], but under our assumptions which are somewhat more general it should rely on Theorem 2.5.

2.7. Corollary. *Assume that functions* $F_n \in \Phi(G, k_n, l_n, \omega)$, $n = 0, 1, 2, \ldots$ *satisfy* (2.2), (2.8) *and* (2.9). *Let* x_0 *be a solution of the equation* (2.7) *on* $[\alpha, \beta] \subset (-T, T)$ *such that* (2.10), (2.11) *hold. Then for every* $\varepsilon > 0$ *there is* $n_0 \in \mathbb{N}$ *and* $\sigma > 0$ *such* *that it holds: If x is a solution of the equation $(2.6)_n$ on $[\alpha, \beta]$ for some $n \geqq n_0$ and if $|x(\alpha) - x_0(\alpha)| < \sigma$, then $\|x - x_0\| < \varepsilon$.*

Proof. Assume that there is such $\varepsilon_0 > 0$ that for every $k \in \mathbb{N}$ there is $n_k \geqq k$ and such a solution x_k of $(2.6)_{n_k}$ on $[\alpha, \beta]$ that $|x_k(\alpha) - x_0(\alpha)| < 1/k$ and $\|x_k - x_0\| \geqq \varepsilon_0$. Then $x_k(\alpha) \to x_0(\alpha)$; by Theorem 2.6 the sequence (x_k) converges to x_0 uniformly on $[\alpha, \beta]$, which is a contradiction.

3. THE R-EMPHATIC CONVERGENCE AND THE CONVERGENCE UNDER SUBSTITUTION

3.1. The concept of R-emphatic convergence of right-hand sides of generalized differential equations

$$(3.1)_n \qquad \frac{\mathrm{d}x}{\mathrm{d}\tau} = \mathrm{D}F_n(x, t)$$

was introduced by J. Kurzweil in [K2]:

Let a set $R \subset G$ be given. Assume that for every $n = 0, 1, 2, \ldots$ a function $F_n \in \mathscr{F}(G, h_n, \omega)$ is given. The sequence $(F_n)_{n=1}^{\infty}$ *converges R-emphatically* to the function F_0, if the following conditions are fulfilled:

(3.2) $\limsup_{n \to \infty} [h_n(t_2) - h_n(t_1)] \leqq h_0(t_2) - h_0(t_1)$ if the function h_0 is continuous at t_1 and t_2, $-T < t_1 < t_2 < T$;

(3.3) there is a function $F^*: G \to \mathbb{R}^N$ such that

$$|F^*(x, t_2) - F^*(x, t_1)| \leqq |h^*(t_2) - h^*(t_1)| \quad \text{for} \quad (x, t_1), (x, t_2) \in G$$

where h^* is the jump part of the function h_0 and $\lim_{n \to \infty} F_n(x, t) = F_0(x, t) + F^*(x, t)$ if $(x, t) \in G$ and t is a point of continuity of h_0;

(3.4) for every $(x_0, t_0) \in R$ the element $x_0 + F_0(x_0, t_0+) - F_0(x_0, t_0)$ belongs to Ω; if, moreover, $h_0(t_0+) > h_0(t_0)$, then for every $\varepsilon > 0$ there is $\delta > 0$ such that for each $\delta' \in (0, \delta)$ there is $n_0 \in \mathbb{N}$ with the following property: if x is a solution of $(3.1)_n$ on $[t_0 - \delta', t_0 + \delta']$ for some $n \geqq n_0$ and if $|x(t_0 - \delta') - x_0| \leqq \delta$, then

$$|x(t_0 + \delta') - x(t_0 - \delta') - [F_0(x_0, t_0+) - F_0(x_0, t_0)]| < \varepsilon .$$

The definition of R-emphatic convergence was invented so as to cover the problem of pointwise convergence of solutions of $(3.1)_n$ to a solution of a limit equation

$$(3.5) \qquad \frac{\mathrm{d}x}{\mathrm{d}\tau} = \mathrm{D}F_0(x, t) .$$

In this chapter another type of convergence will be defined which will cover a similar convergence effect.

3.2. Definition. Assume that functions $F_n \in \Phi(G, k_n, l_n, \omega)$ are given for every $n \in \mathbb{N}$. Let functions $H: G \to \mathbb{R}^N$ and $v \in V^-$ be given such that $H(x, \cdot)$ is left-continuous on $(-T, T]$ and right-continuous at $-T$.

We say that the functions F_n *converge under substitution* to the pair (H, v) if there exists a sequence of continuous increasing functions $v_n \in \Lambda$, $n \in \mathbb{N}$ such that the following conditions hold:

(i) $v_n(t) \to v(t)$ for every $t \in (-T, T)$ such that $v(t) = v(t+)$;

(ii) there is $c > 0$ such that $l_n(T) - l_n(-T) \leqq c$ for every $n \in \mathbb{N}$;

(iii) there is a continuous increasing function $\eta: [0, \infty) \to [0, \infty)$, $\eta(0) = 0$ and an increasing function $K: [-T, T] \to \mathbb{R}$ which is left-continuous on $(-T, T]$, right-continuous at $-T$ and such that

$$(3.6) \qquad k_n(v_n^{-1}(s_2)) - k_n(v_n^{-1}(s_1)) \leqq \eta(K(s_2) - K(s_1)) \quad \text{for every} \quad n \in \mathbb{N}\,, \\ -T \leqq s_1 < s_2 \leqq T\,;$$

(iv) for every $n \in \mathbb{N}$ let us denote by H_n the prolongation of the function F_n along the function v_n; then

$$H_n(x, t) \to H(x, t) \quad \text{for every} \quad (x, t) \in G\,.$$

3.3. Proposition. *Let a sequence $F_n \in \Phi(G, k_n, l_n, \omega)$, $n \in \mathbb{N}$ converge under substitution to a pair (H, v). Then there are functions $\varkappa, \lambda$ such that $H \in \Phi(G, \varkappa, \lambda, \omega)$ and*

$$(3.7) \qquad \varkappa(s_2) - \varkappa(s_1) \leqq \eta(K(s_2) - K(s_1)) \quad \text{if} \quad -T \leqq s_1 < s_2 \leqq T\,, \\ \lambda(T) - \lambda(-T) \leqq c\,.$$

Proof. Denote $\varkappa_n(s) = k_n(v_n^{-1}(s)) - k_n(-T)$, $\lambda_n(s) = l_n(v_n^{-1}(s)) - l_n(-T)$; then $\varkappa_n(-T) = \lambda_n(-T) = 0$ and from (3.6) we get the inequality $\varkappa_n(s_2) - \varkappa_n(s_1) \leqq \leqq \eta(K(s_2) - K(s_1))$ for $n \in \mathbb{N}$, $-T \leqq s_1 < s_2 \leqq T$.

As was stated in 1.4, this inequality implies that the sequence $(\varkappa_n)_{n=1}^{\infty}$ contains a subsequence $(\varkappa_{n_k})$ which converges uniformly on $[-T, T]$ to a function $\varkappa$; the relation (3.7) obviously holds.

Since the functions λ_n are nondecreasing and bounded by the constant c, by Helly's Choice Theorem the sequence $(\lambda_{n_k})_{k=1}^{\infty}$ contains a subsequence, for simplicity denoted again by (λ_{n_k}), such that $\lambda_{n_k}(s) \to \chi(s)$ for every $s \in [-T, T]$. Define $\lambda(s) = = \chi(s-)$ for $s \in (-T, T]$, $\lambda(-T) = \chi((-T)+)$. Obviously λ is nondecreasing, $\lambda(T) - \lambda(-T) \leqq c$.

Since $H_n \in \Phi(G, \varkappa_n, \lambda_n, \omega)$ for every $n \in \mathbb{N}$, we have $|H_{n_k}(x, s_2) - H_{n_k}(x, s_1)| \leqq \leqq \varkappa_{n_k}(s_2) - \varkappa_{n_k}(s_1)$ for every $k \in \mathbb{N}$ and $-T < s_1 < s_2 < T$, $x \in \Omega$. Passing to infinity we get the inequality $|H(x, s_2) - H(x, s_1)| \leqq \varkappa(s_2) - \varkappa(s_1)$.

Similarly $|H(x, s_2) - H(x, s_1) - H(y, s_2) + H(y, s_1)| \leqq \omega(|x - y|)\,(\chi(s_2) - \chi(s_1))$ provided $x, y \in \Omega$, $-T < s_1 < s_2 < T$. If the function χ is left-continuous at the points s_1 and s_2 then $\lambda(s_1) = \chi(s_1)$, $\lambda(s_2) = \chi(s_2)$, hence the inequality

(3.8) $$|H(x, s_2) - H(x, s_1) - H(y, s_2) + H(y, s_1)| \leqq \omega(|x - y|)(\lambda(s_2) - \lambda(s_1))$$

holds. Since the functions $H(x, \cdot)$, $H(y, \cdot)$, λ are left-continuous on $(-T, T]$ and right-continuous at $-T$, we conclude that (3.8) holds for arbitrary s_1, s_2, $-T \leqq s_1 < < s_2 \leqq T$.

3.4. Proposition. *Assume that a sequence $F_n \in \Phi(G, k_n, l_n, \omega)$, $n \in \mathbb{N}$ converges under substitution to a pair (H, v).*

For every $(x, t) \in G$ let us define $F(x, t) = H(x, v(t))$. Then $F_n(x, t) \to F(x, t)$ for every $(x, t) \in G$ such that the function $K \circ v$ is continuous at t (the notation from Definition 3.2 is used).

Proof. If the function $K \circ v$ is continuous at t then v is continuous at t and K is continuous at $v(t)$.

Let $\varepsilon > 0$ be given. There is $\xi > 0$ such that $\omega(\xi) < \varepsilon$, further there is $\delta > 0$ such that $|K(s) - K(v(t))| < \xi$ for every $s \in [-T, T]$ such that $|s - v(t)| < \delta$.

There is an integer $n_0 \in \mathbb{N}$ such that $|v_n(t) - v(t)| < \delta$ and $|H_n(x, v(t)) - - H(x, v(t))| < \varepsilon$ for every $n \geqq n_0$.

We have the estimate

$$\begin{aligned} |F_n(x, t) - F(x, t)| &= |H_n(x, v_n(t)) - H(x, v(t))| \leqq \\ &\leqq |H_n(x, v_n(t)) - H_n(x, v(t))| + |H_n(x, v(t)) - H(x, v(t))| < \\ &< \eta(|K(v_n(t)) - K(v(t))|) + \varepsilon < \eta(\xi) + \varepsilon < 2\varepsilon \quad \text{for} \quad n \geqq n_0\,. \end{aligned}$$

3.5. Proposition. *Assume that a sequence $F_n \in \Phi(G, k_n, l_n, \omega)$, $n \in \mathbb{N}$ converges under substitution to a pair (H, v). Define $F(x, t) = H(x, v(t))$ for $(x, t) \in G$. If $F_0: G \to \mathbb{R}^N$ is the reduction of the function H by v and $F^*(x, t) = F(x, t) - - F_0(x, t)$ for $(x, t) \in G$, then there is a nondecreasing jump function $h: [-T, T] \to \to \mathbb{R}$ such that*

(3.9) $$|F^*(x, t_2) - F^*(x, t_1)| \leqq |h(t_2) - h(t_1)| \quad \text{for} \quad (x, t_1), (x, t_2) \in G$$

and

(3.10) $$h(t_2) - h(t_1) \leqq 2\eta(K(v(t_2)) - K(v(t_1)))\,, \quad -T \leqq t_1 < t_2 \leqq T\,.$$

Proof. By Proposition 3.3 the function H belongs to $\Phi(G, \varkappa, \lambda, \omega)$ where the function $\varkappa$ satisfies (3.7). Then the function $h(t) = 2 \sum_{\substack{-T < s < t \\ v(s) < v(s+)}} [\varkappa(v(s+)+) - \varkappa(v(s))]$ satisfies (3.10) and the relation (3.9) follows immediately from Proposition 1.21.

Remark. (i) h is the jump part of the function $2\varkappa \circ v$.

(ii) (3.10) implies that if the function $K \circ v$ is continuous at t then h is as well.

3.6. Theorem. *Assume that a sequence $F_n \in \Phi(G, k_n, l_n, \omega)$, $n \in \mathbb{N}$ converges under substitution to a pair (H, v). Let $F_0: G \to \mathbb{R}^N$ be the reduction of the function*

H by the function v, denote $R = R_{(H,v)}$. Then the sequence F_n satisfies (3.3), (3.4) *and*

(3.11) *there is a continuous increasing function $\eta: [0, \infty) \to [0, \infty)$, $\eta(0) = 0$ and an increasing function $h_0: [-T, T] \to \mathbb{R}$ which is left-continuous on $(-T, T]$ and such that*

$$\limsup_{n\to\infty} [k_n(t_2) - k_n(t_1)] \leqq \eta(h_0(t_2) - h_0(t_1))$$

if the function h_0 is continuous at t_1 and t_2, $-T \leqq t_1 < t_2 \leqq T$.

Proof. We use the notation from Definition 3.2; let us define $F(x, t) = H(x, v(t))$, $F^*(x, t) = F(x, t) - F_0(x, t)$ and $h_0(t) = K(v(t)) + 2\varkappa(v(t))$, where $\varkappa$ has the same meaning as in Prop. 3.3.

If the function h_0 is continuous at t_1 and t_2, $-T \leqq t_1 < t_2 \leqq T$, then the function v is continuous at t_1, t_2 and the functions K and $\varkappa$ are continuous at $v(t_1)$, $v(t_2)$. As in the proof of Prop. 3.3 let us put $\varkappa_n(s) = k_n(v_n^{-1}(s)) - k_n(-T)$ for $b \in \mathbb{N}$ and $s \in [-T, T]$. Then $k_n(t_i) = \varkappa_n(v_n(t_i)) + k_n(-T)$ $(i = 1, 2)$ and by (3.6) we have

$$\limsup_{n\to\infty} [k_n(t_2) - k_n(t_1)] = \limsup_{n\to\infty} [\varkappa_n(v_n(t_2)) - \varkappa_n(v_n(t_1))] \leqq$$
$$\leqq \lim_{n\to\infty} \eta(K(v_n(t_2)) - K(v_n(t_1))) = \eta(K(v(t_2)) - K(v(t_1)) \leqq \eta(h_0(t_2) - h_0(t_1)) .$$

If h has the same meaning as in the proof of Prop. 3.5 (i.e., h is the jump part of $2\varkappa \circ v$), then by Prop. 3.5 we have

$$|F^*(x, t_2) - F^*(x, t_1)| \leqq h(t_2) - h(t_1) \leqq h^*(t_2) - h^*(t_1) \quad \text{for} \quad x \in \Omega ,$$

$-T < t_1 < t_2 < T$, where h^* is the jump part of the function h_0. This completes the proof of (3.11).

The condition (3.3) follows from Propositions 3.4 and 3.5.

Let a pair $(x_0, t_0) \in R = R_{(H,v)}$ be given such that $h_0(t_0+) > h_0(t_0)$. Let $\varepsilon > 0$ be given.

In case that $v(t_0) = v(t_0+)$, let us find such $\Delta > 0$ that $\eta(K(v(t_0) + \Delta) - K(v(t_0) +)) < \varepsilon/24$. There is such an integer n_1 that

$$(3.12) \qquad |H_n(x_0, v(t_0)) - H(x_0, v(t_0))| < \varepsilon/8 \quad \text{and}$$
$$|H_n(x_0, v(t_0) + \Delta) - H(x_0, v(t_0) + \Delta)| < \varepsilon/24 \quad \text{for every} \quad n \geqq n_1 .$$

Then $|H_n(x_0, v(t_0) +) - H(x_0, v(t_0) +)| \leqq |H_n(x_0, v(t_0) + \Delta) - H(x_0, v(t_0) + \Delta)| + 2\eta(K(v(t_0) + \Delta) - K(v(t_0) +)) < \varepsilon/8$.

In case that $v(t_0) < v(t_0+)$, by the definition of $R_{(H,v)}$ in 1.18 there is $\sigma > 0$ and a solution y_0 of the equation

$$(3.13) \qquad \frac{dy}{d\tau} = DH(y, t)$$

on $[v(t_0), v(t_0 + \sigma)]$ such that $y_0(v(t_0)) = x_0$. By Corollary 2.7 there is $n_1 \in N$ and $\alpha > 0$ such that, if $\sigma' \in (0, \sigma]$ and y is a solution of the equation

$$\text{(3.14)}_n \qquad \frac{dy}{d\tau} = DH_n(y, t)$$

on the interval $[v(t_0), v(t_0 + \sigma')]$ for some $n \geqq n_1$ and if $|y(v(t_0)) - y_0(v(t_0))| < \alpha$, then

$$\text{(3.15)} \qquad |y(s) - y_0(s)| < \varepsilon/4 \quad \text{for} \quad s \in [v(t_0), v(t_0 + \sigma')] .$$

There is $r \in (0, \varepsilon/2)$ such that $\omega(r)\, c < \varepsilon/4$; in case $v(t_0) < v(t_0+)$ let us assume that also $r \leqq \alpha$. There is $\varrho > 0$ such that

$$\eta(K(v(t_0+) + \varrho) - K(v(t_0+)+)) < r/2 , \quad \eta(K(v(t_0)) - K(v(t_0) - \varrho)) < r/2 .$$

There is such $\delta \in (0, r/2)$ that the function v is continuous at the points $t_0 - \delta$, $t_0 + \delta$ and $v(t_0 + \delta) - v(t_0+) < \varrho/2$, $v(t_0) - v(t_0 - \delta) < \varrho/2$.

Let $\delta' \in (0, \delta)$ be given. Find $\delta'' \in (0, \delta']$ such that the function v is continuous at $t_0 - \delta''$, $t_0 + \delta''$.

Then $v_n(t_0 - \delta'') \to v(t_0 - \delta'')$, $v_n(t_0 + \delta'') \to v(t_0 + \delta'')$; since $v(t_0 + \delta'') > v(t_0 + \delta''/2) > v(t_0) > v(t_0 - \delta'')$, there is an integer $n_2 \geqq n_1$ such that $v_n(t_0 + \delta'') > v(t_0 + \delta''/2) > v(t_0) > v_n(t_0 - \delta'')$ for every $n \geqq n_2$. Consequently $[v(t_0), v(t_0+)] \subset (v_n(t_0 - \delta''), v_n(t_0 + \delta''))$ for $n \geqq n_2$. Since $v_n(t_0 - \delta'') \to v(t_0 - \delta'')$ and $v_n(t_0 + \delta'') \to v(t_0 + \delta'')$, there is $n_0 \geqq n_2$ such that $v_n(t_0 - \delta) - v(t_0 - \delta) > -\varrho/2$ and $v_n(t_0 + \delta) - v(t_0 + \delta) < \varrho/2$. For $n \geqq n_0$ we have $0 < K(v(t_0)) - K(v_n(t_0 - \delta')) \leqq K(v(t_0)) - K(v(t_0 - \delta') - \varrho/2) \leqq K(v(t_0)) - K(v(t_0) - \varrho)$; hence

$$\text{(3.16)} \qquad \eta(K(v(t_0)) - K(v_n(t_0 - \delta'))) \leqq \eta(K(v(t_0)) - K(v(t_0) - \varrho) < r/2 .$$

Similarly it can be proved that

$$\text{(3.17)} \quad \eta(K(v_n(t_0 + \delta')) - K(v(t_0+)+)) \leqq \eta(K(v(t_0+) + \varrho) - K(v(t_0+)+)) < r/2$$

for every $n \geqq n_0$.

Let x_n be a solution of $(3.1)_n$ on $[t_0 - \delta', t_0 + \delta']$ for $n \geqq n_0$ such that $|x_n(t_0 - \delta') - x_0| \leqq \delta$.

If we define $y_n(\tau) = x_n(v_n^{-1}(\tau))$ for $\tau \in [v_n(t_0 - \delta'), v_n(t_0 + \delta')]$, then by Theorem 1.17 the function y_n is a solution of $(3.14)_n$ on $[v_n(t_0 - \delta'), v_n(t_0 + \delta')]$.

Denote $y_n = y_n(v(t_0))$, $n \geqq n_0$. Then

$$\text{(3.18)} \qquad |y_n - x_0| = |y_n(v(t_0)) - x_0| \leqq |y_n(v(t_0)) - y_n(v_n(t_0 - \delta'))| +$$
$$+ |x_n(t_0 - \delta) - x_0| \leqq \eta(K(v(t_0)) - K(v_n(t_0 - \delta'))) + \delta \leqq r .$$

a) Assume that $v(t_0) = v(t_0+)$. Using Lemma 2.8 in [S1] for functions y_n, H_n and for $s = v(t_0)$, we get the equality

$$y_n(v(t_0)+) - y_n(v(t_0)) = y_n(s+) - y_n(s) =$$
$$= H_n(y_n(s), s+) - H_n(y_n(s), s) = H_n(y_n, v(t_0)+) - H_n(y_n, v(t_0)) .$$

Lemma 2.6 in [S1] implies that

$$|y_n(s_2) - y_n(s_1)| \leqq \varkappa_n(s_2) - \varkappa_n(s_1) \leqq \eta(K(s_2) - K(s_1))$$

holds for $s_1 < s_2$; hence for $s_1 \to v(t_0)+$ and $s_2 = v_n(t_0 + \delta')$ we get the inequality

$$|y_n(v_n(t_0 + \delta')) - y_n(v(t_0) +)) \leqq \eta(K(v_n(t_0 + \delta')) - K(v(t_0) +));$$

similarly for $s_1 = v_n(t_0 - \delta')$ and $s_2 = v(t_0)$ we have

$$|y_n(t_0)) - y_n(v_n(t_0 - \delta'))| \leqq \eta(K(v(t_0)) - K(v_n(t_0 - \delta'))).$$

From (3.12), (3.16), (3.17) and (3.18) we get the inequality

$$\begin{aligned}
&|x(t_0 + \delta') - x(t_0 - \delta') - [F_0(x_0, t_0 +) - F_0(x_0, t_0)]| = \\
&= |y_n(v_n(t_0 + \delta')) - y_n(v_n(t_0 - \delta')) - [H(x_0, v(t_0) +) - H(x_0, v(t_0))]| = \\
&= |[y_n(v_n(t_0 + \delta')) - y_n(v(t_0) +)] + [H_n(y_n, v(t_0) +) - H_n(y_n, v(t_0))] + \\
&+ [y_n(v(t_0)) - y_n(v_n(t_0 - \delta'))] - [H(x_0, v(t_0) +) - H(x_0, v(t_0))]| \leqq \\
&\leqq |y_n(v_n(t_0 + \delta')) - y_n(v(t_0) +)| + |y_n(v(t_0)) - y_n(v_n(t_0 - \delta'))| + \\
&+ |H_n(y_n, v(t_0) +) - H_n(y_n, v(t_0)) - H_n(x_0, v(t_0) +) + H_n(x_0, v(t_0))| + \\
&+ |H_n(x_0, v(t_0) +) - H(x_0, v(t_0) +)| + |H_n(x_0, v(t_0)) - H(x_0, v(t_0))| \leqq \\
&\leqq \eta(K(v_n(t_0 + \delta')) - K(v(t_0) +)) + \eta(K(v(t_0)) - K(v_n(t_0 - \delta'))) + \\
&+ \omega(|y_n - x_0|) (\lambda_n(v(t_0) +) - \lambda_n(v(t_0))) + \varepsilon/8 + \varepsilon/8 < \\
&< r/2 + r/2 + \omega(r) c + \varepsilon/4 < \varepsilon .
\end{aligned}$$

b) Assume that $v(t_0) < v(t_0+)$. For $n \geqq n_0$ the function y_n is defined on $[v(t_0), v(t_0 + \delta''/2)]$ and according to (3.18) the inequality $|y_n(v(t_0)) - x_0| < r \leqq \alpha$ holds; then (3.15) is satisfied for $y = y_n$ and $\sigma' = \delta''/2$. We have the inequality

$$\begin{aligned}
&|x_n(t_0 + \delta') - x_n(t_0 - \delta') - [F(x_0, t_0+) - F(x_0, t_0)]| = \\
&= |[y_n(v_n(t_0 + \delta')) - y_n(v_n(t_0 - \delta'))] - [y_0(v(t_0+) +) - y_0(v(t_0))]| \leqq \\
&\leqq |y_n(v_n(t_0 + \delta')) - y_n(v(t_0 +) +)| + |y_n(v(t_0+) +) - y_0(v(t_0+) +)| + \\
&+ |y_0(v(t_0)) - y_n(v(t_0))| + |y_n(v(t_0)) - y_n(v_n(t_0 - \delta'))| \leqq \\
&\leqq \eta(K(v_n(t_0 + \delta')) - K(v(t_0+) +)) + \eta(K(v(t_0)) - K(v_n(t_0 - \delta'))) + \\
&+ 2\|y_n - y_0\| < r/2 + r/2 + 2 . \varepsilon/4 < \varepsilon .
\end{aligned}$$

Consequently, the condition (3.4) is verified.

Remark. Taking into account that (3.11) is a certain "generalization" of the condition (3.2) and the function F_0 need not belong to $\mathscr{F}(G, h_0, \omega)$, we can say that the sequence (F_n) in Theorem 3.6 "converges R-emphatically to F_0" in a little more general setting.

3.7. Lemma. *Let a sequence of functions $H_n \in \Phi(G, \varkappa_n, \lambda_n, \omega)$, $n \in \mathbb{N}$ be given such that*

(3.19) *there is such* $c > 0$ *that* $\lambda_n(T) - \lambda_n(-T) \leqq c$, $n \in \mathbb{N}$;

(3.20) *there is a continuous increasing function $\eta: [0, \infty) \to [0, \infty)$, $\eta(0) = 0$, and an increasing function $K: [-T, T] \to \mathbb{R}$ which is left-continuous on $(-T, T]$, right-continuous at $-T$ and such that $\varkappa_n(s_2) - \varkappa_n(s_1) \leqq \leqq \eta(K(s_2) - K(s_1))$ for every $n \in \mathbb{N}$, $-T \leqq s_1 < s_2 \leqq T$;*

(3.21) *there is such $\sigma \in (-T, T)$ that $H_n(x, \sigma) = 0$ for every $x \in \Omega$, $n \in \mathbb{N}$.*

Then $(H_n)_{n=1}^{\infty}$ contains a pointwise convergent subsequence.

Proof. For every $n \in \mathbb{N}$ let us define a function $\hat{H}_n: \Omega \times [K(-T), K(T)] \to \mathbb{R}^N$ in the following way: $\hat{H}_n(x, \tau) = H_n(x, t)$ for every $x \in \Omega$ and $\tau \in (K(-T), K(T))$ having the form $\tau = K(t)$. If $t \in (-T, T)$ is such a point that $K(t) < K(t+)$ then $\hat{H}_n(x, K(t+)) = H_n(x, t+)$ and the function $\hat{H}_n(x, \cdot)$ is defined linearly on $[K(t), K(t+)]$ (in terms of the notions from [F2], the function $\hat{H}_n(x, \cdot)$ is the linear prolongation of the function $H_n(x, \cdot)$ along the function K).

By [F2], Prop. 1.22 there is a continuous concave increasing function $\hat{\eta}: [0, \gamma] \to \to [0, \infty)$ where $\gamma = K(T) - K(-T)$, such that $\hat{\eta}(0) = 0$ and $\eta(r) \leqq \hat{\eta}(r)$ for every $r \in [0, \gamma]$. Then for every $n \in \mathbb{N}$, $x \in \Omega$ the inequality

$$|H_n(x, t_2) - H_n(x, t_1)| \leqq \varkappa_n(t_2) - \varkappa_n(t_1) \leqq \hat{\eta}(K(t_2) - K(t_1)), \quad -T < t_1 < t_2 < T$$

holds. From [F2], Prop. 2.9 it follows that

$$|\hat{H}_n(x, \tau_2) - \hat{H}_n(x, \tau_1)| \leqq \hat{\eta}(\tau_2 - \tau_1) \tag{3.22}$$

for every $x \in \Omega$, $K(-T) < \tau_1 < \tau_2 < K(T)$, $n \in \mathbb{N}$.

The inequality (3.22) implies that the limits $\hat{H}_n(x, K(-T)+)$, $\hat{H}_n(x, K(T)-)$ exist for every $x \in \Omega$, $n \in \mathbb{N}$. Let us define $\hat{H}_n(x, K(-T)) = \lim_{\tau \to K(-T)+} \hat{H}_n(x, \tau)$, $\hat{H}_n(x, K(T)) = \lim_{\tau \to K(T)-} \hat{H}_n(x, \tau)$. Then the inequality (3.22) holds if $K(-T) \leqq \leqq \tau_1 < \tau_2 \leqq K(T)$.

Let $t \in (-T, T)$ be given, denote $\tau = K(t)$. For every $x, y \in \Omega$, $n \in \mathbb{N}$ we have

$$\begin{gathered}|\hat{H}_n(x, \tau) - \hat{H}_n(y, \tau)| = |H_n(x, t) - H_n(y, t)| = \\ = |H_n(x, t) - H_n(x, \sigma) - H_n(y, t) + H_n(y, \sigma)| \leqq \omega(|x - y|)\,|\lambda_n(t) - \lambda_n(\sigma)| \leqq \\ \leqq \omega(|x - y|)\,(\lambda_n(T) - \lambda_n(-T)) \leqq \omega(|x - y|)\,c\,.\end{gathered}$$

If $K(t_0) < K(t_0+)$, then passing to the limit with $t \to t_0+$ we get the inequality

$$|\hat{H}_n(x, \tau) - \hat{H}_n(y, \tau)| \leqq \omega(|x - y|)\,c \tag{3.23}$$

also for $\tau = K(t_0+)$. Since (3.23) holds for $\tau = K(t_0)$ and for $\tau = K(t_0+)$ and the function $\hat{H}_n(x, \cdot)$ is linear on the interval $[K(t_0), K(t_0+)]$, the inequality (3.23) holds for every $\tau \in [K(t_0), K(t_0+)]$. Consequently, (3.23) is valid for every $\tau \in \in [K(-T), K(T)]$, $x \in \Omega$, $n \in \mathbb{N}$. From (3.22) and (3.23) we get

$$|\hat{H}_n(x, \tau_2) - \hat{H}_n(y, \tau_1)| \leqq |\hat{H}_n(x, \tau_2) - \hat{H}_n(x, \tau_1)| +$$
$$+ |\hat{H}_n(x, \tau_1) - \hat{H}_n(y, \tau_1)| \leqq \hat{\eta}(\tau_2 - \tau_1) + \omega(|x - y|)\, c \quad \text{for} \quad x, y \in \Omega\,, \quad n \in \mathbb{N}\,,$$
$$K(-T) \leqq \tau_1 < \tau_2 \leqq K(T)\,;$$

hence the functions $\hat{H}_n$ are equicontinuous on $\Omega \times [K(-T), K(T)]$. By (3.21), (3.22) we have $|\hat{H}_n(x, \tau)| = |\hat{H}_n(x, \tau) - \hat{H}_n(x, K(\sigma))| \leqq \hat{\eta}(|\tau - K(\sigma)|) \leqq \hat{\eta}(K(T) - K(-T))$, hence the functions $\hat{H}_n$ are bounded. It follows from the Arzelà-Ascoli Theorem that for every compact subset A of Ω the sequence $(\hat{H}_n)_{n=1}^{\infty}$ contains a subsequence which is uniformly convergent on $A \times [K(-T), K(T)]$; using the diagonalization we can find a subsequence $(\hat{H}_{n_k})_{k=1}^{\infty}$ which converges pointwise to a function $\hat{H}: \Omega \times \times [K(-T), K(T)] \to \mathbb{R}^N$. If we define $H(x, t) = \hat{H}(x, K(t))$ for every $(x, t) \in G$, then $H_n(x, t) = \hat{H}_n(x, K(t)) \to \hat{H}(x, K(t)) = H(x, t)$.

3.8. Lemma. *Let functions $F_0, \tilde{F}: G \to \mathbb{R}^N$ be given such that*
(i) *there is a nondecreasing left-continuous function $h: [-T, T] \to \mathbb{R}$ which has zero continuous part, such that*

$$|F_0(x, t_2) - \tilde{F}(x, t_2) - F_0(x, t_1) + \tilde{F}(x, t_1)| \leqq h(t_2) - h(t_1)$$
$$\textit{for every} \quad x \in \Omega\,, \quad -T < t_1 < t_2 < T\,;$$

(ii) *there is a set $\tilde{R} \subset G$ such that for every $(x, t) \in \tilde{R}$ the identity $F_0(x, t+) - - F_0(x, t) = \tilde{F}(x, t+) - \tilde{F}(x, t)$ holds.*

If $x: [\alpha, \beta] \to \mathbb{R}^N$, $[\alpha, \beta] \subset (-T, T)$ is such a function that $(x(t), t) \in \tilde{R}$ for every $t \in [\alpha, \beta)$, then

$$\int_{t_1}^{t_2} \mathrm{D}F_0(x(\tau), t) = \int_{t_1}^{t_2} \mathrm{D}\tilde{F}(x(\tau), t)$$

for every $t_1, t_2 \in [\alpha, \beta]$ provided at least one of the integrals exists.

Proof. Let us denote $N(x, t) = F_0(x, t) - \tilde{F}(x, t)$, $(x, t) \in G$. Then

$$(3.24) \qquad |N(x, t_2) - N(x, t_1)| \leqq h(t_2) - h(t_1) \quad \text{for every} \quad x \in \Omega\,, \quad -T < t_1 < t_2 < T\,;$$

$$(3.25) \qquad N(x, t+) - N(x, t) = 0 \quad \text{for every} \quad (x, t) \in \tilde{R}\,.$$

Assume that $\alpha \leqq t_1 < t_2 \leqq \beta$.

By Lemma 1.6 we have

$$\int_{t_1}^{t_2} \mathrm{D}N(x(\tau), t) = \sum_{t_1 \leqq s < t_2} [N(x(s), s+) - N(x(s), s)]\,.$$

Since $(x(s), s) \in \tilde{R}$ for every $s \in [t_1, t_2)$, (3.25) implies that $N(x(s), s+) - N(x(s), s) = 0$. Consequently

$$\int_{t_1}^{t_2} \mathrm{D}[F_0(x(\tau), t) - \tilde{F}(x(\tau), t)] = \int_{t_1}^{t_2} \mathrm{D}N(x(\tau), t) = 0\,.$$

The rest of the Lemma follows from [S1], Th. 1.6.

3.9. Theorem. *Assume that a sequence of functions $F_n \in \mathscr{F}(G, h_n, \omega)$, $n \in \mathbb{N}$ converges R-emphatically to a function $F_0\colon G \to \mathbb{R}^N$. Then*

(i) *there is a subsequence $(F_{n_k})_{k=1}^{\infty}$ which converges under substitution to a pair (H, v);*

(ii) *if we denote by $\tilde{F}$ the reduction of the function H by the function v, then for every $x \in \Omega$ the continuous part of the function $F_0(x, \cdot) - \tilde{F}(x, \cdot)$ is constant and for every $(x, t) \in R \cap R_{(H,v)}$ the identity $F_0(x, t+) - F_0(x, t) = F(x, t+) - F(x, t)$ holds;*

(iii) *let $[\alpha, \beta] \subset (-T, T)$; a function $x\colon [\alpha, \beta] \to \mathbb{R}^N$ such that $(x(t), t) \in R \cap R_{(H,v)}$ for every $t \in [\alpha, \beta)$ is a solution of the equation (3.5), if and only if it is a solution of the equation*

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \mathrm{D}\tilde{F}(x, t) \tag{3.26}$$

on the interval $[\alpha, \beta]$.

Proof. (i) By [F2], Th. 1.21 there is a subsequence $(h_{n_k})_{k=1}^{\infty}$ for which there exists a sequence of continuous increasing functions $(v_k)_{k=1}^{\infty} \subset \Lambda$ and an increasing function $v \in V^-$ such that $v_k(t) \to v(t)$ for every $t \in (-T, T)$ at which v is continuous, and the functions $h_{n_k} \circ v_k^{-1}$ are equiregulated.

Since the functions $h_{n_k} \circ v_k^{-1}$ are equiregulated, by the property 1.4 there is an increasing continuous function $\eta\colon [0, \infty) \to [0, \infty)$, $\eta(0) = 0$ and an increasing function $K\colon [-T, T] \to \mathbb{R}$ which is left-continuous on $(-T, T]$ and right-continuous at $-T$ so that

$$h_{n_k}(v_k^{-1}(s_2)) - h_{n_k}(v_k^{-1}(s_1)) \leqq \eta(K(s_2) - K(s_1)) \quad \text{for every} \quad k \in \mathbb{N}\,,$$
$$-T \leqq s_1 < s_2 \leqq T\,.$$

Then the inequality (3.6) is satisfied provided k_n, v_n are replaced by h_{n_k}, v_k.

Fix such a point $t_0 \in (-T, T)$ that the functions h_0, v are continuous at t_0 and K is continuous at $v(t_0)$. Let $H_k\colon G \to \mathbb{R}^N$ be the prolongation of the function F_{n_k} along v_k, denote $H'_k(x, \tau) = H_k(x, \tau) - H_k(x, v(t_0))$ for every $(x, \tau) \in G$.

By Lemma 3.7 there is a subsequence of (H'_k) which for simplicity will be denoted again by (H'_k), such that $H'_k(x, \tau) \to H'(x, \tau)$ for every $(x, \tau) \in G$. Define $H(x, \tau) = H'(x, \tau) + \lim_{n \to \infty} F_n(x, t_0)$.

By (3.3) for every $\varepsilon > 0$ there is such an integer k_0 that $|F_{n_k}(x, t_0) - \lim_{n\to\infty} F_n(x, t_0)| < \varepsilon$ for every $k \geqq k_0$. For $k \geqq k_0$ we have $|H_k(x, v(t_0)) - \lim_{n\to\infty} F_n(x, t_0)| = |F_{n_k}(x, v_k^{-1}(v(t_0)) - \lim_{n\to\infty} F_n(x, t_0)| \leqq |F_{n_k}(x, v_k^{-1}(v(t_0)) - F_{n_k}(x, t_0)| + |F_{n_k}(x, t_0) - \lim_{n\to\infty} F_n(x, t_0)| < \varepsilon + h_{n_k}(v_k^{-1}(v(t_0)) - h_{n_k}(t_0) \leqq \varepsilon + \eta(|K(v(t_0)) - K(v_k(t_0))|)$.

Since v is continuous at t_0, we conclude that $v_k(t_0) \to v(t_0)$; further $K(v_k(t_0)) \to K(v(t_0))$ because K is continuous at $v(t_0)$. Consequently $H_k(x, t) \to H(x, t)$, $(x, t) \in G$, and the subsequence (F_{n_k}) converges under substitution to (H, v).

(ii) By (3.3) there is a function $F(x, t) = F_0(x, t) + F^*(x, t)$ which is left-continuous in t and such that $F_n(x, t) \to F(x, t)$ for every $x \in \Omega$ and $t \in (-T, T)$ at which the function h_0 is continuous. Proposition 3.4 yields that $F(x, t) = H(x, v(t))$, $(x, t) \in G$.

By Proposition 3.5 there is such a jump function h that (3.9), (3.10) hold when $F^*(x, t)$ is replaced by $\tilde{F}^*(x, t) = F(x, t) - \tilde{F}(x, t)$. Then

$$\begin{aligned} &|[F_0(x, t_2) - \tilde{F}(x, t_2)] - [F_0(x, t_1) - \tilde{F}(x, t_1)]| = \\ &= |[F^*(x, t_2) - F^*(x, t_1)] - [\tilde{F}^*(x, t_2) - \tilde{F}^*(x, t_1)]| \leqq \\ &\leqq [h^*(t_2) - h^*(t_1)] + [h(t_2) - h(t_1)] = \tilde{h}(t_2) - \tilde{h}(t_1) \end{aligned}$$

where $\tilde{h}(t) = h^*(t) + h(t)$.

If $(x, t) \in R$ then the value $F_0(x, t+) - F_0(x, t)$ is evaluated by (3.4). Since the subsequence (F_{n_k}) converges R-emphatically to F_0 and $R_{(H,v)}$-emphatically to $\tilde{F}$, we have

$$F_0(x, t+) - F_0(x, t) = \tilde{F}(x, t+) - \tilde{F}(x, t) \quad \text{for every} \quad (x, t) \in R \cap R_{(H,v)}\,.$$

Part (iii) is an evident consequence of Lemma 3.8.

3.10. Theorem. *Assume that a sequence $F_n \in \Phi(G, k_n, l_n, \omega)$, $n \in \mathbb{N}$ converges under substitution to a pair (H, v), let $F_0\colon G \to \mathbb{R}^N$ be the reduction of the function H by v. Assume that the function $K \circ v$ is continuous at α, β, $-T < \alpha < \beta < T$.*

(i) *If x_n is a solution of the equation $(3.1)_n$ on $[\alpha, \beta]$ for every $n \in \mathbb{N}$ and if the set $\{x_n(\alpha); n \in \mathbb{N}\}$ is bounded, then there is a function $x_0\colon [\alpha, \beta] \to \mathbb{R}^N$ with bounded variation and left-continuous on $(\alpha, \beta]$, and a subsequence $(x_{n_k})_{k=1}^{\infty}$ such that $x_{n_k}(t) \to x_0(t)$ for every $t \in [\alpha, \beta]$ at which the function $K \circ v$ is continuous* (notation from Def. 3.2 is used).

(ii) *Assume that $x_0(\alpha) \in \Omega$. Then either the function x_0 is a solution of the generalized differential equation with a substitution*

$$x(t) = y(v(t))\,, \quad \frac{dy}{d\tau'} = DH(y, t') \tag{3.27}$$

on the interval $[\alpha, \beta]$, or there is such $\beta' \in (\alpha, \beta]$ that x_0 is a maximal solution *on* $[\alpha, \beta')$ *or* $[\alpha, \beta']$, *or there is such* $\beta'' \in (\alpha, \beta)$ *that* x_0 *is a solution of* (3.27) *on* $[\alpha, \beta'']$ *and disappears at* β''.

(iii) *If* $(x_0(t), t) \in R_{(H,v)}$ *for every* $t \in [\alpha, \beta]$, *then the function* x_0 *is a solution of* (3.27) *on* $[\alpha, \beta]$, *as well as of the equation* (3.5).

Proof. (i) By Corollary 2.7 in [S1] the function x_n has bounded variation on $[\alpha, \beta]$ for every $n \in \mathbb{N}$ and

$$\operatorname{var}_\alpha^\beta x_n \leqq k_n(\beta) - k_n(\alpha) \leqq \eta(K(v_n(\beta)) - K(v_n(\alpha))) \leqq \eta(K(T) - K(-T))$$

for every $n \in \mathbb{N}$. By Helly's Choice Theorem there is a function $x\colon [\alpha, \beta] \to \mathbb{R}^N$ of bounded variation and a subsequence $(x_{n_k})_{k=1}^\infty$ such that $x_{n_k}(t) \to x(t)$ for every $t \in [\alpha, \beta]$.

Since $|x_{n_k}(t_2) - x_{n_k}(t_1)| \leqq k_{n_k}(t_2) - k_{n_k}(t_1) \leqq \eta(K(v_{n_k}(t_2)) - K(v_{n_k}(t_1)))$ for every $k \in \mathbb{N}$, $\alpha \leqq t_1 < t_2 \leqq \beta$, passing to the limit with $k \to \infty$ we get the inequality

$$|x(t_2) - x(t_1)| \leqq \eta(K(v(t_2)) - K(v(t_1))) \tag{3.28}$$

for every t_1, t_2 at which the function $K \circ v$ is continuous, $\alpha \leqq t_1 < t_2 \leqq \beta$.

Let us define $x_0(\alpha) = x(\alpha)$, $x_0(t) = x(t-)$ for $t \in (\alpha, \beta]$. Then the function x_0 has bounded variation and is left-continuous on $(\alpha, \beta]$. Since the function x has one-sided limits on $[\alpha, \beta]$, the inequality (3.28) yields

$$\begin{aligned} |x(\alpha+) - x(\alpha)| &\leqq \eta(K(v(\alpha+)+) - K(v(\alpha))) = 0\,; \\ |x(t) - x(t-)| &\leqq \eta(K(v(t)) - K(v(t-)-)) = 0 \end{aligned} \tag{3.29}$$

for every $t \in (\alpha, \beta]$ at which $K \circ v$ is continuous.

If the function $K \circ v$ is continuous at $t \in [\alpha, \beta]$ then the function x is continuous at t, which implies that $x_0(t) = x(t) = \lim_{k\to\infty} x_{n_k}(t)$.

(ii) For every $t \in [\alpha, \beta]$, $n \in \mathbb{N}$ we have the estimate $|x_n(t)| \leqq |x_n(\alpha)| + \eta(K(T) - K(-T))$, hence there is $d > 0$ such that

$$|x_n(t)| \leqq d \quad \text{for every} \quad t \in [\alpha, \beta]\,, \quad n \in \mathbb{N}\,. \tag{3.30}$$

For every $k \in \mathbb{N}$ let us define $y_k(\tau) = x_{n_k}(v_{n_k}^{-1}(\tau))$, $\tau \in [v_{n_k}(\alpha), v_{n_k}(\beta)]$. By Theorem 1.17 the function y_k is a solution of the equation $(3.14)_{n_k}$ on $[v_{n_k}(\alpha), v_{n_k}(\beta)]$.

Since the function v is continuous at α, β, we have $v_{n_k}(\alpha) \to v(\alpha)$, $v_{n_k}(\beta) \to v(\beta)$. Hence for every $[\gamma, \delta] \subset (v(\alpha), v(\beta))$ there is such $k_0 \in \mathbb{N}$ that $[\gamma, \delta] \subset [v_{n_k}(\alpha), v_{n_k}(\beta)]$ for every $k \geqq k_0$. By Theorem 2.5 the sequence y_k contains a subsequence which is uniformly convergent on $[\gamma, \delta]$. By a diagonalization process we can find a function $y_0\colon (v(\alpha), v(\beta)) \to \mathbb{R}^N$ and a subsequence of (y_k) — which will be denoted again by (y_k) — so that $y_k \rightrightarrows y_0$ on $[\gamma, \delta]$ for every $[\gamma, \delta] \subset (v(\alpha), v(\beta))$.

From (3.2) and Lemma 2.6 in [S1] it follows that

$$|y_k(s_2) - y_k(s_1)| \leqq \eta(K(s_2) - K(s_1))\,, \quad v_{n_k}(\alpha) \leqq s_1 < s_2 \leqq v_{n_k}(\beta)\,, \quad k \in \mathbb{N}\,;$$

then

$$(3.31)\qquad |y_0(s_2) - y_0(s_1)| \leqq \eta(K(s_2) - K(s_1)), \quad v(\alpha) < s_1 < s_2 < v(\beta).$$

Let us define $y_0(v(\alpha)) = y_0(v(\alpha)+)$, $y_0(v(\beta)) = y_0(v(\beta)-)$. If the function $K \circ v$ is continuous at $t \in (\alpha, \beta)$ then

$$|x_0(t) - y_0(v(t))| \leqq |x_0(t) - x_{n_k}(t)| + |y_{n_k}(v_{n_k}(t)) - y_0(v_{n_k}(t))| +$$
$$+ |y_0(v_{n_k}(t)) - y_0(v(t))| \leqq |x_0(t) - x_{n_k}(t)| +$$
$$+ \|y_{n_k} - y_0\|_{[v(t)-\delta, v(t)+\delta]} + \eta(|K(v_{n_k}(t)) - K(v(t))|)$$

where $\delta > 0$ is so small that $[v(t) - \delta', v(t) + \delta] \subset (v(\alpha), v(\beta))$. The expression at the end of the inequalities tends to zero with $k \to \infty$, hence $x_0(t) = y_0(v(t))$ for every $t \in (\alpha, \beta)$ at which the function $K \circ v$ is continuous. Since the functions x_0, y_0, v are left-continuous, the equality $x_0(t) = y_0(v(t))$ holds for every $t \in (\alpha, \beta]$. The continuity of $K \circ v$ at α implies that the functions x_0, v are right-continuous at α and y_0 is right-continuous at $v(\alpha)$. Hence

$$(3.32)\qquad x_0(t) = y_0(v(t)) \quad \text{for every} \quad t \in [\alpha, \beta].$$

Since $x_0(\alpha) = y_0(v(\alpha)) \in \Omega$ and the function y_0 is right-continuous at $v(\alpha)$, there is such $\delta > 0$ that $y_0(\tau) \in \Omega$ for every $\tau \in [v(\alpha), v(\alpha) + \delta]$.

If $y_0(\tau) \in \Omega$ for every $\tau \in [v(\alpha), v(\beta)]$, by Theorem 2.5 the function y_0 is a solution of the equation (3.13); then the function $x_0 = y_0 \circ v$ is a solution of (3.27) on $[\alpha, \beta]$. Assume that there is such $\gamma \in (v(\alpha), v(\beta)]$ that $y_0(\tau) \in \Omega$ for every $\tau \in [v(\alpha), \gamma)$ and $y_0(\gamma) \notin \Omega$. If $\gamma = v(\beta')$ for some $\beta' \in (\alpha, \beta]$ then the function x_0 is a maximal solution of (3.27) on $[\alpha, \beta')$. If there is such $\beta'' \in (\alpha, \beta)$ that $\gamma \in (v(\beta''), v(\beta''+)]$ then the function x_0 is a solution of (3.27) on $[\alpha, \beta'']$ and disappears at β''.

Finally, assume that there is such $\bar{\gamma} \in (v(\alpha), v(\beta))$ that $y_0(\tau) \in \Omega$ for every $\tau \in [v(\alpha), \bar{\gamma}]$ but there is no $s > \bar{\gamma}$ such that $y_0(\tau) \in \Omega$ for every $\tau \in [v(\alpha), s]$. Let us find such $\bar{\beta}$ that $\gamma \in [v(\bar{\beta}), v(\bar{\beta}+)]$. If $v(\bar{\beta}) = v(\bar{\beta}+)$ then x_0 is a maximal solution on $[\alpha, \bar{\beta}]$. If $v(\bar{\beta}) < v(\bar{\beta}+)$ then x_0 is a solution on $[\alpha, \bar{\beta}]$ and disappears at $\bar{\beta}$.

(iii) If $(x_0(t), t) \in R_{(H,v)}$ for every $t \in [\alpha, \beta)$ then $y_0(\tau) \in \Omega$ for every $\tau \in [v(\alpha), v(\beta)]$, hence the function x_0 is a solution of (3.27). By Theorem 1.24 the function x_0 is also a solution of (3.5) on $[\alpha, \beta]$.

3.11. Theorem. *Assume that functions $F_n \in \Phi(G, k_n, l_n, \omega)$, $n \in \mathbb{N}$ converge under substitution to a pair (H, v), let $F_0: G \to \mathbb{R}^N$ be the reduction of the function H by v. Assume that the function $K \circ v$ is continuous at α, β, $-T < \alpha < \beta < T$.*

Let $x_0: [\alpha, \beta] \to \mathbb{R}^N$ be a solution of the equation (3.5) on $[\alpha, \beta]$ which has the uniqueness property (2.10) when (2.7) is replaced by (3.27). Assume that $(x_0(t), t) \in R_{(H,v)}$ for every $t \in [\alpha, \beta)$.

Assume that any solution $y: [v(\alpha), v(\beta)] \to \mathbb{R}^N$ of the equation (3.13) such that $y(v(\alpha)) = x_0(\alpha)$ satisfies

(3.33) *there is such* $\varrho > 0$ *that if* $z \in \mathbb{R}^N$, $s \in [v(\alpha), v(\beta)]$ *and* $|z - y(s)| \leqq \varrho$ *then* $z \in \Omega$.

Let a sequence $(z_n)_{n=1}^{\infty} \subset \Omega$ *be given such that* $\lim_{n \to \infty} z_n = x_0(\alpha)$.

Then there is an integer n_0 *such that for every* $n \geqq n_0$ *there is a solution* x_n *of the equation* $(3.1)_n$ *on* $[\alpha, \beta]$ *such that* $x_n(\alpha) = z_n$, *and* $\lim_{n \to \infty} x_n(t) = x_0(t)$ *for every* $t \in [\alpha, \beta]$ *at with the function* $K \circ v$ *is continuous.*

Proof. We will use the notation from Definition 3.2. By Theorem 1.24 the function x_0 is a solution of (3.27) on $[\alpha, \beta]$, hence there is a solution y_0 of the equation (3.13) on $[v(\alpha), v(\beta)]$ such that $y_0(v(t)) = x_0(t)$ for every $t \in [\alpha, \beta]$.

If $\delta \in (v(\alpha), v(\beta)]$ and y is a solution of (3.13) on $[v(\alpha), \delta]$ such that $y(v(\alpha)) = x_0(\alpha)$, let us find such $\gamma \in [\alpha, \beta]$ that $\delta \in [v(\gamma), v(\gamma+)]$ and define $x(t) = y(v(t))$, $t \in [\alpha, \gamma]$. Then the function x is a solution of the equation (3.27) on $[\alpha, \gamma]$ and by the uniqueness property (2.10) we obtain that $x(t) = x_0(t)$ for every $t \in [\alpha, \gamma]$.

For every $\tau \in [v(\alpha), \delta]$ let us find such $t \in [\alpha, \gamma]$ that $\tau \in [v(t), v(t+)]$. If $\tau = v(t)$ then $y(\tau) = x(t) = x_0(t) = y_0(\tau)$. If $v(t) < v(t+)$ and $\tau \in (v(t), v(t+)]$, then the definition of the set $R_{(H,v)}$ in 1.18 implies that $y(\tau) = y_0(\tau)$. We have proved that

(3.34) if $y: [v(\alpha), \delta] \to \mathbb{R}^N$ is a solution of (3.13) such that $y(v(\alpha)) = y_0(v(\alpha))$ then $y(\tau) = y_0(\tau)$, $\tau \in [v(\alpha), \delta]$.

Since the function $K \circ v$ is continuous at α, there is such $d > 0$ that $\eta(K(v(\alpha) + d) - K(v(\alpha) - d)) < \varrho/2$ (the number ϱ is taken from (3.33)). Since $v_n(\alpha) \to v(\alpha)$ and $z_n \to x_0(\alpha)$, there is such an integer n_0 that $v_n(\alpha) \in [v(\alpha) - d, v(\alpha) + d]$ and $|z_n - x_0(\alpha)| < \varrho/2$ for every $n \geqq n_0$. Let $n \geqq n_0$ be fixed.

By A let us denote the set of all functions y from $\mathscr{R}_N[v(\alpha) - d, v(\alpha) + d]$ satisfying $|y(\tau) - z_n| \leqq \eta(|K(\tau) - K(v_n(\alpha))|)$ for every $\tau \in [v(\alpha) - d, v(\alpha) + d]$. The set A is closed in $\mathscr{R}_N[v(\alpha) - d, v(\alpha) + d]$ and $y(v_n(\alpha)) = z_n$ for $y \in A$. If $y \in A$ then $|y(\tau) - y_0(v(\alpha))| \leqq |y(\tau) - y(v_n(\alpha))| + |y(v_n(\alpha)) - x_0(\alpha)| \leqq \eta(|K(\tau) - K(v_n(\alpha))|) + |z_n - x_0(\alpha)| \leqq \eta(K(v(\alpha) + d) - K(v(\alpha) - d) + \varrho/2 < \varrho$ for every $\tau \in [v(\alpha) - d, v(\alpha) + d]$, hence $y(\tau) \in \Omega$ owing to (3.33).

For every $y \in A$ the function

$$T y(\sigma) = z_n + \int_{v_n(\alpha)}^{\sigma} \mathrm{D}H_n(y(\tau), t), \quad \sigma \in [v(\alpha) - d, v(\alpha) + d]$$

is defined. For $y \in A$ we have

$$(3.35) \qquad |T y(\sigma_2) - T y(\sigma_1)| \leqq \varkappa_n(\sigma_2) - \varkappa_n(\sigma_1) \leqq \eta(K(\sigma_2) - K(\sigma_1)), \quad v(\alpha) - d \leqq \sigma_1 < \sigma_2 \leqq v(\alpha) + d,$$

consequently the set $T(A)$ is relatively compact in $\mathscr{R}_N[v(\alpha) - d, v(\alpha) + d]$ and $T(A) \subset A$. According to Lemma 2.1 the operator T is continuous. By the Schauder-Tichonov fixed point theorem there is such a function $y_n' \in A$ that $y_n'(\tau) = T y_n'(\tau)$,

$\tau \in [v(\alpha) - d, v(\alpha) + d]$. By Theorem 2.5 there is a subsequence (y'_{n_k}) which converges uniformly to a function y on $[v(\alpha) - d, v(\alpha) + d]$. Since $|y'_n(\tau) - y_0(v(\alpha))| < \varrho$, $n \geqq n_0$, we have $|y(\tau) - y_0(v(\alpha))| \leqq \varrho$ for every $\tau \in [v(\alpha) - d, v(\alpha) + d]$; the assumption (3.33) implies that $y(\tau) \in \Omega$. By Theorem 2.5 the function y is a solution of (3.13) on $[v(\alpha) - d, v(\alpha) + d]$. From (3.34) it follows that $y(\tau) = y_0(\tau)$ for every $\tau \in [v(\alpha), v(\alpha) + d]$, consequently $y'_n \rightrightarrows y_0$ on $[v(\alpha), v(\alpha) + d]$.

Since $y'_n(v(\alpha) + d) \to y_0(v(\alpha) + d)$ and the solution y_0 satisfies the assumptions of Theorem 2.6 on $[v(\alpha) + d, v(\beta)]$, there is such $n_1 \geqq n_0$ that for every $n \geqq n_1$ there is a solution y_n of $(3.14)_n$ on the interval $[v(\alpha) + d, v(\beta)]$ such that $y_n(v(\alpha) + d) = y'_n(v(\alpha) + d)$, and $y_n \rightrightarrows y_0$ on $[v(\alpha) + d, v(\beta)]$. If we define $y_n(\tau) = y'_n(\tau)$, $\tau \in [v(\alpha) - d, v(\alpha) + d]$, then y_n is a solution of $(3.14)_n$ on $[v(\alpha), v(\beta)]$, $y_n(v_n(\alpha)) = z_n$ and $y_n \rightrightarrows y_0$ on $[v(\alpha), v(\beta)]$.

Since $y_0(v(\beta)) \in \Omega$, $y_n(v(\beta)) \to y_0(v(\beta))$ and the function $K \circ v$ is continuous at β, it can be proved similarly as above that there are such $d' > 0$ and $n_2 \geqq n_1$ that the solutions y_n can be continued on $[v(\alpha) - d, v(\beta) + d']$ and $v_n(\beta) \in [v(\beta) - d', v(\beta) + d']$, $n \geqq n_2$.

For every $n \geqq n_2$ let us define $x_n(t) = y_n(v_n(t))$, $t \in [\alpha, \beta]$; by Theorem 1.17 the function x_n is a solution of the equation $(3.1)_n$ on $[\alpha, \beta]$, $x_n(\alpha) = y_n(v_n(\alpha)) = z_n$.

If the function $K \circ v$ is continuous at $t \in (\alpha, \beta)$ and $n \geqq n_2$, then there is such $n' \geqq n_2$ that $v_n(t) \in [v(\alpha), v(\beta)]$, $n \geqq n'$. We have $|x_n(t) - x_0(t)| = |y_n(v_n(t)) - y_0(v(t))| \leqq |y_n(v_n(t)) - y_0(v_n(t))| + |y_0(v(t)) - y_0(v(t))| \leqq \|y_n - y_0\|_{[v(\alpha),v(\beta)]} + \eta(|K(v_n(t)) - K(v(t))|)$; the last expression tends to zero with $n \to \infty$.

Let $\varepsilon > 0$ be given; there is such $\delta > 0$ that $\eta(\delta) < \varepsilon$. Find such $t \in (\alpha, \beta)$ that the function $K \circ v$ is continuous at t and $K(v(\beta)) - K(v(t)) < \delta/2$. There is such $n'' \geqq n_2$ that

$$|K(v_n(\beta)) - K(v(\beta))| < \delta/4\,, \quad |K(v_n(t)) - K(v(t))| < \delta/4$$

and

$$|x_n(t) - x_0(t)| < \varepsilon \quad \text{for every} \quad n \geqq n''\,.$$

Then

$$\begin{aligned}|x_n(\beta) - x_0(\beta)| &\leqq |x_n(\beta) - x_n(t)| + |x_0(t) - x_c(\beta)| + |x_n(t) - x_0(t)| < \\ &< \eta(K(v_n(\beta)) - K(v_n(t))) + \eta(K(v(\beta) - K(v(t))) + \varepsilon < \\ &< \eta(K(v(\beta)) - K(v(t)) + \delta/2) + \eta(\delta) + \varepsilon < 3\varepsilon\,.\end{aligned}$$

Consequently $x_n(t) \to x_{)}(t)$ for every $t \in [\alpha, \beta]$ at which the function $K \circ v$ is continuous.

3.12. Example. Assume that functions $F \in \mathscr{F}(G, h, \omega)$, $g\colon \Omega \to R^N$ and $\Phi_n\colon [-T, T] \to R$, $n \in N$ are given such that

(i) the function h is continuous on $[-T, T]$ and the function g is uniformly continuous on Ω;

(ii) the functions Φ_n, $n \in N$ are continuous on $[-T, T]$ and there is such $c > 0$ that $\operatorname{var}_{-T}^{T} \Phi_n \leqq c$ for every $n \in N$;

(iii) for every $\varepsilon > 0$ there is such $\delta > 0$ that

$$\operatorname{var}_{-T}^{-T+\delta} \Phi_n + \operatorname{var}_{T-\delta}^{T} \Phi_n < \varepsilon \quad \text{for every} \quad n \in \mathbb{N} ;$$

(iv) there is a function $\Phi \in BV[-T, T]$ which is left-continuous on $(-T, T]$, right-continuous at $-T$ and such that $\Phi_n(t) \to \Phi(t)$ for every $t \in [-T, T]$ at which Φ is continuous (including $-T, T$).

Our aim is to find a limit equation for the sequence of generalized differential equations

$$\frac{dx}{d\tau} = D[F(x, t) + g(x) \Phi_n(t)] \tag{3.36$_n$}$$

(see also [S2], Example 4.7).

Denote $c_n = 1 + (1/2T) \operatorname{var}_{-T}^{T} \Phi_n$ for $n \in \mathbb{N}$, $\tilde{c} = 1 + (1/2T) c$. By (ii), for any $n \in \mathbb{N}$ the inequality $1 \leqq c_n \leqq \tilde{c}$ holds.

For every $n \in \mathbb{N}$ let us define functions $v_n(t) = (1/c_n) [t + T + \operatorname{var}_{-T}^{t} \Phi_n] - T$, $t \in [-T, T]$. Then $v_n \in \Lambda$. Since the functions v_n, $n \in \mathbb{N}$ are increasing and bounded, there is a subsequence (v_{n_k}) and a nondecreasing function $v_0 \colon [-T, T] \to \mathbb{R}$ such that $v_{n_k}(t) \to v_0(t)$ for every $t \in [-T, T]$. Evidently $v_0(-T) = -T$, $v_0(T) = T$.

Let us prove that v_0 is continuous at the endpoints of $[-T, T]$. For $\varepsilon > 0$ given let us find $\delta \in (0, \varepsilon]$ by the assumption (iii). For any $t \in (-T, -T + \delta]$ we have

$$v_0(t) - v_0(-T) = \lim_k [v_{n_k}(t) - v_{n_k}(-T)] =$$

$$= \lim_k \left[\frac{1}{c_{n_k}} (t + T + \operatorname{var}_{-T}^{t} \Phi_{n_k})\right] \leqq (t + T) + \limsup_k \operatorname{var}_{-T}^{T} \Phi_{n_k} < 2\varepsilon ;$$

hence v_0 is right-continuous at $-T$. The left-continuity at T can be proved similarly.

We have

$$v_n(t_2) - v_n(t_1) \geqq \frac{1}{c_n} (t_2 - t_1) \geqq \frac{1}{\tilde{c}} (t_2 - t_1) \quad \text{for} \quad n \in \mathbb{N}, \tag{3.37}$$

$$n \in \mathbb{N}, \quad -T \leqq t_1 < t_2 \leqq T,$$

and consequently

$$v_0(t_2) - v_0(t_1) \geqq \frac{1}{\tilde{c}} (t_2 - t_1) \quad \text{for} \quad -T \leqq t_1 < t_2 \leqq T.$$

Let us define $v(-T) = -T$, $v(T) = T$, $v(t) = v_0(t-)$ for $t \in (-T, T)$. Then obviously $v \in V^-$ and $v_{n_k}(t) \to v(t)$ for every $t \in [-T, T]$ at which v is continuous.

Let us define $\Psi_n(\tau) = \Phi_n(v_n^{-1}(\tau))$ for $\tau \in [-T, T]$, $n \in \mathbb{N}$. If $-T \leqq \tau_1 < \tau_2 \leqq T$ and $\tau_1 = v_n(t_1)$, $\tau_2 = v_n(t_2)$ for some $n \in \mathbb{N}$, then

$$|\Psi_n(\tau_2) - \Psi_n(\tau_1)| = |\Phi_n(t_2) - \Phi_n(t_1)| \leqq \operatorname{var}_{t_1}^{t_2} \Phi_n \leqq$$
$$\leqq c_n(v_n(t_2) - v_n(t_1)) = c_n(\tau_2 - \tau_1) \leqq \tilde{c}(\tau_2 - \tau_1).$$

Then the sequence (Ψ_n) is equicontinuous; it is also bounded, because $|\Phi_n(t)| \leqq$ $\leqq |\Phi_n(t_0)| + \operatorname{var}_{-T}^{T} \Phi_n$ and the sequence $(\Phi_n(t_0))$ is convergent for some t_0.

By the Arzelà-Ascoli Theorem the sequence (Ψ_{n_k}) contains a subsequence – which will be denoted again by (Ψ_{n_k}) – such that $\Psi_{n_k} \rightrightarrows \Psi$ on $[-T, T]$. Evidently $\Psi(v(t)) =$ $= \Phi(t)$. From (3.37) it follows that the functions $v_{n_k}^{-1}$ are Lipschitzian with the constant $1/\tilde{c}$, hence they converge uniformly to the function u defined by $u(\tau) = t$ if $v(t) \leqq \tau \leqq v(t+)$ (see [F2], Def. 1.10 and Prop. 1.11).

By Theorem 1.17 the equation (3.36) has the same solutions as the generalized differential equation with a substitution

$$\text{(3.38)} \qquad x(t) = y(v_n(t)), \quad \frac{\mathrm{d}y}{\mathrm{d}\tau'} = \mathrm{D}[F(y, v_n(t')) + g(y)\, \Psi_n(t')] .$$

Since the function u is continuous, we have $F(x, v_{n_k}^{-1}(t)) \to F(x, u(t))$ for every $x \in \Omega$, $t \in [-T, T]$. It is simple to verify that the sequence of functions $F_k(x, t) =$ $= F(x, t) + g(x)\, \Phi_n(t)$ converges under substitution to the pair (H, v) where $H(x, t) =$ $= F(x, u(t)) + g(x)\, \Psi(t)$; then the equation with a substitution

$$\text{(3.39)} \qquad x(t) = y(v(t)), \quad \frac{\mathrm{d}y}{\mathrm{d}\tau'} = \mathrm{D}[F(y, u(t)) + g(y)\, \Psi(t)]$$

is a limit equation for the sequence $(3.36)_{n_k}$.

In case the functions Φ_n satisfy the condition $\operatorname{var}_{-T}^{-\delta} \Phi_n + \operatorname{var}_{\delta}^{T} \Phi_n \to 0$ for every $\delta > 0$, the function Ψ will be constant on $[T, v(0))$ and on $(v(0+), T]$, and the function v will have a unique discontinuity at 0; then the function u will be constant on $[v(0), v(0+)]$. The equation (3.39) has the form

$$\text{(3.40)} \quad x(t) = y(v(t)), \quad \frac{\mathrm{d}y}{\mathrm{d}\tau'} = \mathrm{D}F(y, u(t)) \quad \text{on} \quad [-T, v(0)] \cup [v(0+), T],$$

$$\frac{\mathrm{d}y}{\mathrm{d}\tau'} = \mathrm{D}[g(y)\, \Psi(t)] \quad \text{on} \quad [v(0), v(0+)] .$$

Since the function Ψ is Lipschitzian, it is absolutely continuous and has a.e. a derivative $\Psi'(t) = \psi(t)$. Using Theorem 4A.1 in [S1] and the fact that the function v is continuous on $[-T, 0)$ and $(0, T]$, we find an equivalent form for (3.40):

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \mathrm{D}F(x, t) \quad \text{on} \quad [-T, 0) \cup (0, T] ;$$

$$x(0) = y(v(0)) \quad \text{and} \quad x(0+) = y(v(0+)), \quad \frac{\mathrm{d}y}{\mathrm{d}t} = g(y)\, \psi(t) .$$

Notice that the equation $\mathrm{d}y/\mathrm{d}t = g(y)\, \psi(t)$ need not have the uniqueness property and the function ψ may depend on the subsequence (Ψ_{n_k}).

Let us return to the former case of an arbitrary sequence (Φ_n) satisfying the condition (iii), moreover assuming that for every $x \in \Omega$ the ordinary differential equation

$$\dot{y} = g(y) \tag{3.41}$$

has a unique maximal solution such that $y(0) = x$, and this will be denoted by $\chi(t, x)$. Denoting $H(x, t) = F(x, u(t)) + g(x)\,\psi(t)$, let us describe the set $R_{(H,v)}$:

Let $(x, t) \in G$ be given such that $v(t) < v(t+)$. Since the function u is constant on $[v(t), v(t+)]$, the equation $\mathrm{d}y/\mathrm{d}\tau' = \mathrm{D}H(y, t')$ will have the form $\mathrm{d}y/\mathrm{d}t' = g(y)\,\psi(t')$ on $[v(t), v(t+)]$, (all solutions of the equation $\mathrm{d}y/\mathrm{d}\tau' = \mathrm{D}H(y, t')$ are continuous, consequently we do not need the solution y on an interval $[v(t), v(t+\delta)]$). The function $y(s) = \chi(\Psi(s) - \Psi(v(t)), x)$ is a unique solution of the initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}s} = g(y)\,\psi(s)\,, \quad y(v(t)) = x\,; \tag{3.42}$$

the function y is defined on $[v(t), v(t+)]$ if the function $\chi(\cdot, x)$ is defined on the set

$$\{\tau \in R;\ \tau = \Psi(s) - \Psi(v(t)) \quad \text{for some} \quad s \in [v(t), v(t+)]\}\,.$$

If y is the unique solution of (3.42) on $[v(t), v(t+)]$ then (x, t) belongs to $R_{(H,v)}$ and we denote

$$p(x, t) = \chi(\Psi(v(t+)) - \Psi(v(t)), x) - x = \chi(\Phi(t+) - \Phi(t), x) - x\,.$$

Then the reduction $\tilde{F}$ of the function H by v which is defined in (1.10) will have the following form:

For every $x \in \Omega$ the continuous part of $\tilde{F}(x, \cdot)$ is equal to $F(x, \cdot) + g(x)\,\Phi^C(\cdot)$, and $\tilde{F}(x, t+) - \tilde{F}(x, t) = \chi(\Phi(t+) - \Phi(t), x) - x$, $(x, t) \in R_{(H,v)}$.

Let us mention that the set $R_{(H,v)}$ can depend on the choice of the subsequence Ψ_{n_k} which converges to Ψ, but the values $\tilde{F}(x, t+) - \tilde{F}(x, t)$ do not.

Let us denote by R the set of all pairs $(x, t) \in G$ such that the function $\chi(\cdot, x)$ is defined on the interval $[-\tilde{c}(v(t+) - v(t)), \tilde{c}(v(t+) - v(t))]$. Let us define

$$F_0(x, t) = F(x, t) + g(x)\,\Phi^C(t) + \sum_{\substack{-T<\tau<t \\ (x,\tau)\in R}} [\chi(\Phi(\tau+) - \Phi(\tau), x) - x]\,.$$

Then the sequence of functions $F_n(x, t) = F(x, t) + g(x)\,\Phi_n(t)$ converges R-ephatically to the function F_0.

References

[F1] *D. Fraňková:* A discontinuous substitution in the generalized Perron integral (to appear).

[F2] *D. Fraňková:* Regulated functions (to appear).

[K1] *J. Kurzweil:* Generalized ordinary differential equations and continuous dependence on a parameter. Czechoslovak Math. J. *7* (*82*) (1957), 418—449.

[K2] *J. Kurzweil:* Generalized ordinary differential equations. Czechoslovak Math. J. *8* (*83*) (1958), 360—388.

[K3] *J. Kurzweil:* Nichtabsolut konvergente Integrale. BSB B. G. Teubner Verlagsgesellschaft, Leipzig 1980.

[S1] *Š. Schwabik:* Generalized differential equations: Fundamental results. Rozpravy ČSAV (*95*) (1985), No 6.

[S2] *Š. Schwabik:* Generalized differential equations: Special results (to appear).

Souhrn

SPOJITÁ ZÁVISLOST ŘEŠENÍ ZOBECNĚNÝCH DIFERENCIÁLNÍCH ROVNIC NA PARAMETRU

DANA FRAŇKOVÁ

V teorii zobecněných diferenciálních rovnic se vyskytuje zajímavý konvergenční efekt, který byl popsán J. Kurzweilem jako R-emfatická konvergence. S použitím pojmu zobecněné diferenciální rovnice se substitucí bude definována tzv. konvergence se substitucí, o níž se ukáže, že je velmi podobná R-emfatické konvergenci. Posloupnost rovnic, která je konvergentní se substitucí, se dá převést na jinou posloupnost rovnic, která ke své limitní rovnici konverguje klasickým způsobem, tj. se stejnoměrnou konvergencí řešení a pravých stran těchto rovnic.

Резюме

НЕПРЕРЫВНАЯ ЗАВИСИМОСТЬ ОТ ПАРАМЕТРА РЕШЕНИЙ ОБОБЩЕННЫХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ

DANA FRAŇKOVÁ

В теории обобщенных дифференциальных уравнений появляется интересный эффект, который был описан Я. Курцвейлом как R-эмфатическая сходимость. В статье при помощи понятия обобщенного дифференциального уравнения с подстановкой определяется так называемая сходимость с подстановкой и показывается, что она очень похожа на R-эмфатическую сходимость. Последовательность уравнений, которая сходится с подстановкой, можно перевести на другую последовательность уравнений, которая сходится к своему предельному уравнению в классическом смысле, т.е. решения и правые части этих уравнений сходятся равномерно.

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