Časopis pro pěstování matematiky

Włodzimierz M. Mikulski

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Časopis pro pěstování matematiky, Vol. 114 (1989), No. 1, 57--59

Persistent URL: `http://dml.cz/dmlcz/118368`

THERE EXISTS A PROLONGATION FUNCTOR OF INFINITE ORDER

W. M. MIKULSKI, Krakow

(Received November 20, 1986)

Summary. An example of a prolongation functor of infinite order is given.

Keywords: prolongation functor, order of prolongation functor.

The concept of a natural bundle, which is due to A. Nijenhuis, originated the study of a wider class of geometric functors. A prolongation functor F in the sense of [3] is a covariant functor defined on the category $\mathcal{M}$ of all smooth finite dimensional manifolds and smooth maps with values in the category $\mathcal{FM}$ of smooth fibred manifolds and their morphisms satisfying the following two conditions:

(1) The composition $B \circ F$ of F with the base functor $B: FM \to M$ is the identity on $\mathcal{M}$.

(2) If $M \in \operatorname{Obj} \mathcal{M}$ and $i: U \to M$ is the inclusion of an open subset, then $Fi: FU \to \pi_M^{-1}(U)$ is an $\mathcal{FM}$-isomorphism, where $\pi_M: \mathcal{FM} \to \mathcal{M}$ is the bundle projection of FM.

Let r be a natural number or infinity. A prolongation functor F is said to be of the order r if for any two manifolds M, N, any maps $f, g: M \to N$ and any point $x \in M$, the condition $j_x^r f = j_x^r g$ implies $F f(y) = F g(y)$ for all points $y \in \pi_M^{-1}(x)$, and r is the smallest number with this property.

The restriction of an arbitrary prolongation functor F to the subcategory $\mathcal{M}_n$ of n-dimensional manifolds and their embeddings is a natural bundle in the sense of A. Nijenhuis. Here a well known result is, [1], that $F \mid \mathcal{M}_n$ has a finite order, provided FR^n has a countable basis.

By a recent description of a general class of geometric functors by means of Weil algebras, which is due to G. Kainz and P. W. Michor, [2], all product-preserving prolongation functors also have finite orders.

Hence it seems to be interesting to discuss the following question: "Has any prolongation functor a finite order?" In this paper we give a counter-example of a prolongation functor of infinite order.

I would like to thank Prof. I. Kolář for valuable suggestions and Dr. J. Slovák for correction.

Example. A class of well known functors in differential geometry consists of the so-called r-th order tangent functors, which can be constructed as follows, see e.g.

[3]. Given an integer $r \geqq 1$ and a manifold M, we set $T^{r*}M = J^r(M, R)_0$ (i.e. the set of all r-jets of M into R with target 0). One easily sees that $T^{r*}M$ is a vector bundle with standard fibre $J_0^r(R^m, R)_0$, provided $\dim M = m$. Let T^rM be the dual vector bundle of $T^{r*}M$. Given any r-jet $A \in J_x^r(M, N)_y$, the composition of jets determines a linear map from the fibre $(T^{r*}N)_y$ over $y \in N$ into $(T^{r*}M)_x$. Hence any smooth map $f: M \to N$ induces a linear $\mathcal{FM}$-morphism $T^{r*}f: f^!T^{r*}N \to T^{r*}M$, where $f^!T^{r*}N$ means the pull-back of $T^{r*}N$ with respect to f. Then we define T^rf: $T^rM \to T^rN$ to be the dual map of $T^{r*}f$ and obtain an r-th order prolongation functor T^r with values in the subcategory $\mathcal{VB} \subset \mathcal{FM}$ of smooth vector bundles.

Now, put $d_k = \dim((T^kR^k)_0)$. For any smooth manifold M we define FM to be the (formally infinite) fibred product over M

$$FM = \mathop{\times_M}_{k \geqq 1} (\Lambda^{d_k}T^kM),$$

and for any smooth map $f: M \to N$ we define Ff to be the fibre product of morphisms

$$Ff = \mathop{\times_M}_{k \geqq 1} (\Lambda^{d_k}T^kf): FM \to FN.$$

Clearly, if $k > \dim M$, then $\Lambda^{d_k}T^kM = M \times \{0\}$, so that we deal in fact with a finite fibred product over every manifold M. Hence F is a prolongation functor.

A simple consideration shows that F is of infinite order. It is sufficient to deduce that the order of $\Lambda^{d_k}T^k$ is at least k. Let $f: R^{d_k} \to R^{d_k}$ be defined by $(x_1, \ldots, x_{d_k}) \mapsto (x_1^k, \ldots, x_{d_k}^k)$. Recalling that the map $T^kf \mid (T^kR^{d_k})_0$ is dual to the map $\tilde{f}: J_0^k(R^{d_k}, R)_0 \to J_0^k(R^{d_k}, R)_0$, $\tilde{f}(j_0^k\gamma) = j_0^k(\gamma \circ f)$, we find (since $\tilde{f}(j_0^k(x_j)) = j_0^k(x_j^k)$) that $\operatorname{rank}(T^kf \mid (T^kR^{d_k})_0) = \operatorname{rank}(\tilde{f}) \geqq d_k$. Therefore $\Lambda^{d_k}(T^kf \mid (T^kR^{d_k})_0) \neq 0$. But $j_0^{k-1}f$ coincides with the $(k-1)$-jet of a constant map, which implies that the order of $\Lambda^{d_k}T^k$ is at least k.

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Souhrn

PROLONGAČNÍ FUNKTOR NEKONEČNÉHO ŘÁDU EXISTUJE

W. M. Mikulski

Je podán příklad prolongačního funktoru nekonečného řádu.

Резюме

ПРОДОЛЖАЮЩИЙ ФУНКТОР БЕСКОНЕЧНОГО ПОРЯДКА СУЩЕСТВУЕТ

W. M. Mikulski

Приведен пример продолжающего функтора бесконечного порядка.

Author's address: Institute of Mathematics, Jagellonian University, ul. Reymonta 4, Kraków, Poland.