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A REMARK ON CANCELLATION IN DIRECT PRODUCTS OF GRAPHS

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Summary. The direct product of two graphs G, G' is the graph $G \times G'$ whose vertex set is the Cartesian product of vertex sets of G and G' and in which two vertices (v_1, v_1'), (v_2, v_2') are adjacent if and only if v_1, v_2 are adjacent in G and v_1', v_2' are adjacent in G'. There exists a family $\mathfrak{F}$ of the power of continuum consisting of pairwise non-isomorphic locally connected non-bipartite graphs with the property that for every bipartite graph G and for any two graphs G_1, G_2 from $\mathfrak{F}$ the graphs $G \times G_1, G \times G_2$ are isomorphic. For every positive integer n there exists such a family of finite graphs which has the cardinality greater than n. This is a negative solution of a problem by V. Puš.

Keywords: direct product of graphs, isomorphism of graphs.

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We consider undirected graphs without loops and multiple edges. If G is a graph, then $V(G)$ denotes its vertex set. The symbol $G + G'$ denotes the union of two vertex-disjoint graphs G and G'. By C_n we denote a circuit of the length n.

The direct product $G \times G'$ of two graphs G and G' is the graph with the vertex set $V(G \times G') = V(G) \times V(G')$ in which two vertices (v_1, v_1'), (v_2, v_2') are adjacent if and only if v_1, v_2 are adjacent in G and v_1', v_2' are adjacent in G'.

The aim of this paper is to show an infinite class of graphs for which the implication

$$(1) \qquad G \times G_1 \cong G \times G_2 \Rightarrow G_1 \cong G_2$$

is not true.

L. Lovász [1, 2] has proved that (1) holds, if G is not bipartite or if all graphs G, G_1, G_2 are bipartite and G is not discrete. Further, for each odd number $k \geqq 3$ and for any bipartite graph we have

$$(2) \qquad G \times C_{2k} \cong G \times (C_k + C_k).$$

At the Czechoslovak Conference on Graph Theory and Combinatorics in Raček Valley in May 1986, V. Puš proposed the following problem [3].

Decide whether (1) *holds provided that*

(i) *neither* G_1 *nor* G_2 *is bipartite*;

(ii) *all graphs* G, G_1, G_2 *are connected.*

We shall extend (2), thus giving the negative answer to this question.

Theorem 1. *Let a finite graph G_1 contain an induced subgraph G_0 isomorphic to the circuit of a length congruent with 2 modulo 4. Let G_0 have the property that any vertex $x \in V(G_1) - V(G_0)$ is adjacent to a vertex $y \in V(G_0)$ if and only if x is adjacent to $\bar{y}$, where $\bar{y}$ is the opposite vertex to y in the circuit G_0. Then there exists a graph G_2 non-isomorphic to G_1 and such that $G \times G_1 \cong G \times G_2$ for any bipartite graph G.*

Proof. As the length of the circuit G_0 is congruent with 2 modulo 4, it is equal to $2k$, where k is an odd integer. Let $V(G_0) = \{u_1, \ldots, u_k, u'_1, \ldots, u'_k\}$, let the edges of G_0 be $u_1u'_k$, $u_ku'_1$ and u_iu_{i+1}, $u'_iu'_{i+1}$ for $i = 1, \ldots, k-1$. The graph G_2 is obtained from G_1 by deleting the edges $u_1u'_k$, $u_ku'_1$ and adding the edges u_1u_k, $u'_1u'_k$.

Now let G be a bipartite graph. Consider the direct products $G \times G_1$, $G \times G_2$. As $V(G_1) = V(G_2)$, also $V(G \times G_1) = V(G \times G_2)$; this is the set of all ordered pairs (v, w) where $v \in V(G)$, $w \in V(G_1)$. Let A, B be the bipartition classes of G. We define a mapping φ of $V(G \times G_1)$ onto $V(G \times G_1)$. If $v \in V(G)$, $w \in V(G_1) - V(G_0)$, then $\varphi((v, w)) = (v, w)$. If $v \in A$, then $\varphi((v, u_i)) = (v, u_i)$, $\varphi((v, u'_i)) = (v, u'_i)$ for i odd and $\varphi((v, u_i)) = (v, u'_i)$, $\varphi((v, u'_i)) = (v, u_i)$ for i even. If $v \in B$, then $\varphi((v, u_i)) = (v, u_i)$, $\varphi((v, u'_i)) = (v, u'_i)$ for i even and $\varphi((v, u_i)) = (v, u'_i)$, $\varphi((v, u'_i)) = (v, u_i)$ for i odd. We shall prove that φ is an isomorphic mapping of $G \times G_1$ onto $G \times G_2$. Let (v_1, w_1), (v_2, w_2) be two vertices of $V(G \times G_1)$. Suppose that they are adjacent in $G \times G_1$. Then v_1, v_2 are adjacent in G and w_1, w_2 are adjacent in G_1. The vertices v_1, v_2 must belong to different bipartition classes of G; without loss of generality we may suppose that $v_1 \in A$, $v_2 \in B$. If both w_1, w_2 are in $V(G_1) - V(G_0)$, then $\varphi((v_1, w_1)) = (v_1, w_1)$, $\varphi((v_2, w_2)) = (v_2, w_2)$; the vertices w_1, w_2 are adjacent also in G_2 and $(v_1, w_1), (v_2, w_2)$ are adjacent also in $G \times G_2$. Suppose that $w_1 \in V(G_1) - V(G_0)$, $w_2 \in V(G_0)$. Then again $\varphi((v_1, w_1)) = (v_1, w_1)$. If $w_2 = u_i$, where i is odd, then $\varphi((v_2, w_2)) = \varphi((v_2, u_i)) = (v_2, u'_i)$. As w_1, u_i are adjacent in G_1, so are w_1, u'_i, because u'_i is the opposite vertex to u_i in G_0. They are adjacent also in G_2 and thus $\varphi((v_1, w_1))$, $\varphi((v_2, w_2))$ are adjacent in $G \times G_2$. Analogously if $w_2 = u'_i$ for i odd. If $w_2 = u_i$ or $w_2 = u'_i$ for i even, then $\varphi((v_2, w_2)) = (v_2, w_2)$ and again $\varphi((v_1, w_1))$, $\varphi((v_2, w_2))$ are adjacent in $G \times G_2$. If $w_1 \in V(G_0)$, $w_2 \in V(G_1) - V(G_0)$, the considerations are analogous. Now let $w_1 \in V(G_0)$, $w_2 \in v(G_0)$. If both w_1, w_2 are in $\{u_1, \ldots, u_k\}$, then $w_1 = u_i$, $w_2 = u_j$, where $j = i + 1$ or $j = i - 1$. If i is odd, then j is even. We have $\varphi((v_1, w_1)) = \varphi((v_1, u_i)) = (v_1, u_i) = (v_1, w_1)$, $\varphi((v_2, w_2)) = \varphi((v_2, u_j)) = (v_2, u_j) = (v_2, w_2)$ and again $\varphi((v_1, w_1))$, $\varphi((v_2, w_2))$ are adjacent in $G \times G_2$. If i is even, then j is odd. We have $\varphi((v_1, w_1)) = \varphi((v_1, u_i)) = (v_1, u'_i)$, $\varphi((v_2, w_2)) = \varphi((v_2, u_j)) = (v_2, u'_j)$. As $j = i + 1$ or $j = i - 1$, the vertices u'_i, u'_j are adjacent in G_1 and in G_2 and the vertices (v_1, u'_i), (v_2, u'_j) are adjacent in $G \times G_2$. Analogously if both w_1, w_2 are in $\{u'_1, \ldots, u'_k\}$. If $w_1 \in \{u_1, \ldots, u_k\}$, $w_2 \in \{u'_1, \ldots, u'_k\}$, then either $w_1 = u_1$, $w_2 = u'_k$, or $w_1 = u_k$, $w_2 = u'_1$. In the former case $\varphi((v_1, w_1)) = \varphi((v_1, u_1)) = (v_1, u_1)$, $\varphi((v_2, w_2)) =$

$= \varphi((v_2, u_k')) = (v_2, u_k)$. As u_1, u_k are adjacent in G_2, the vertices $\varphi((v_1, u_1))$, $\varphi((v_2, u_k'))$ are adjacent in $G \times G_2$. In the latter case $\varphi((v_1, w_1)) = \varphi((v_1, u_k)) = (v_1, u_k)$, $\varphi((v_2, w_2)) = \varphi((v_2, u_1')) = (v_2, u_1)$ and the situation is the same as in the former. Analogously if $w_1 \in \{u_1', \ldots, u_k'\}$, $w_2 \in \{u_1, \ldots, u_k\}$. We have proved that φ maps each pair of vertices adjacent in $G \times G_1$ onto a pair of vertices adjacent in $G \times G_2$. Analogously we may prove that φ^{-1} maps each pair of vertices adjacent in $G \times G_2$ onto a pair of vertices adjacent in $G \times G_1$. The mapping φ is an isomorphism of $G \times G_1$ onto $G \times G_2$.

It remains to prove that G_2 is not isomorphic to G_1. Suppose $G_1 \cong G_2$. The graph G_2 contains an induced subgraph consisting of two vertex-disjoint circuits of the length k with the property that in G_1 none of these circuits exists. As G_1, G_2 are finite, the graph G_1 must also contain an induced subgraph consisting of two vertex-disjoint circuits D_1, D_2 of the length k with the property that in G_2 none of these circuits exists. This implies that one of these circuits, say D_1, contains the edge $u_1 u_k'$ and the other contains the edge $u_k u_1'$. Let x be the vertex of D_1 adjacent to u_1. Then, according to the assumption, x is adjacent to u_1', because this is the opposite vertex to u_1 in G . But u_1' belongs to D_2 and thus there exists an edge joining a vertex of D_1 with a vertex of D_2, which is a contradiction with the assumption that the union of D_1 and D_2 is an induced subgraph of G_1. Hence G_1 and G_2 are not isomorphic. □

Note that the assumption that G_1 is finite was used only in the proof that G_1, G_2 are not isomorphic. Other considerations may be easily extended to the case when G_1, G_2 are infinite. Therefore we may prove another theorem.

Theorem 2. *There exists a family $\mathfrak{F}$ of the power of continuum consisting of pairwise non-isomorphic locally finite connected non-bipartite graphs with the property that for any bipartite graph G and any two graphs G_1, G_2 from $\mathfrak{F}$ we have $G \times G_1 \cong G \times G_2$.*

Proof. Let P be a one-way infinite path whose vertices are x_i and whose edges are $x_i x_{i+1}$ for all positive integers i. Let D_i for all positive integers i be pairwise vertex-disjoint circuits of the length 6 vertex-disjoint with P. In each D_i choose a vertex y_i and by $\bar{y}_i$ denote the vertex of D_i opposite to y_i. Join both y_i and $\bar{y}_i$ by edges with x_i for each i. Denote the graph thus obtained by H. Let $\mathscr{A} = (a_i)_{i=1}^{\infty}$ be a sequence such that $a_i = 0$ or $a_i = 1$ for each i. To the sequence $\mathscr{A}$ we assign the graph $H(\mathscr{A})$ in such a way that for each i such that $a_i = 1$ we perform in H the transformation from the proof of Theorem 1 with D_i, i.e. we replace D_i by two triangles, each of which has one vertex adjacent to x_i. Evidently any two graphs $H(\mathscr{A}_1)$, $H(\mathscr{A}_2)$ for different sequences $\mathscr{A}_1, \mathscr{A}_2$ are non-isomorphic. It follows from the considerations in the proof of Theorem 1 that $G \times H(\mathscr{A}_1) \cong G \times H(\mathscr{A}_2)$ for any bipartite graph G and any two sequences $\mathscr{A}_1, \mathscr{A}_2$ with the described property. As the set of all such sequences is of the power of continuum, the assertion is proved. □

Theorem 3. *For any positive integer n there exists a family $\mathfrak{F}$ of a finite cardinality greater than n consisting of pairwise non-isomorphic finite connected non-bipartite graphs with the property that for any bipartite graph G and any two graphs G_1, G_2 from $\mathfrak{F}$ we have $G \times G_1 \cong G \times G_2$.*

Proof is done analogously as that of Theorem 2 with the only difference that P is a finite path (of an arbitrarily large length). □

Evidently there exists no infinite family of finite graphs with this property, because the vertex sets of all graphs of such a family would have to be of the same cardinality and there are only finitely many non-isomorphic graphs with a given finite number of vertices.

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Souhrn

POZNÁMKA O KRÁCENÍ V DIREKTNÍCH SOUČINECH GRAFŮ

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Existuje systém $\mathfrak{F}$ mohutnosti kontinua skládající se z neisomorfních lokálně konečných souvislých nikoliv sudých grafů té vlastnosti, že pro každý sudý graf G a pro každé dva grafy G_1, G_2 z $\mathfrak{F}$ platí $G \times G_1 \cong G \times G_2$. Pro každé přirozené číslo n existuje takový systém konečných grafů, který má konečnou mohutnost větší než n. Tóto je negativní řešení problému V. Puše.

Резюме

ЗАМЕЧАНИЕ О СОКРАЩЕНИИ В ПРЯМЫХ ПРОИЗВЕДЕНИЯХ ГРАФОВ

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Существуют семейство $\mathfrak{F}$ мощности континуума, состоящее из попарно неизоморфных локально конечных связных недвудольных графов и обладающее тем свойством, что для каждого двудольного графа G и для каждых двух графов G_1, G_2 из $\mathfrak{F}$ имеет место изоморфизм $G \times G_1 \cong G \times G_2$. Для каждого натурального числа n существует аналогичное семейство конечных графов, которое имеет конечную мощность больше чем n. Это решает отрицательно проблему В. Пуша.

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