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Layer potentials on boundaries with corners and edges

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LAYER POTENTIALS ON BOUNDARIES WITH CORNERS AND EDGES

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Summary. Consider a bounded domain (termed rectangular) in 3-space such that each point z on its boundary ∂D has a neighbourhood $U \subset \partial D$ homeomorphic with the plane such that U is contained in the union of the three planes passing through z parallel to the coordinate planes. Let W be the double layer potential operator acting on the space $C(\partial D)$ of all continuous functions on ∂D, let I be the identity operator, and denote by Q the space of all compact linear operators acting on $C(\partial D)$. It may happen that the distance of $W - \alpha I$ from Q, measured by the usual maximum norm, exceeds $|\alpha|$ for each choice of the parameter α. It is shown in the present paper that one can always introduce a new norm p in $C(\partial D)$ inducing the same topology of uniform convergence such that the p-distance of $W - \frac{1}{2}I$ from Q becomes less than $\frac{1}{2}$.

Keywords: double layer potential, integral operators in potential theory.

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We shall adopt the following terminology introduced in [1]. An open set $D \subset \mathbb{R}^3$ is called rectangular if its boundary $\partial D \neq \emptyset$ is compact and each point $z \in \partial D$ has a neighbourhood V in ∂D homeomorphic with an open disc in $\mathbb{R}^2$ such that V is contained in the union of the three planes through z parallel to the coordinate planes.

Roughly speaking, D is rectangular if ∂D is locally a surface and D is built of bricks. Boundary value problems for sets of this type occur frequently in applications. Their treatment by the method of integral equations of the second kind may cause difficulties because of presence of peculiar corners in ∂D (no neighbourhood of which has a $1-1$ orthogonal projection into some plane in $\mathbb{R}^3$ — cf. for instance Example 2 described in [1] and in Lemma 3 below). Our aim in this note is to present a method permitting to overcome these difficulties. For the sake of simplicity we restrict our attention to rectangular sets but we believe that the method will apply to more general piecewise smooth boundaries as well.

If $y \in \partial D$ is not situated on an edge then we denote by $n(y)$ the unit vector of the exterior normal to D at y; for the remaining $y \in \partial D$ we put $n(y) = 0$ (= the zero vector in $\mathbb{R}^3$). The symbol σ will denote the 2-dimensional surface measure. For any fixed $z \in \mathbb{R}^3$ we define the signed measure λ_z on Borel subsets of ∂D by

$$d\lambda_z(y) = \frac{n(y)\,(y - z)}{4\pi|y - z|^3}\, d\sigma(y)\,;$$

note that, in case $z \in \partial D$, $\mathrm{d}\lambda_z/\mathrm{d}\sigma$ vanishes identically in some neighbourhood of z in ∂D, so that λ_z has always bounded variation on ∂D. Geometrical meaning of λ_z is the measure of the spacial angle under which parts of the oriented boundary of D are visible from z.

The symbol $\mathscr{C}(\partial D)$ is used to denote the space of all continuous functions on ∂D endowed with the topology of uniform convergence; the usual maximum norm in $\mathscr{C}(\partial D)$ will be denoted by $\|\ldots\|$. For any $f \in \mathscr{C}(\partial D)$ and $z \in R^3$ the corresponding double layer potential with momentum density f is defined by

$$Wf(z) = \int_{\partial D} f \,\mathrm{d}\lambda_z ;$$

it represents a harmonic function of the variable z on $R^3 \setminus \partial D$. We shall denote by

$$\Omega_r(y) = \{x \in R^3;\ |x - y| < r\}$$

the ball of center y and radius r and by

$$d(y) = \lim_{r \downarrow 0} \frac{\text{volume}\,[\Omega_r(y) \cap D]}{\text{volume}\,[\Omega_r(y)]}$$

the density of D at y. With this notation we have for $y \in \partial D$ and $f \in \mathscr{C}(\partial D)$

$$\lim_{\substack{z \to y \\ z \in D}} Wf(z) = Wf(y) + [1 - d(y)]\, f(y)\,.$$

Defining the so-called direct value of the double layer potential at $y \in \partial D$ by

$$\overline{W}f(y) = Wf(y) + [\tfrac{1}{2} - d(y)]\, f(y)\,,$$

we arrive at a bounded linear operator

$$\overline{W}: f \to \overline{W}f$$

on $\mathscr{C}(\partial D)$. The attempt to represent the solution of the Dirichlet problem with a prescribed boundary condition $g \in \mathscr{C}(\partial D)$ as a double layer potential with an unknown momentum density $f \in \mathscr{C}(\partial D)$ leads to an equation

$$(\tfrac{1}{2}I + \overline{W})f = g\,,$$

where I is the identity operator on $\mathscr{C}(\partial D)$. In connection with applicability of the Riesz-Schauder theory and Fredholm's theorems to this equation it is important to know whether $\overline{W}$ can be sufficiently closely approximated by compact operators (cf. [2]). If the deviation is measured by the maximum norm $\|\ldots\|$, then the distance of $\overline{W}$ from the space $\mathscr{Q}$ of all compact operators acting on $\mathscr{C}(\partial D)$ may exceed the critical value $\frac{1}{2}$, as shown in [1].

If p is a norm on $\mathscr{C}(\partial D)$ inducing the topology of uniform convergence in $\mathscr{C}(\partial D)$ (so that the space of compact operators acting on $\mathscr{C}(\partial D)$ with this norm remains the same) we define the associated essential norm of $\overline{W}$ by

$$\omega_p \overline{W} = \inf \{p(\overline{W} - T);\ T \in \mathscr{Q}\}$$

where, of course, $p(\overline{W} - T)$ denotes the p-norm of the operator $\overline{W} - T$ defined in the usual way. For some choices of p, $\omega_p \overline{W}$ will be smaller than the same quantity corresponding to $\|\ldots\|$; examples showing this are given in [1]. Our main objective in this paper is to present a general construction of a norm p enjoying the properties from the following theorem.

Theorem. *For each rectangular set $D \subset \mathbb{R}^3$ there is a norm p inducting the topology of uniform convergence on $\mathscr{C}(\partial D)$ such that*

$$\omega_p \overline{W} < \tfrac{1}{2}\,.$$

Before going into the proof we shall consider three examples described in the following lemmas 1–3.

Lemma 1. *Put*

$$D_0 = (-\infty, \infty) \times (0, \infty) \times (-\infty, 0) \cup (-\infty, 0) \times (-\infty, \infty) \times$$
$$\times (-\infty, 0) \cup (-\infty, 0) \times (0, \infty) \times (-\infty, \infty)$$

(cf. Fig. 1) and suppose that $\tilde{D} \subset \mathbb{R}^3$ is a rectangular set such that, for suitable $b > 0$,

$$\Omega_{3b}(0) \cap \partial\tilde{D} = \Omega_{3b}(0) \cap \partial D_0\,.$$

Let

$$S: x \to Sx = -x$$

be the symmetry with respect to the origin and put

$$E_b = \{0\} \times (0, b) \times (0, b)\,,$$
$$F_b = \langle 0, b) \times \langle 0, b) \times \{0\}\,,$$
$$G_b = \langle 0, b) \times \{0\} \times (-b, 0)\,,$$
$$\tilde{U}_b = E_b \cup F_b \cup G_b\,,$$
$$U_b = \tilde{U}_b \cup S(\tilde{U}_b)\,.$$

Fix $a \in (1, 2)$ and define the pseudonorm $|f|_b \equiv |f|_{a,b}$ for any $f \in \mathscr{C}(\partial\tilde{D})$ as the maximum of the following three expressions (1)–(3):

(1) $$\sup\{|f(x)|;\ x \in U_b\}\,,$$

(2) $$\sup\{2|f(x)|;\ x \in U_{2b} \setminus U_b\}\,,$$

(3) $$\sup\{|f(x) + f(Sx)|/a;\ x \in \tilde{U}_b\}\,.$$

Clearly, U_{2b} is a neighbourhood of 0 in $\partial\tilde{D}$ and $|\ldots|_b$ just induces the uniform convergence on U_{2b}. Let $\tilde{P}$ be the union of all edges in $\partial\tilde{D}$ and denote by

$$\tilde{\delta}(x) = \operatorname{dist}(x, \tilde{P})$$

the distance of $x \in \partial\tilde{D}$ from $\tilde{P}$. If $\varepsilon \in (0, b\rangle$, $f \in \mathscr{C}(\partial\tilde{D})$ has support in

$$\tilde{P}_\varepsilon = \{x \in \partial\tilde{D};\ \tilde{\delta}(x) < \varepsilon\}$$

and satisfies the conditions

(4) $$\|f\| \leqq 1\,, \quad |f|_{a,b} \leqq 1\,,$$

then

$$|\overline{W}f|_{a,b} \leqq \frac{1}{2}\max\left\{\frac{7 + 2(a-1)}{8}, \frac{3}{4}, \frac{1}{a}\right\} + o_\varepsilon(1)\,;$$

here (and always below) $o_\varepsilon(1)$ is a quantity independent of f (and depending on ε and on the geometry of $\partial\tilde{D}$ only) such that $\lim_{\varepsilon\downarrow 0} o_\varepsilon(1) = 0$.

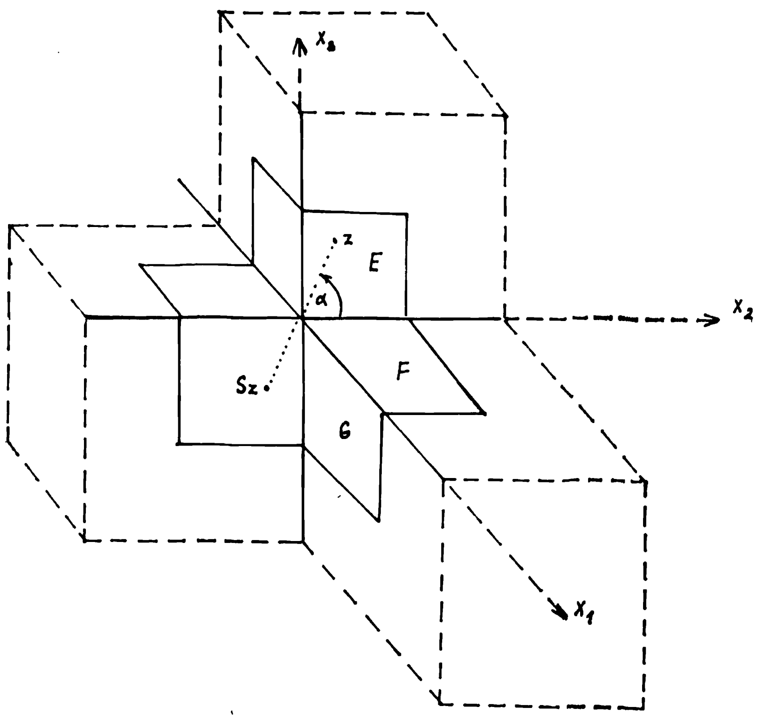

Fig. 1

Proof. Since passing to the complement changes only the sign of $\overline{W}$ we may suppose that

$$\Omega_{3b}(0) \cap \tilde{D} = \Omega_{3b}(0) \cap D_0\,.$$

Let us fix $z = [z_1, z_2, z_3] \in E_b$.

The element

$$\mathrm{d}\lambda_z^0(\zeta) = \frac{n^0(\zeta)\,(\zeta - z)}{4\pi|\zeta - z|^3}\,\mathrm{d}\sigma(\zeta)$$

(here $n^0(\zeta)$ is the exterior normal to D_0 at $\zeta \in \partial D_0$) of the spacial angle under which an elementary oriented area $\mathrm{d}\sigma(\zeta)$ around $\zeta \in S(G_\infty)$ is visible from z exceeds the angle under which a symmetrically situated area around $S\zeta \in G_\infty$ is visible from the same point z. We see that the maximal value of the integral

$$\int_{G_b \cup S(G_b)} f(\zeta)\,\mathrm{d}\lambda_z^0(\zeta)$$

under the conditions

$$|f(\zeta) + f(S\zeta)| \leqq a\,, \quad |f(\zeta)| \leqq 1$$

would be attained for such distribution of values of f for which $f(\zeta) = 1$ when $\zeta \in S(G_b)$ and $f(\zeta) = a - 1$ when $\zeta \in G_b$. Hence we conclude that (4) implies

$$\left|\int_{G_b \cup S(G_b)} f\,\mathrm{d}\lambda_z^0\right| \leqq |\lambda_z^0(S(G_\infty))| + (a - 1)\,|\lambda_z^0(G_\infty)|\,.$$

Similar reasoning yields the estimate

$$\left|\int_{F_b \cup S(F_b)} f\,\mathrm{d}\lambda_z^0\right| \leqq |\lambda_z^0(F_\infty)| + (a - 1)\,|\lambda_z^0(S(F_\infty))|\,.$$

It is also clear that for any $f \in \mathscr{C}(\partial\tilde{D})$ with

(5) $$\operatorname{spt} f \subset \tilde{P}_\varepsilon$$

satisfying $\|f\| \leqq 1$ we have the estimate

$$\left|\int_{\partial\tilde{D}\setminus U_{2b}} f\,\mathrm{d}\tilde{\lambda}_z\right| \leqq \mathrm{o}_\varepsilon(1)\,,$$

where $\mathrm{o}_\varepsilon(1)$ is independent of $z \in E_b$ and f; here, of course,

$$\mathrm{d}\tilde{\lambda}_z(\zeta) = \frac{\tilde{n}(\zeta)\,(\zeta - z)}{4\pi|\zeta - z|^3}\,\mathrm{d}\sigma(\zeta)\,,$$

where $\tilde{n}(\zeta)$ is the unit vector of the exterior normal to $\tilde{D}$ at $\zeta \in \partial\tilde{D}$.

Finally we get for any $z \in E_b$ and f submitted to (4), (5)

$$\left|\int_{U_{2b}\setminus U_b} f\,\mathrm{d}\tilde{\lambda}_z\right| \leqq \frac{1}{2}\,|\tilde{\lambda}_z(\tilde{P}_\varepsilon \cap [U_{2b} \setminus U_b])| \leqq \frac{1}{2}\,\frac{\pi/2}{2\pi}\,\frac{1}{2} + \mathrm{o}_\varepsilon(1) = \frac{1}{16} + \mathrm{o}_\varepsilon(1)\,.$$

Summarizing we have for $z \in E_b$ and all $f \in \mathscr{C}(\partial\tilde{D})$ submitted to (4), (5) the estimate

$$|\overline{W}f(z)| \leqq |\lambda_z^0(F_\infty)| + |\lambda_z^0(S(G_\infty))| + (a - 1)\,[|\lambda_z^0(G_\infty)| + |\lambda_z^0(S(F_\infty))|] + \\ + 1/16 + \mathrm{o}_\varepsilon(1)\,.$$

Writing $\alpha = \operatorname{arctg}(z_3/z_2)$ we can evaluate the spacial angles

(6) $$|\lambda_z^0(F_\infty)| = \frac{\pi - \alpha}{2\pi}\,\frac{1}{2}\,,$$

(7) $$|\lambda_z^0(S(F_\infty))| = \frac{\alpha}{2\pi}\frac{1}{2},$$

(8) $$|\lambda_z^0(S(G_\infty))| = \frac{\pi/2 + \alpha}{2\pi}\frac{1}{2},$$

(9) $$|\lambda_z^0(G_\infty)| = \frac{\pi/2 - \alpha}{2\pi}\frac{1}{2},$$

whence

$$|\overline{W}f(z)| \leqq \frac{\pi - \alpha}{2\pi}\frac{1}{2} + \frac{\pi/2 + \alpha}{2\pi}\frac{1}{2} + (a - 1)\left(\frac{\pi/2 - \alpha}{2\pi}\frac{1}{2} + \frac{\alpha}{2\pi}\frac{1}{2}\right) +$$
$$+ \frac{1}{16} + o_\varepsilon(1) = \frac{7 + 2(a - 1)}{16} + o_\varepsilon(1).$$

By symmetry and continuity the same estimate holds for all $z \in U_b$. We have thus verified for $f \in \mathscr{C}(\partial D)$ satisfying (4), (5) the inequality

(10) $$\sup\{|\overline{W}f(z)|;\ z \in U_b\} \leqq \frac{7 + 2(a - 1)}{16} + o_\varepsilon(1).$$

Observe now that the elementary spacial angle $d\lambda_z^0(\zeta)$ under which the oriented element of area $d\sigma(\zeta)$ around a point $\zeta \in S(G_\infty)$ is visible from $z \in E_b$ has opposite sign in comparison with the elementary spacial angle $d\lambda_{Sz}(\zeta)$ under which the same area is visible from Sz, their magnitudes being related as follows:

$$d\lambda_z^0(\zeta) \geqq -d\lambda_{Sz}^0(\zeta) = |d\lambda_{Sz}^0(\zeta)|.$$

If $f \in \mathscr{C}(\partial\tilde{D})$ satisfies (5) and $\|f\| \leqq 1$, we have thus for any $z \in E_b$

$$|\overline{W}f(z) + \overline{W}f(Sz)| = |\textstyle\int_{S(G_{2b})} f\, d\lambda_z^0 + \int_{S(G_{2b})} f\, d\lambda_{Sz}^0 + \int_{G_{2b}} f\, d\lambda_{Sz}^0 +$$
$$+ \textstyle\int_{G_{2b}} f\, d\lambda_z^0 + \int_{S(F_{2b})} f\, d(\lambda_{Sz}^0 + \lambda_z^0) + \int_{F_{2b}} f\, d(\lambda_z^0 + \lambda_{Sz}^0)| + o_\varepsilon(1) \leqq$$
$$\leqq (\lambda_z^0 + \lambda_{Sz}^0)(S(G_\infty)) + (\lambda_{Sz}^0 + \lambda_z^0)(G_\infty) + (\lambda_{Sz}^0 + \lambda_z^0)(S(F_\infty)) +$$
$$+ |(\lambda_z^0 + \lambda_{Sz}^0)(F_\infty)| + o_\varepsilon(1) \leqq |\lambda_z^0(S(G_\infty))| - |\lambda_{Sz}^0(S(G_\infty))| + |\lambda_{Sz}^0(G_\infty)| -$$
$$- |\lambda_z^0(G_\infty)| + |\lambda_{Sz}^0(S(F_\infty))| - |\lambda_z^0(S(F_\infty))| + |\lambda_z^0(F_\infty)| - |\lambda_{Sz}^0(F_\infty)| + o_\varepsilon(1).$$

If α has the meaning described above, the relations (6)–(9) can be completed by evaluation of the following spacial angles:

(11) $$|\lambda_{Sz}^0(F_\infty)| = \frac{\alpha}{2\pi}\frac{1}{2},$$

(12) $$|\lambda_{Sz}^0(S(F_\infty))| = \frac{\pi - \alpha}{2\pi}\frac{1}{2},$$

$$(13) \qquad |\lambda^0_{Sz}(G_\infty)| = \frac{\pi/2 + \alpha}{2\pi}\frac{1}{2},$$

$$(14) \qquad |\lambda^0_{Sz}(S(G_\infty))| = \frac{\pi/2 - \alpha}{2\pi}\frac{1}{2}.$$

Hence we get

$$|\overline{W}f(z) + \overline{W}f(Sz)| \leqq 2\left(\frac{\pi - \alpha}{2\pi}\frac{1}{2} - \frac{\alpha}{2\pi}\frac{1}{2}\right) + 2\left(\frac{\pi/2 + \alpha}{2\pi}\frac{1}{2} - \frac{\pi/2 - \alpha}{2\pi}\frac{1}{2}\right) + \\ + o_\varepsilon(1) = \tfrac{1}{2} + o_\varepsilon(1).$$

By symmetry and continuity such estimates hold for all $z \in U_b$. We have thus for $f \in \mathscr{C}(\partial\tilde{D})$ satisfying (5) and $\|f\| \leqq 1$ the following inequality:

$$(15) \qquad \sup\{|\overline{W}f(z) + \overline{W}f(Sz)|;\ z \in U_b\} \leqq \tfrac{1}{2} + o_\varepsilon(1).$$

Next we shall consider $z \in E_{2b} \setminus E_b$. Assuming that $f \in \mathscr{C}(\partial\tilde{D})$ fulfils (4), (5) we get for such z

$$|\overline{W}f(z)| \leqq |\tilde{\lambda}_z|(\tilde{P}_\varepsilon \cap U_b) + |\tilde{\lambda}_z|(\tilde{P}_\varepsilon \cap (\partial\tilde{D} \setminus U_{2b})) + \tfrac{1}{2}|\tilde{\lambda}_z|(\tilde{P}_\varepsilon \cap (U_{2b} \setminus U_b)) \leqq \\ \leqq \frac{1}{2}\frac{\pi/2}{2\pi} + o_\varepsilon(1) + \frac{1}{2}\frac{\pi/2}{2\pi}\frac{1}{2} = \frac{3}{16} + o_\varepsilon(1).$$

In view of symmetry and continuity we have for our f

$$(16) \qquad \sup\{|\overline{W}f(z)|;\ z \in U_{2b} \setminus U_b\} \leqq \tfrac{3}{16} + o_\varepsilon(1).$$

Taking into account the relations (10), (15), (16) we obtain for all $f \in \mathscr{C}(\partial\tilde{D})$ satisfying (4), (5) the estimate

$$|\overline{W}f|_{a,b} \leqq \max\left(\frac{7 + 2(a-1)}{16}, \frac{1}{2a}, \frac{3}{8}\right) + o_\varepsilon(1)$$

and our lemma is established.

Remark 1. If $a > 1$ is fixed sufficiently close to 1, then $2(a - 1) < 1$ and for all sufficiently small $\varepsilon > 0$ the right-hand side of the last inequality becomes less than a constant $< 1/2$; this will be crucial for later application of Lemma 1.

The following two lemmas modify the examples considered in [1] into the form which will be needed in the proof of our theorem.

Lemma 2. *Put*

$$D_1 = (-\infty, \infty) \times (0, \infty) \times (-\infty, 0) \cup (0, \infty) \times (-\infty, \infty) \times (-\infty, 0)$$

and suppose that $\tilde{D} \subset R^3$ is a rectangular set such that, for suitable $b > 0$,

$$\Omega_{3b}(0) \cap \partial\tilde{D} = \Omega_{3b}(0) \cap \partial D_1.$$

Put

$$E_{2b} = \langle 0, 2b) \times \langle 0, 2b) \times \{0\},$$
$$\hat{E}_{2b} = \langle 0, 2b) \times (-2b, 0) \times \{0\} \cup (-2b, 0) \times \langle 0, 2b) \times \{0\},$$
$$C_b = \{0\} \times (-b, 0\rangle \times (-b, 0),$$
$$B_b = (-b, 0) \times \{0\} \times (-b, 0),$$

so that

$$U_{2b} = E_{2b} \cup \hat{E}_{2b} \cup B_b \cup C_b$$

is a neighbourhood of the origin in $\partial \tilde{D}$ (cf. Fig. 2). *Fix* $q \in (1, 4/3)$ *and define*

$$w(y) = \begin{cases} \frac{1}{4}q & \text{for } y \in E_{2b}, \\ \frac{3}{4}q & \text{for } y \in \hat{E}_{2b}, \\ 1 & \text{for } y \in U_{2b} \setminus (E_{2b} \cup \hat{E}_{2b}). \end{cases}$$

(*Note that* w *is lower semicontinuous on* U_{2b}, $1/4 < w \leqq 1$.) *Let us define the pseudonorm*

$$|f|_b \equiv |f|_{w,b} = \sup \{|f(y)|/w(y);\ y \in U_{2b}\}$$

on $\mathscr{C}(\partial \tilde{D})$; *clearly* $|\ldots|_b$ *induces uniform convergence on* U_{2b}. *For any* $\varepsilon \in (0, b\rangle$ *define* $\tilde{P}_\varepsilon$ *as in Lemma* 1. *If* $f \in \mathscr{C}(\partial \tilde{D})$ *has support in* $\tilde{P}_\varepsilon$ *and satisfies the conditions*

$$(17) \qquad \|f\| \leqq 1, \quad |f|_{w,b} \leqq 1,$$

then

$$|\overline{W}f|_{w,b} \leqq \frac{1}{2} \max \left(\frac{7q + 8}{16}, \frac{1}{q} \right) + o_\varepsilon(1),$$

where $o_\varepsilon(1)$ *has a similar meaning as in Lemma* 1.

Proof. We may suppose that

$$\Omega_{3b}(0) \cap \tilde{D} = \Omega_{3b}(0) \cap D_1$$

(otherwise we could replace $\tilde{D}$ by the complement of its closure).

If $z \in E_{2b}$, then

$$|\tilde{\lambda}_z(B_b)| + |\tilde{\lambda}_z(C_b)| \leqq \frac{1}{8} = \frac{w(z)}{2q},$$

where $\tilde{\lambda}_z$ has the meaning described in the proof of Lemma 1. If $f \in \mathscr{C}(\partial \tilde{D})$ has support in $\tilde{P}_\varepsilon$ and $\|f\| \leqq 1$, then

$$(18) \qquad \sup \{|\overline{W}f(z)|/w(z);\ z \in E_{2b}\} \leqq \frac{1}{2q} + o_\varepsilon(1),$$

where $o_\varepsilon(1)$ does not depend on f.

If $z \in \hat{E}_{2b}$, then

$$|\tilde{\lambda}_z(B_{3b})| + |\tilde{\lambda}_z(C_{3b})| \leqq \frac{3}{8} = \frac{w(z)}{2q},$$

whence we get again for the same f as above

$$\sup\{|\overline{W}f(z)|/w(z);\ z \in \hat{E}_{2b}\} \leqq \frac{1}{2q} + o_\varepsilon(1). \tag{19}$$

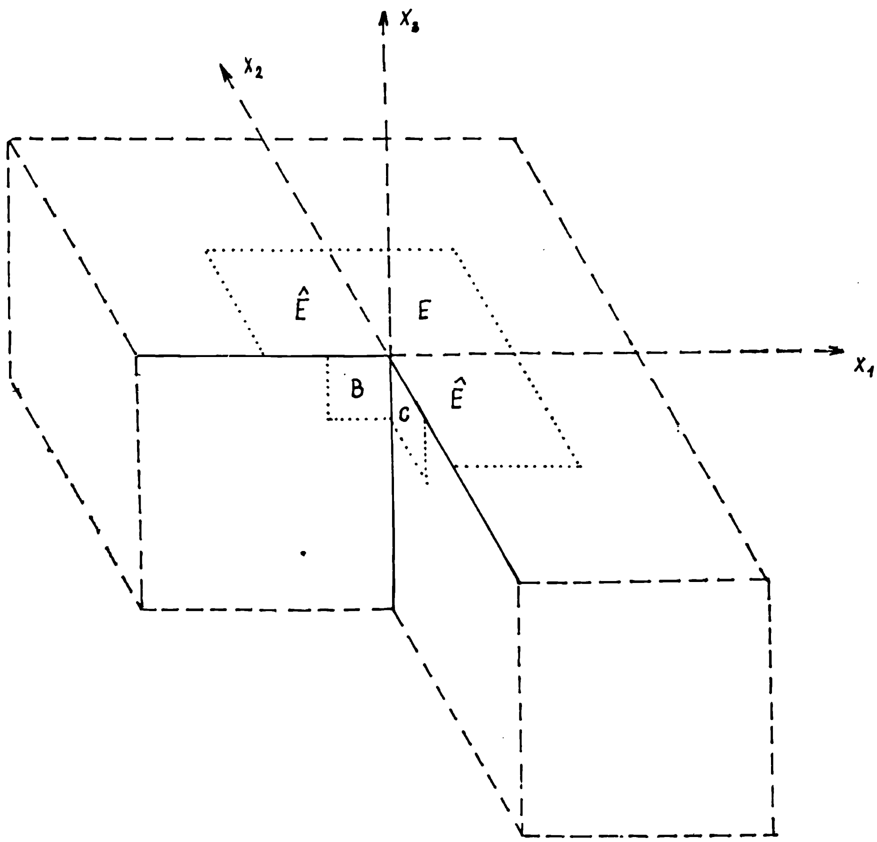

Fig. 2

Consider now $z \in B_b$. The detailed discussion of Example 1 in [1], case (II), shows that the following estimates hold:

$$|\tilde{\lambda}_z(\hat{E}_{2b})| \leqq \tfrac{1}{4}, \quad |\tilde{\lambda}_z(E_{2b})| \leqq \tfrac{1}{8}, \quad |\tilde{\lambda}_z(C_b)| \leqq \tfrac{1}{4}.$$

Hence we get for $f \in \mathscr{C}(\partial \tilde{D})$ satisfying (17) with support in $\tilde{P}_\varepsilon$ the estimate

$$|\overline{W}f(z)| \leqq \frac{1}{4}\frac{3q}{4} + \frac{1}{8}\frac{q}{4} + \frac{1}{4} + o_\varepsilon(1) = \frac{1}{2}\frac{7q+8}{16} + o_\varepsilon(1) .$$

In view of the symmetry, the same estimate holds for $z \in C_b$, whence

$$\sup\{|\overline{W}f(z)|;\ z \in B_b \cup C_b\} \leqq \frac{1}{2}\frac{7q+8}{16} + o_\varepsilon(1) . \tag{20}$$

Combining (18)–(20) we obtain for the same f as above

$$|\overline{W}f|_{w,b} \leqq \frac{1}{2}\max\left(\frac{1}{q}, \frac{7q+8}{16}\right) + o_\varepsilon(1) \tag{21}$$

and the proof is complete.

Remark 2. If q_0 is the positive root of the equation

$$1/q = (7q + 8)/16 ,$$

then $q_0 \in (1, 8/7)$ and we may choose $q = q_0$ in Lemma 2 to guarantee that the right hand side in the inequality (21) is below a fixed constant $< 1/2$ for all sufficiently small $\varepsilon > 0$.

Lemma 3. *Put*

$$\begin{aligned}
D_2 &= (-\infty, \infty) \times (-\infty, 0) \times (-\infty, 0) \cup (-\infty, 0) \times (0, \infty) \times (-\infty, \infty) \cup \\
&\quad \cup (-\infty, 0) \times \{0\} \times (-\infty, 0) , \\
C &= \{0\} \times (0, \infty) \times (-\infty, \infty) , \\
C_1 &= \{[0, x_2, x_3]\} \in C;\ |x_3| \leqq \sqrt{3}\, x_2\} , \\
C_2 &= C \setminus C_1 , \\
E &= (-\infty, \infty) \times (-\infty, 0\rangle \times \{0\} , \\
E_1 &= \{[x_1, x_2, 0] \in E;\ |x_1| \leqq -\sqrt{3}\, x_2\}
\end{aligned}$$

(cf. Fig. 3). *Fix a constant* $q \in (1, 6/5)$ *and define*

$$w(y) = \begin{cases} \frac{5}{6}q & \text{for } y \in C_1 \cup E_1 , \\ 1 & \text{for } y \in \partial D_2 \setminus (C_1 \cup E_1) . \end{cases}$$

(Note that w *is lower semicontinuous on* ∂D_2, $5/6 < w \leqq 1$.*) Let* $\tilde{D} \subset \mathbb{R}^3$ *be a rectangular set such that, for suitable* $b > 0$,

$$\Omega_{3b}(0) \cap \partial \tilde{D} = \Omega_{3b}(0) \cap \partial D_2 .$$

Writing

$$U_b = \{x = [x_1, x_2, x_3] \in \partial \tilde{D};\ |x_k| < b,\ 1 \leqq k \leqq 3\}$$

we define the pseudonorm

$$|f|_b \equiv |f|_{w,b} = \sup \{|f(y)|/w(y);\ y \in U_{2b}\}$$

for $f \in \mathscr{C}(\partial \tilde{D})$; *clearly* $|\ldots|_b$ *just induces uniform convergence on* U_{2b}. *For any* $\varepsilon \in (0, b\rangle$ *define* $\tilde{P}_\varepsilon$ *as in Lemma* 1. *If* $f \in \mathscr{C}(\partial \tilde{D})$ *has support in* $\tilde{P}_\varepsilon$ *and satisfies* (17), *then*

$$|\overline{W}f|_{w,b} \leqq \frac{1}{2} \max \left\{ 1/q,\ (9 + 5q)/18 + \frac{1}{\pi} \operatorname{arctg} \sqrt{2/3} \right\} + o_\varepsilon(1),$$

where $o_\varepsilon(1)$ *has the meaning described in Lemma* 1.

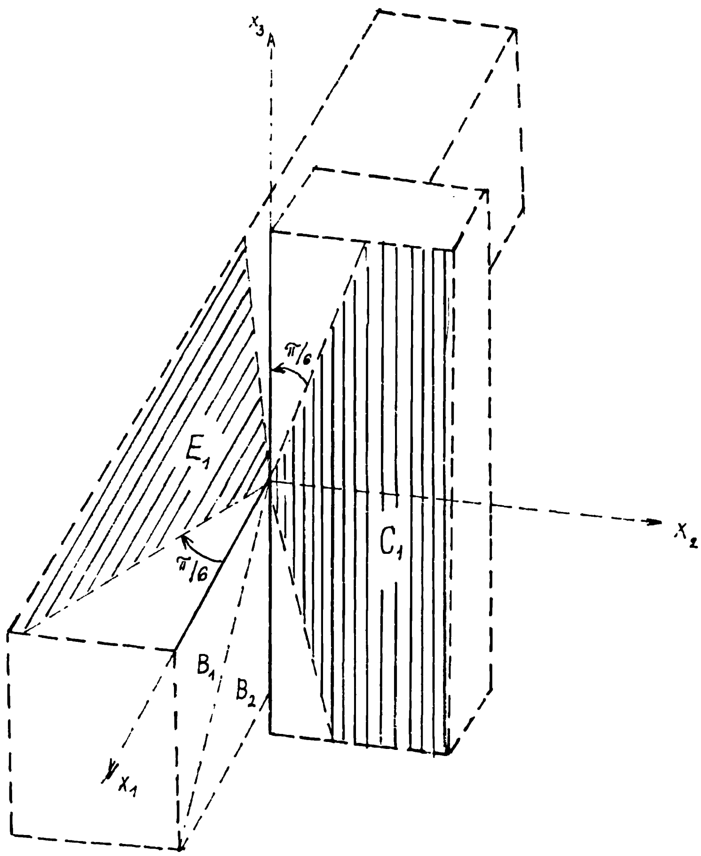

Fig. 3

Proof. We shall again suppose, as we may, that

$$\Omega_{3b} \cap \tilde{D} = \Omega_{3b} \cap D_2 .$$

Put

$$B = (0, \infty) \times \{0\} \times (-\infty, 0),$$
$$B_1 = \{[x_1, 0, x_3] \in B;\ x_3 \geqq -x_1\},$$
$$B_2 = B \setminus B_1 .$$

Consider first $y \in B_1$. If $\lambda_y^2(M)$ denotes the normalized spacial angle unter which a Borel set $M \subset \partial D_2$ is visible from y, then we have the estimates

$$|\lambda_y^2(E)| \leqq 1/4,$$
$$|\lambda_y^2(C_1)| \leqq 1/6 .$$

Writing

$$c = \frac{9}{2\pi} \operatorname{arctg} \sqrt{(2/3)} \in (0, 1)$$

we have also

$$|\lambda_y^2(C \setminus C_1)| \leqq c/9$$

(cf. the discussion of Example 2 in [1], case (A)). Let now $\varepsilon \in (0, b\rangle$. If $f \in \mathscr{C}(\partial\tilde{D})$ has its support in $\tilde{P}_\varepsilon$ and satisfies (17), then we have for $y \in B_1 \cap U_{2b}$

$$|\overline{W}f(y)| \leqq \frac{1}{4} + \frac{c}{9} + \frac{5q}{6}\frac{1}{6} + o_\varepsilon(1) = \frac{1}{2}\frac{9 + 4c + 5q}{18} + o_\varepsilon(1),$$

where $o_\varepsilon(1)$ does not depend on f.

Next consider $y \in B_2$. We have then symmetrically

$$|\lambda_y^2(C)| \leqq \frac{1}{4}, \quad |\lambda_y^2(E_1)| \leqq \frac{1}{6}, \quad |\lambda_y^2(E \setminus E_1)| \leqq \frac{c}{9},$$

so that we obtain again for the same f as above and for $y \in B_2 \cap U_{2b}$

$$|\overline{W}f(y)| \leqq \frac{1}{2}\frac{9 + 4c + 5q}{18} + o_\varepsilon(1) . \tag{22}$$

We see that (22) holds for all $y \in B \cap U_{2b}$ and $f \in \mathscr{C}(\partial\tilde{D})$ described above.

Suppose now that $y \in C_2$ and put

$$\tilde{B} = \{x = [x_1, x_2, x_3] \in \partial D_2,\ x_2 = 0\} .$$

Then we have the inequalities

$$|\lambda_y^2(\tilde{B})| \leqq 1/4, \quad |\lambda_y^2(E_1)| \leqq 1/6, \quad |\lambda_y^2(E \setminus E_1)| \leqq 1/12,$$

whence we get for $f \in \mathscr{C}(\partial\tilde{D})$ with support in $\tilde{P}_\varepsilon$ satisfying (17) and for $y \in C_2 \cap U_{2b}$

$$|\overline{W}f(y)| \leqq \frac{1}{2}\left(\frac{1}{2} + \frac{5q}{6}\frac{1}{3} + \frac{1}{6}\right) + o_\varepsilon(1) = \frac{1}{2}\frac{12 + 5q}{18} + o_\varepsilon(1), \tag{23}$$

where $o_\varepsilon(1)$ has the usual meaning.

Now let $y \in C_1$. Then

$$|\lambda_y^2(\tilde{B})| \leqq 1/4\,, \quad |\lambda_y^2(E)| \leqq 1/6\,,$$

so that we obtain for f specified above and for $y \in C_1 \cap U_{2b}$

$$(24) \qquad |\overline{W}f(y)| \leqq \frac{1}{2}\left(\frac{1}{2} + \frac{1}{3}\right) + o_\varepsilon(1) = \frac{1}{2}\frac{5}{6} + o_\varepsilon(1) = \frac{w(y)}{2q} + o_\varepsilon(1)\,.$$

In view of the symmetry, (24) holds also for $y \in E_1 \cap U_{2b}$, (22) holds for $y \in (\tilde{B} \setminus B) \cap \cap U_{2b}$ and (23) holds for $y \in (E \setminus E_1) \cap U_{2b}$ as well.

Summarizing we get for $y \in U_{2b}$ and f described above

$$|\overline{W}f(y)|/w(y) \leqq \frac{1}{2} \max\left(1/q,\ \frac{12 + 5q}{18},\ \frac{9 + 4c + 5q}{18}\right) + o_\varepsilon(1) =$$
$$= \frac{1}{2} \max\left(1/q,\ \frac{9 + 4c + 5q}{18}\right) + o_\varepsilon(1)$$

and the lemma is established.

Remark 3. If q_1 is the positive root of the equation

$$1/q = \frac{9 + 4c + 5q}{18}\,,$$

where c has the meaning described in the above proof, then $1 < q_1 < 6/5$ and choosing $q = q_1$ we have for our f

$$|\overline{W}f|_{w,b} \leqq \frac{1}{2q_1} + o_\varepsilon(1)\,,$$

whence we infer that $|\overline{W}f|_{w,b}$ is below a fixed constant $< 1/2$ for all sufficiently small $\varepsilon > 0$.

Now we are in position to present the following

Proof of Theorem. Let P be the union of all edges in ∂D, denote by $\delta(x)$ the distance of $x \in \partial D$ from P and put

$$P_\varepsilon = \{x \in \partial D;\ \delta(x) < \varepsilon\}\,.$$

We shall say that a point $z \in \partial D$ is critical if, for each $\varepsilon > 0$,

$$\limsup_{\substack{y \to z \\ y \in \partial D}} |\lambda_y|\,(P_\varepsilon) \geqq \tfrac{1}{2}\,.$$

Let us denote by H the set of all critical points in ∂D; clearly, all points in H are vertices. It turns out that

$$H = \bigcup_{i=0}^{2} H_i\,,$$

where H_i denotes the set of those $z \in H$ for which D can be mapped isometrically onto a set $\tilde{D}$ in such away that z is mapped into the origin and, for suitable $b > 0$,

$$\Omega_{3b}(0) \cap \partial\tilde{D} = \Omega_{3b}(0) \cap \partial D_i \,, \tag{25}$$

where D_i is the set described in Lemma numbered $(i + 1)$ above $(0 \leqq i \leqq 2)$; for $z \in H_i$ we denote by τ_z the corresponding isometric mapping of $\partial\tilde{D}$ onto ∂D. We may suppose that $b > 0$ has been fixed so small that $6b$ is less than the distance of any two different vertices in ∂D. We have seen in the above lemmas that there is a distinguished pseudonorm $|\ldots|_b$ inducing uniform convergence on some neighbourhood U_{2b} of the origin in $\partial\tilde{D}$. We put $U^z = \tau_z(U_{2b})$ and, denoting by $f \circ \tau_z$ the composition of τ_z and f, define

$$p_z(f) = |f \circ \tau_z|_b \,, \quad f \in \mathscr{C}(\partial D) \,.$$

Clearly, U^z is a neighbourhood of z in ∂D and p_z is a pseudonorm inducing uniform convergence on U^z.

For any $f \in \mathscr{C}(\partial D)$ we shall now define $p(f)$ as the maximum of all $p_z(f)$ $(z \in H)$ and of

$$\sup \{|f(x)|; \; x \in \partial D \setminus \bigcup_{z \in H} U^z\} \,.$$

It is easily seen that $p(\cdot)$ is a norm on $\mathscr{C}(\partial D)$ inducing the topology of uniform convergence such that

$$(f \in \mathscr{C}(\partial D), \; p(f) \leqq 1) \Rightarrow \|f\| \leqq 1 \,.$$

It is not difficult to verify that

$$\limsup_{\varepsilon \downarrow 0} \{|\lambda_x| \, (P_\varepsilon); \; x \in \partial D \setminus \bigcup_{z \in H} U^z\} \leqq \tfrac{3}{8} \,. \tag{26}$$

Let us fix, for each $\varepsilon > 0$, a symmetric function ϕ_ε on $\mathbb{R}$ such that $0 \leqq \phi_\varepsilon \leqq 1$,

$$\phi_\varepsilon(t) = 0 \quad \text{for} \quad |t| \geqq \tfrac{3}{4}\varepsilon \,, \quad \phi_\varepsilon(t) = 1 \quad \text{for} \quad |t| \leqq \tfrac{1}{2}\varepsilon$$

and define the function Φ_ε on ∂D by

$$\Phi_\varepsilon(x) = 1 - \phi_\varepsilon(\delta(x)) \,, \quad x \in \partial D \,.$$

Since all the functions $\Phi_\varepsilon \,.\, f$ vanish in some neighbourhood of P, one easily verifies that all the functions in

$$\{\overline{W}(\Phi_\varepsilon \,.\, f); \; f \in \mathscr{C}(\partial D), \; p(f) \leqq 1\}$$

are equicontinuous and uniformly bounded.

In other words, the operator

$$T_\varepsilon : f \to \overline{W}(\Phi_\varepsilon \,.\, f)$$

is compact in $\mathscr{C}(\partial D)$. We are now going to estimate the norm of the operator

$$\overline{W} - T_\varepsilon : f \to \overline{W}((1 - \Phi_\varepsilon) f) \,.$$

Fix an arbitrary $f \in \mathscr{C}(\partial D)$ with $p(f) \leqq 1$ and observe that, for sufficiently small $\varepsilon \in (0, b\rangle$ and any $z \in H$,

$$\operatorname{spt}(1 - \Phi_\varepsilon) \subset P_\varepsilon\,, \quad \|(1 - \Phi_\varepsilon) f\| \leqq 1\,, \quad p_z((1 - \Phi_\varepsilon) f) \leqq p_z(f)\,;$$

the last inequality holds even if $z \in H_0$ thanks to the fact that Φ_ε is symmetric on U_z with respect to z.

Using Lemmas 1–3 (cf. also Remarks 1–3) we conclude that, for all $z \in H$ and a suitable constant c we have

$$p_z(\overline{W}((1 - \Phi_\varepsilon) f)) \leqq c + o_\varepsilon(1) < \tfrac{1}{2}\,,$$

where c, $o_\varepsilon(1)$ are independent of f. As observed in (26), we have also

$$\sup \{|\overline{W}((1 - \Phi_\varepsilon) f)(x)|;\ x \in \partial D \setminus \bigcup_{z \in H} U^z\} \leqq$$

$$\leqq \sup \{|\lambda_x|(P_\varepsilon);\ x \in \partial D \setminus \bigcup_{z \in H} U^z\} \leqq \tfrac{3}{8} + o_\varepsilon(1)\,.$$

We conclude that, for all $f \in \mathscr{C}(\partial D)$ with $p(f) \leqq 1$, and suitable c

$$p(\overline{W}((1 - \Phi_\varepsilon) f)) \leqq c + o_\varepsilon(1) < \tfrac{1}{2}\,.$$

Since c, $o_\varepsilon(1)$ do not depend on f and $o_\varepsilon(1)$ tends to zero as $\varepsilon \downarrow 0$ we see that

$$p(\overline{W} - T_\varepsilon) < \tfrac{1}{2}$$

for all sufficiently small $\varepsilon > 0$, so that

$$\omega_p(\overline{W}) < \tfrac{1}{2}\,.$$

Thus the proof is complete.

Remark. The above proved theorem implies that, for any bounded rectangular set $D \subset \mathbb{R}^3$ whose complement is connected, the solution of the Dirichlet problem with an arbitrarily prescribed boundary condition $g \in \mathscr{C}(\partial D)$ is always representable as a double layer potential Wf in D with a uniquely determined $f \in \mathscr{C}(\partial D)$. More generally, Corollary 2 established in [1] for admissible multiply connected rectangular sets remains in force for arbitrary multiply connected rectangular sets.

Similarly, Corollary 1 from [1] dealing with representability of the solution of the generalized Neumann problem by a potential of a signed measure supported in the boundary remains valid for arbitrary rectangular sets.

We refer the reader to [1] for further references concerning applicability of layer potentials to boundary value problems in domains with irregular boundaries.

References

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Souhrn

POTENCIÁLY DVOJVRSTVY NA HRANICÍCH S HRANAMI A VRCHOLY

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Nechť D je omezená (tzv. rektangulární) oblast v trojrozměrném prostoru, pro niž každý bod z na hranici ∂D má takové okolí $U \subset \partial D$ homeomorfní s rovinou, že U je obsaženo ve sjednocení tří rovin vedených bodem z rovnoběžně se souřadnými rovinami. Buď W operátor potenciálu dvojvrstvy na prostoru $C(\partial D)$ všech spojitých funkcí na ∂D, nechť I značí identický operátor a Q prostor všech kompaktních lineárních operátorů na $C(\partial D)$. Je známo, že vzdálenost $W - \alpha I$ od Q (měřená pomocí supremové normy) může převyšovat $|\alpha|$ pro každou volbu parametru α. V článku je dokázáno, že na prostoru $C(\partial D)$ lze vždy zavést novou normu p (indukující stejnou topologii stejnoměrné konvergence) takovým způsobem, aby p-vzdálenost operátoru $W - \frac{1}{2}I$ od Q byla menší než $\frac{1}{2}$.

Резюме

ПОТЕНЦИАЛЫ ДВОЙНОГО СЛОЯ НА ГРАНИЦАХ С ВЕРШИНАМИ И РЕБРАМИ

T. S. ANGELL, R. E. KLEINMAN, J. KRÁL

Рассмотрим ограниченную область D в трёхмерном пространстве (называемую прямоугольной) такую, что для каждой точки z границы ∂D существует такая гомеоморфная плоскости окрестность $U \subset \partial D$, что U содержится в соединении трех плоскостей, проходящих через z и параллельных координатным плоскостям. Пусть W— оператор потенциала двойного слоя, действующий на пространстве $C(\partial D)$ всех непрерывных функций на ∂D. Обозначим через I тождественный оператор и через Q пространство всех компактных линейных операторов на $C(\partial D)$. Известно, что расстояние от $W - \alpha I$ до Q относительно обыкновенной максимум-нормы может превосходить $|\alpha|$ для каждого параметра α. В настоящей статье доказывается, что в $C(\partial D)$ всегда можно ввести новую норму p, индуцирующую ту же самую топологию равномерной сходимости, таким образом, чтобы расстояние от $W - \frac{1}{2}I$ до Q относительно p стало меньше чем $\frac{1}{2}$.

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