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ON ORTHOGONALITY IN NETS

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Summary. The orthogonality of (ordinary) lines in the Desargues net satisfying the quadrangle closure condition is investigated; the order and the degree of the net are infinite cardinals. An algebraic equivalent of the net with the quadrangle closure condition satisfied is derived.

Orthogonality of lines in a net is understood as that characterized by the so-called trivial axioms of orthogonality. As a closure condition for the orthogonality of lines, the Reidemeister theorem on orthologic quadrangles is used.

The main result is an algebraic equivalent of the net in which the quadrangle closure condition and the orthologic quadrangles theorem hold simultaneously.

Keywords: Desargues net, coordinate algebra, closure condition, orbit of a (nonzero) element, orthogonality of lines, orthological quandrangles.

Classification AMS: 51A15.

INTRODUCTION

Papers [6] and [7] deal with closure conditions for the orthogonality of lines in an affine plane and with the corresponding properties of the coordinate algebra of the affine plane with orthogonal lines. The present paper follows up with [6] and [7]. It deals with the orthogonality of lines in a net – more precisely in a projective net $\mathcal{N} = (\mathbf{P}, \mathcal{L}, (\mathrm{V}_\iota)_{\iota \in I})$ in which all singular points lie on the singular line. The definition of such a net can be found, e.g., in [2] and therefore it will be not repeated here. Also, [2] contains the definitions of other notions to be dealt with: ordinary and singular points of the net, ordinary lines and the singular line of the net, joinable and non-joinable points of the net, the degree and the order of the net, the frame of the net, etc. Our notation follows [2].

As the coordinate algebra of $\mathcal{N}$ (with respect to a given frame) we shall use an admissible algebra $\mathfrak{A} = (\mathbf{S}, 0, (\sigma_\iota)_{\iota \in J}, (+_\iota)_{\iota \in J})$. The definition of an admissible algebra as well as the construction of the coordinate algebra $\mathfrak{A}(\mathcal{N})$ (with respect to a given frame) for the given net $\mathcal{N}$ and the construction of the net $\mathcal{N}(\mathfrak{A})$ for the given admissible algebra $\mathfrak{A}$, can be found in [2] or [3].

By orthogonality of lines in net $\mathcal{N}$ we understand a binary relation "$\perp$" in the set of all ordinary lines. The relation "$\perp$" is required to satisfy only the so-called trivial axioms of orthogonality. The case when at least one ordinary line of $\mathcal{N}$ is isotropic (a line $\mathbf{g}$ is isotropic if $\mathbf{g} \perp \mathbf{g}$) will be excluded from our considerations.

When studying the orthogonality of lines in a net we shall employ the Reidemeister theorem on orthological quadrangles (as a closure condition for the orthogonality of lines in the net). The validity of the orthologic quadrangles theorem assumes the validity of another, the so-called quadrangle closure condition in the net in question. The nets satisfying the quadrangle closure conditions are studied in [1]. In Section 2 of the present paper we point out the relationship between the validity of the quadrangle closure condition in the net and the validity of the minor (affine) and the major (affine) Desargues conditions in the net. (For these conditions see Section 1.) This yields an algebraic equivalent of a net satisfying the quadrangle closure condition.

In Section 3 we define the notion of an orbit of a nonzero element of the support $\mathbf{S}$ of the coordinate algebra $\mathfrak{A}(\mathcal{N})$ of $\mathcal{N}$. Theorem 9 describes the relationship between an arbitrary orbit (enlarged by the zero element 0 and equipped with two binary operations "$+$" and "$\bullet$") and the skewfield $(\mathbf{J}', +_0, \circ)$ of indexes of $\mathfrak{A}(\mathcal{N})$.

Using the diagonal, the generalized diagonal and the Reidemeister closure condition (see Section 1) in $\mathcal{N}$ with the orthogonality of lines, in Section 4 we derive some properties of the mapping $\mathbf{I} \to \mathbf{I}$, $\varkappa \mapsto (\varkappa)\,\mathbf{p}$, where $\mathsf{OV}_{(\varkappa)\mathbf{p}} \perp \mathsf{OV}_{\varkappa}$ and O is an ordinary point. These properties are then exploited to determine an algebraic equivalent of $\mathcal{N}$ in which the quadrangle closure condition and the orthologic quadrangles theorem hold.

1. CLOSURE CONDITIONS IN A NET

When studying the orthogonality of lines in a net $\mathcal{N}$ we shall often use some known closure conditions and the corresponding properties of the coordinate algebra $\mathfrak{A}(\mathcal{N})$ of this net. We have in mind the minor (affine) Desargues condition, the major (affine) Desargues condition, the diagonal and the generalized diagonal conditions, and the Reidemeister condition. These conditions are defined and studied in [2], [4] and [5].

Throughout the paper we shall assume that the net $\mathcal{N} = (\mathbf{P}, \mathcal{L}, (\mathsf{V}_\iota)_{\iota\in I})$ satisfies the following conditions:

1. The degree k of $\mathcal{N}$ is an infinite cardinal number;
2. $\mathcal{N}$ has the following property:
   (rs) $\mathcal{N}$ possesses a frame $(\mathsf{O}; \alpha, \beta, \gamma)$ and a coordinate algebra $\mathfrak{A}(\mathcal{N}) = (\mathbf{S}, 0, (\boldsymbol{\sigma}_\iota)_{\iota\in\mathbf{J}}, (+_\iota)_{\iota\in\mathbf{J}})$ with respect to this frame;
3. $\mathcal{N}$ is a Desargues net, i.e., $\mathcal{N}$ has the following properties:
   (mD) $\mathcal{N}$ satisfies (universally) the minor (affine) Desargues condition;
   (VD) $\mathcal{N}$ satisfies the major (affine) Desargues condition.

If $\mathcal{N}$ has the properties (rs) and (mD) then the following theorem holds for $\mathfrak{A}(\mathcal{N})$ (cf. [2]).

Theorem 1. *Let $\mathcal{N}$ be a net such that* (rs) *holds. Then for each index $\iota \in \mathbf{J}$ we have $+_\iota = +_\gamma =: +$, $(\mathbf{S}, +)$ is an abelian group and $\boldsymbol{\sigma}_\iota$ is an automorphism of $(\mathbf{S}, +)$ for each index $\iota \in \mathbf{J}$ provided $\mathcal{N}$ has the property* (mD).

Let $\mathcal{N} = (\mathbf{P}, \mathcal{L}, (\mathrm{V}_\iota)_{\iota \in I})$ be a net which possesses the property (rs). Define:
(1) $\boldsymbol{J}' := \boldsymbol{J} \cup \{\beta\}$;
(2) $\boldsymbol{\sigma}_\beta : \boldsymbol{S} \to \boldsymbol{S},\ x \mapsto (x)\,\boldsymbol{\sigma}_\beta = 0$;
(3) $\boldsymbol{\Sigma}' := \{\boldsymbol{\sigma}_\iota \mid \iota \in \boldsymbol{J}'\}$;
(4) for each two elements $\boldsymbol{\sigma}_\varkappa, \boldsymbol{\sigma}_\lambda \in \boldsymbol{\Sigma}'$ define their sum "$\mathring{+}$" by

$$(x)\,(\boldsymbol{\sigma}_\varkappa \mathring{+} \boldsymbol{\sigma}_\lambda) = (x)\,\boldsymbol{\sigma}_\varkappa + (x)\,\boldsymbol{\sigma}_\lambda \ \forall x \in \boldsymbol{S};$$

(5) for each two elements $\varkappa, \lambda \in \boldsymbol{J}'$ define their sum "$+_0$" by

$$\varkappa +_0 \lambda = \varrho \Leftrightarrow \boldsymbol{\sigma}_\varkappa \mathring{+} \boldsymbol{\sigma}_\lambda = \boldsymbol{\sigma}_\varrho\,.$$

In the Desargues net $\mathcal{N}$ which has an infinite degree k and possesses the property (rs), the diagonal and the generalized diagonal conditions (defined and investigated in [4]) are special cases of the major (affine) Desargues condition. Hence, by the results obtained in [4], the following holds:

Theorem 2. *Let $\mathcal{N}$ be a Desargues net with an infinite degree k which possesses the property* (rs). *Then* $(\boldsymbol{\Sigma}', \mathring{+})$ *is an (abelian) group with $\boldsymbol{\sigma}_\beta$ as its neutral element.*

Since the mapping $\boldsymbol{\Sigma}' \to \boldsymbol{J}',\ \boldsymbol{\sigma}_\xi \mapsto \xi$ is an isomorphism of the groupp $(\boldsymbol{\Sigma}', \mathring{+})$ onto $(\boldsymbol{J}', +_0)$, Theorem 2 yields the following

Corollary. *Let $\mathcal{N}$ be a Desargues net with an infinite degree k which has the property* (rs). *Then* $(\boldsymbol{J}', +_0)$ *is an (abelian) group with β as its neutral element.*

Remark 1. a) In $(\boldsymbol{J}', +_0)$, the inverse element to ξ is denoted by $-_0\,\xi$.
b) In $(\boldsymbol{\Sigma}', \mathring{+})$, the inverse element to $\boldsymbol{\sigma}_\xi$ is denoted by $-^0\,\boldsymbol{\sigma}_\xi$ and we have $-^0\,\boldsymbol{\sigma}_\xi = \boldsymbol{\sigma}_{-_0\xi}$.

If $\mathcal{N} = (\mathbf{P}, \mathcal{L}, (\mathrm{V}_\iota)_{\iota \in I})$ has the property (rs) then define:

(6) $$\boldsymbol{\Sigma} := \boldsymbol{\Sigma}' \setminus \{\boldsymbol{\sigma}_\beta\}\ ;$$

(7) for each two elements $\boldsymbol{\sigma}_\varkappa, \boldsymbol{\sigma}_\lambda \in \boldsymbol{\Sigma}$ define their product – composition by

$$(x)\,(\boldsymbol{\sigma}_\varkappa \boldsymbol{\sigma}_\lambda) = ((x)\,\boldsymbol{\sigma}_\varkappa)\,\boldsymbol{\sigma}_\lambda \quad \forall x \in \boldsymbol{S}\ ;$$

(8) for each two elements $\varkappa, \lambda \in \boldsymbol{J}$ define their product "$\circ$" by

$$\varkappa \circ \lambda = \varrho \Leftrightarrow \boldsymbol{\sigma}_\varkappa \boldsymbol{\sigma}_\lambda = \boldsymbol{\sigma}_\varrho\,.$$

In a Desargues net $\mathcal{N}$ the degree k of which is an infinite cardinal, the validity of the so-called major Reidemeister condition (for the definition, see [5]) follows from (VD). The special case of the major Reidemeister condition from [5] is the Reidemeister condition defined in [4]. Thus, according to the results of [4], the following holds:

Theorem 3. *Let $\mathcal{N}$ be a Desargues net with an infinite degree k which possesses the property* (rs). *Then the set $\mathbf{\Sigma}$ together with the operation of the composition of maps is a group* $(\mathbf{\Sigma},\ )$ *with $\boldsymbol{\sigma}_\gamma$ as its neutral element.*

From the fact that the mapping $\mathbf{\Sigma} \to \mathbf{J}$, $\boldsymbol{\sigma}_\xi \mapsto \xi$ is an isomorphism of the group $(\mathbf{\Sigma},\ )$ onto $(\mathbf{J}, \circ)$ we get the following corollary to Theorem 3.

Corollary. *Let $\mathcal{N}$ be a Desargues net with an infinite degree k which possesses the property* (rs). *Then $(\mathbf{J}, \circ)$ is a group with γ as its neutral element.*

Remark 2. a) In $(\mathbf{J}, \circ)$, the inverse element to ξ is the element ξ^{-1} defined by

$$\boldsymbol{\sigma}_\xi \boldsymbol{\sigma}_{\xi^{-1}} = \boldsymbol{\sigma}_{\xi^{-1}} \boldsymbol{\sigma}_\xi = \boldsymbol{\sigma}_\gamma\,.$$

b) In $(\mathbf{\Sigma},\ )$, the inverse element to $\boldsymbol{\sigma}_\xi$ is the element $\boldsymbol{\sigma}_\xi^{-1} = \boldsymbol{\sigma}_{\xi^{-1}}$.

The above two Corollaries (of Theorem 2 and Theorem 3) yield

Theorem 4. *Let $\mathcal{N}$ be a Desargues net with an infinite degree k which possesses the property* (rs). *Then $(\mathbf{J}', +_0, \circ)$ is a skewfield.*

Proof. For each permutation $\boldsymbol{\sigma}_\iota$, $\iota \in \mathbf{J}$, of the coordinate algebra $\mathfrak{A}(\mathcal{N})$, the zero element $0 \in \mathbf{S}$ is a fixed point. From this and from (2), (7) and (8) we get $\beta \circ \iota = \iota \circ \beta = \beta$ for each index $\iota \in \mathbf{J}$. Consequently, it suffices to prove the distributivity laws

$$\varkappa \circ (\lambda +_0 \mu) = \varkappa \circ \lambda +_0 \varkappa \circ \mu\,,$$
$$(\varkappa +_0 \lambda) \circ \mu = \varkappa \circ \mu +_0 \lambda \circ \mu$$

for each three indexes $\varkappa, \lambda, \mu \in \mathbf{J}'$.

Let $x \in \mathbf{S} \setminus \{0\}$ be an arbitrary element. According to (4), (5), (7), (8) and to the above two Corollaries we have

$$\begin{aligned}(x)\, \sigma_{\varkappa \circ (\lambda +_0 \mu)} &= (x)\,(\sigma_\varkappa \sigma_{\lambda +_0 \mu}) = ((x)\, \boldsymbol{\sigma}_\varkappa)\,(\boldsymbol{\sigma}_\lambda \mathbin{\mathring{+}} \boldsymbol{\sigma}_\mu) = \\ &= (x)\,(\boldsymbol{\sigma}_\varkappa \boldsymbol{\sigma}_\lambda) + (x)\,(\boldsymbol{\sigma}_\varkappa \boldsymbol{\sigma}_\mu) = (x)\, \boldsymbol{\sigma}_{\varkappa \circ \lambda} + (x)\, \boldsymbol{\sigma}_{\varkappa \circ \mu} = \\ &= (x)\,(\boldsymbol{\sigma}_{\varkappa \circ \lambda} \mathbin{\mathring{+}} \boldsymbol{\sigma}_{\varkappa \circ \mu}) = (x)\, \boldsymbol{\sigma}_{\varkappa \circ \lambda +_0 \varkappa \circ \mu}\end{aligned}$$

and also

$$\begin{aligned}(x)\, \boldsymbol{\sigma}_{(\varkappa +_0 \lambda) \circ \mu} &= (x)\,(\boldsymbol{\sigma}_{\varkappa +_0 \lambda} \boldsymbol{\sigma}_\mu) = ((x)\, \boldsymbol{\sigma}_{\varkappa +_0 \lambda})\, \boldsymbol{\sigma}_\mu = \\ &= ((x)\,(\boldsymbol{\sigma}_\varkappa \mathbin{\mathring{+}} \boldsymbol{\sigma}_\lambda))\, \boldsymbol{\sigma}_\mu = ((x)\, \boldsymbol{\sigma}_\varkappa + (x)\, \boldsymbol{\sigma}_\lambda)\, \boldsymbol{\sigma}_\mu = \\ &= ((x)\, \boldsymbol{\sigma}_\varkappa)\, \boldsymbol{\sigma}_\mu + ((x)\, \boldsymbol{\sigma}_\lambda)\, \boldsymbol{\sigma}_\mu = (x)\,(\boldsymbol{\sigma}_\varkappa \boldsymbol{\sigma}_\mu) + (x)\,(\boldsymbol{\sigma}_\lambda \boldsymbol{\sigma}_\mu) = \\ &= (x)\, \boldsymbol{\sigma}_{\varkappa \circ \mu} + (x)\, \boldsymbol{\sigma}_{\lambda \circ \mu} = (x)\,(\boldsymbol{\sigma}_{\varkappa \circ \mu} \mathbin{\mathring{+}} \boldsymbol{\sigma}_{\lambda \circ \mu}) = \\ &= (x)\, \boldsymbol{\sigma}_{\varkappa \circ \mu} +_{0 \lambda \circ \mu}\,,\end{aligned}$$

where we also used the fact that, by Theorem 1, $\boldsymbol{\sigma}_\mu$ is an automorphism of the group $(\mathbf{S}, +)$. ∎

According to [1], in $\mathcal{N} = (\mathbf{P}, \mathcal{L}, (\mathsf{V}_\iota)_{\iota\in\mathbf{I}})$ the quadrangle closure condition holds if $\mathcal{N}$ has the property

(Q) For each quadruple of points $\mathsf{A}_i \in \mathbf{P} \setminus \mathbf{v}$, $i \in \{1, 2, 3, 4\}$, no three of which lie on one line, the following holds: whenever any five of the six lines $\mathsf{A}_i\mathsf{A}_k$ joining the points A_i and A_k exist $(i, k \in \{1, 2, 3, 4\})$, then the sixth joining line exists, too.

For instance if "$\mathsf{A}_1\mathsf{A}_2\mathsf{V}_\iota$" $\wedge$ "$\mathsf{A}_1\mathsf{A}_3\mathsf{V}_\varkappa$" $\wedge$ "$\mathsf{A}_1\mathsf{A}_4\mathsf{V}_\lambda$" $\wedge$ "$\mathsf{A}_2\mathsf{A}_3\mathsf{V}_\mu$" $\wedge$ "$\mathsf{A}_2\mathsf{A}_4\mathsf{V}_\varrho$" holds, then there exists an index $\tau \in \mathbf{I}$ such that "$\mathsf{A}_3\mathsf{A}_4\mathsf{V}_\tau$" also holds if $\mathcal{N}$ has the property (Q). The index $\tau \in \mathbf{I}$ is then uniquely determined by the indexes $\iota, \varkappa, \lambda, \mu, \varrho \in$ $\in \mathbf{I}$.

Theorem 5. *Assume that $\mathcal{N}$ has the property* (rs). *Then in $\mathcal{N}$ the quadrangle closure condition* (Q) *holds if $\mathcal{N}$ is a Desargues net* (*with an infinite degree k*).

Proof. a) Let $\mathcal{N}$ satisfy the quadrangle closure condition (Q). By Corollary 1 and Corollary 2 of Theorem 5 in [1], $\mathcal{N}$ is a Desargues net.

b) Let $\mathcal{N}$ be a Desargues net with an infinite degree k which possesses the property (rs). By Theorem 4, $(\mathbf{J}', +_0, \circ)$ is a skewfield. Further, let $\mathsf{A}_1, \mathsf{A}_2, \mathsf{A}_3, \mathsf{A}_4 \in \mathbf{P} \setminus \mathbf{v}$ be arbitrary four points no three of which lie on the same line, and such that "$\mathsf{A}_1\mathsf{A}_2\mathsf{V}_\iota$" $\wedge$ $\wedge$ "$\mathsf{A}_1\mathsf{A}_3\mathsf{V}_\varkappa$" $\wedge$ "$\mathsf{A}_1\mathsf{A}_4\mathsf{V}_\lambda$" $\wedge$ "$\mathsf{A}_2\mathsf{A}_3\mathsf{V}_\mu$" $\wedge$ "$\mathsf{A}_2\mathsf{A}_4\mathsf{V}_\varrho$". We shall prove that there exists a uniquely determined index $\tau \in \mathbf{I}$ such that "$\mathsf{A}_3\mathsf{A}_4\mathsf{V}_\tau$". The proof will be carried out only in the case when the indexes $\iota, \varkappa, \lambda, \mu, \varrho \in \mathbf{I} \setminus \{\alpha, \beta\}$ are mutually distinct. In all other cases the proof can be carried out analogously; in some cases it might be even simpler.

Assume that the points A_i, $i \in \{1, 2, 3, \dots, 4\}$ are $\mathsf{A}_1 = (a_1, a_2)$, $\mathsf{A}_2 = (b_1, b_2)$, $\mathsf{A}_3 = (c_1, c_2)$, $\mathsf{A}_4 = (d_1, d_2)$. From the assumption that "$\mathsf{A}_1\mathsf{A}_2\mathsf{V}_\iota$" holds and that $\mathcal{N}$ has the property (mD) it follows that $b_2 - a_2 = (b_1 - a_1)\,\boldsymbol{\sigma}_\iota$. Similarly,

$$\begin{aligned}
&\text{“}\mathsf{A}_1\mathsf{A}_3\mathsf{V}_\varkappa\text{”} \quad \text{implies} \quad c_2 - a_2 = (c_1 - a_1)\,\boldsymbol{\sigma}_\varkappa\,,\\
&\text{“}\mathsf{A}_1\mathsf{A}_4\mathsf{V}_\lambda\text{”} \quad \text{implies} \quad d_2 - a_2 = (d_1 - a_1)\,\boldsymbol{\sigma}_\lambda\,,\\
&\text{“}\mathsf{A}_2\mathsf{A}_3\mathsf{V}_\mu\text{”} \quad \text{implies} \quad c_2 - b_2 = (c_1 - b_1)\,\boldsymbol{\sigma}_\mu \quad \text{and}\\
&\text{“}\mathsf{A}_2\mathsf{A}_4\mathsf{V}_\varrho\text{”} \quad \text{implies} \quad d_2 - b_2 = (d_1 - b_1)\,\boldsymbol{\sigma}_\varrho\,.
\end{aligned}$$

Transforming both sides of the fourth equality we have

$$c_2 - b_2 = (c_2 - a_2) - (b_2 - a_2) = (c_1 - a_1)\,\boldsymbol{\sigma}_\varkappa - (b_1 - a_1)\,\boldsymbol{\sigma}_\iota\,,$$

or

$$(c_1 - b_1)\,\boldsymbol{\sigma}_\mu = [(c_1 - a_1) - (b_1 - a_1)]\,\boldsymbol{\sigma}_\mu = (c_1 - a_1)\,\boldsymbol{\sigma}_\mu - (b_1 - a_1)\,\boldsymbol{\sigma}_\mu\,.$$

The fourth equality now has the form

$$(c_1 - a_1)\,\boldsymbol{\sigma}_{\varkappa -_0 \mu} = (b_1 - a_1)\,\boldsymbol{\sigma}_{\iota -_0 \mu}\,;$$

we have used Theorem 2, its Corollary, and the relations (4), (5). Since $\varkappa \neq \mu$, there exists $(\varkappa -_0 \mu)^{-1} \in \boldsymbol{J}$ and hence we have

$$c_1 - a_1 = (b_1 - a_1)\,\boldsymbol{\sigma}_{(\iota -_0 \mu)\circ(\varkappa -_0 \mu)^{-1}}\,.$$

Analogously, from the fifth equality we get

$$d_1 - a_1 = (b_1 - a_1)\,\boldsymbol{\sigma}_{(\iota -_0 \varrho)\circ(\lambda -_0 \varrho)^{-1}}\,.$$

Now it can be easily shown that we have

$$d_2 - c_2 = (d_1 - c_1)\,\boldsymbol{\sigma}_\tau\,, \quad \text{i.e.,} \quad \text{“}\mathsf{A}_3\mathsf{A}_4\mathsf{V}_\tau\text{”}\,.$$

Indeed, put

$$\tau = [(\iota -_0 \varrho)\circ(\lambda -_0 \varrho)^{-1} -_0 (\iota -_0 \mu)\circ(\varkappa -_0 \mu)^{-1}]^{-1} \circ$$
$$\circ [(\iota -_0 \varrho)\circ(\lambda -_0 \varrho)^{-1}\circ \lambda -_0 (\iota -_0 \mu)\circ(\varkappa -_0 \mu)^{-1}\circ \varkappa]\,.$$

Since the indexes $\iota, \varkappa, \lambda, \mu, \varrho \in \boldsymbol{J}$ are mutually distinct and $(\boldsymbol{J}', +_0, \circ)$ is a skewfield the index τ is uniquely determined. Hence we have

$$\begin{aligned}
(d_1 - c_1)\,\boldsymbol{\sigma}_\tau &= [(d_1 - a_1) - (c_1 - a_1)]\,\boldsymbol{\sigma}_\tau = \\
&= [(b_1 - a_1)\,\boldsymbol{\sigma}_{(\iota -_0 \varrho)\circ(\lambda -_0 \varrho)^{-1}} - (b_1 - a_1)\,\boldsymbol{\sigma}_{(\iota -_0 \mu)\circ(\varkappa -_0 \mu)^{-1}}]\,\boldsymbol{\sigma}_\tau = \\
&= (b_1 - a_1)\,\boldsymbol{\sigma}_{[(\iota -_0 \varrho)\circ(\lambda -_0 \varrho)^{-1} -_0 (\iota -_0 \mu)\circ(\varkappa -_0 \mu)^{-1}]\circ\tau} = \\
&= (b_1 - a_1)\,\boldsymbol{\sigma}_{(\iota -_0 \varrho)\circ(\lambda -_0 \varrho)^{-1}\circ\lambda -_0 (\iota -_0 \mu)\circ(\varkappa -_0 \mu)^{-1}\circ\varkappa} = \\
&= ((b_1 - a_1)\,\boldsymbol{\sigma}_{(\iota -_0 \varrho)\circ(\lambda -_0 \varrho)^{-1}})\,\boldsymbol{\sigma}_\lambda - ((b_1 - a_1)\,\boldsymbol{\sigma}_{(\iota -_0 \mu)\circ(\varkappa -_0 \mu)^{-1}})\,\boldsymbol{\sigma}_\varkappa = \\
&= (d_1 - a_1)\,\boldsymbol{\sigma}_\lambda - (c_1 - a_1)\,\boldsymbol{\sigma}_\varkappa = (d_2 - a_2) - (c_2 - a_2) = d_2 - c_2. \;\blacksquare
\end{aligned}$$

Theorem 5 yields

Corollary. *The algebraic equivalent of the net* $\mathcal{N} = (\mathbf{P}, \mathcal{L}, (\mathsf{V}_\iota)_{\iota\in I})$ *satisfying the quadrangle closure condition* (Q) *is the geometry over an admissible algebra* $\mathfrak{A}(\mathcal{N}) = (\boldsymbol{S}, 0, (\boldsymbol{\sigma}_\iota)_{\iota\in J}, (+_\iota)_{\iota\in J})$ *as the coordinate algebra of* $\mathcal{N}$, *with respect to some frame* $(\mathsf{O}; \alpha, \beta, \gamma)$. *Further, for* $\mathfrak{A}(\mathcal{N})$ *the following holds*:

(i) *For each index* $\iota \in \boldsymbol{J}$ *we have* $+_\iota = +_\gamma =: +$, $(\boldsymbol{S}, +)$ *is an abelian group, and for each* $\iota \in \boldsymbol{J}$ $\boldsymbol{\sigma}_\iota$ *is an automorphism of the group* $(\boldsymbol{S}, +)$.

(ii) $(\boldsymbol{J}', +_0, \circ)$ *is a skewfield, where* $\boldsymbol{J}'$ *is determined by* (1), *and the operations* “$+_0$” *and* “$\circ$” *are determined by* (5) *and* (8), *respectively.*

3. THE ORBITS OF ELEMENTS OF THE SET S

Definition 1. *Let* $\mathcal{N}$ *be a net which has the property* (rs). *Let* $a \in \boldsymbol{S}\setminus\{0\}$ *be an arbitrary element. The set*

$$\boldsymbol{S}_a := \{x \in \boldsymbol{S} \mid x = (a)\,\boldsymbol{\sigma}_\xi,\ \xi \in \boldsymbol{J}\}$$

is called the orbit of the element a (*with respect to the set* $\boldsymbol{\Sigma}$ *of permutations* $\boldsymbol{\sigma}_\iota$).

Lemma 6. *Let $\mathcal{N}$ be a Desargues net with an infinite degree k which possesses the property* (rs). *Then for each two elements $a, b \in \boldsymbol{S} \setminus \{0\}$ we have*

$$b \in \boldsymbol{S}_a \Leftrightarrow \boldsymbol{S}_a = \boldsymbol{S}_b .$$

Proof. a) If $\boldsymbol{S}_a = \boldsymbol{S}_b$ then $b = (b)\,\boldsymbol{\sigma}_\gamma \in \boldsymbol{S}_b$. Thus $b \in \boldsymbol{S}_a$.

b) Let $b \in \boldsymbol{S}_a$. According to Definition 1 there exists an index $\varkappa \in \boldsymbol{J}$ such that $b = (a)\,\boldsymbol{\sigma}_\varkappa$.

If $y \in \boldsymbol{S}_b$ is an arbitrary element then $y = (b)\,\boldsymbol{\sigma}_\eta$ for some $\eta \in \boldsymbol{J}$. By Theorem 3 and the Corollary following Theorem 3 we have

$$y = (b)\,\boldsymbol{\sigma}_\eta = ((a)\,\boldsymbol{\sigma}_\varkappa)\,\boldsymbol{\sigma}_\eta = (a)\,(\boldsymbol{\sigma}_\varkappa \boldsymbol{\sigma}_\eta) = (a)\,\boldsymbol{\sigma}_{\varkappa \circ \eta} .$$

Hence $y \in \boldsymbol{S}_a$ and $\boldsymbol{S}_b \subseteq \boldsymbol{S}_a$.

If $x \in \boldsymbol{S}_a$ is an arbitrary element then $x = (a)\,\boldsymbol{\sigma}_\xi$, $\xi \in \boldsymbol{J}$. According to Theorem 3 and its Corollary we have $a = (b)\,\boldsymbol{\sigma}_\varkappa^{-1}$, and hence

$$x = (a)\,\boldsymbol{\sigma}_\xi = ((b)\,\boldsymbol{\sigma}_\varkappa^{-1})\,\boldsymbol{\sigma}_\xi = (b)\,(\boldsymbol{\sigma}_\varkappa^{-1}\boldsymbol{\sigma}_\xi) = (b)\,\boldsymbol{\sigma}_{\varkappa^{-1} \circ \xi} .$$

Thus $x \in \boldsymbol{S}_b$ and $\boldsymbol{S}_a \subseteq \boldsymbol{S}_b$. Consequently, $\boldsymbol{S}_a = \boldsymbol{S}_b$. ∎

It follows from Lemma 6 that the support $\boldsymbol{S}$ of the coordinate algebra $\mathfrak{A}(\mathcal{N})$ of a Desargues net $\mathcal{N}$ which has an infinite degree k and possesses the property (rs) can be partitioned (decomposed) into mutually disjoint orbits of elements of $\boldsymbol{S}$.

Let $\boldsymbol{S}_a$ be the orbit of an element $a \in \boldsymbol{S} \setminus \{0\}$. Denote

$$\boldsymbol{S}_a' = \boldsymbol{S}_a \cup \{0\} = \{x \in \boldsymbol{S} \mid x = (a)\,\boldsymbol{\sigma}_\xi,\ \xi \in \boldsymbol{J}'\} . \tag{9}$$

Lemma 7. *Let $\mathcal{N}$ be a Desargues net with an infinite degree k which possesses the property* (rs). *Then for each element $a \in \boldsymbol{S} \setminus \{0\}$, $(\boldsymbol{S}_a', +)$ is an (abelian) group.*

Proof. According to Theorem 1, $(\boldsymbol{S}, +)$ is an abelian group and for each element $a \in \boldsymbol{S} \setminus \{0\}$ we have $\boldsymbol{S}_a' \subset \boldsymbol{S}_a$. By Theorem 2 and its Corollary, for each two elements $x, y \in \boldsymbol{S}_a'$, $x = (a)\,\boldsymbol{\sigma}_\xi$, $y = (a)\,\boldsymbol{\sigma}_\eta$ we have

$$x + y = (a)\,\boldsymbol{\sigma}_\xi + (a)\,\boldsymbol{\sigma}_\eta = (a)\,(\boldsymbol{\sigma}_\xi \mathbin{\mathring{+}} \boldsymbol{\sigma}_\eta) = (a)\,\boldsymbol{\sigma}_{\xi +_0 \eta} \in \boldsymbol{S}_a'$$

and also

$$-x = -(a)\,\boldsymbol{\sigma}_\xi = (a)\,(-{}^0\boldsymbol{\sigma}_\xi) = (a)\,\boldsymbol{\sigma}_{-_0 \xi} \in \boldsymbol{S}_a' .$$

Hence $(\boldsymbol{S}_a', +)$ is a subgroup of the group $(\boldsymbol{S}, +)$. The neutral element of $(\boldsymbol{S}_a', +)$ is the element $(a)\,\boldsymbol{\sigma}_\beta = 0$. ∎

In the set $\boldsymbol{S}_a'$, $a \in \boldsymbol{S} \setminus \{0\}$ we define another binary operation "●" – product – as follows:

(10) For each two elements $x, y \in \boldsymbol{S}_a'$, $x = (a)\,\boldsymbol{\sigma}_\xi$, $y = (a)\,\boldsymbol{\sigma}_\eta$, put

$$x \bullet y = (a)\,\boldsymbol{\sigma}_\xi \bullet (a)\,\boldsymbol{\sigma}_\eta = (a)\,\boldsymbol{\sigma}_\xi \boldsymbol{\sigma}_\eta .$$

Lemma 8. *Let $\mathcal{N}$ be a Desargues net which has an infinite degree k and possesses the property* (rs). *Then for each element $a \in \boldsymbol{S} \setminus \{0\}$, $(\boldsymbol{S}_a, \bullet)$ is a group and for* *each three elements $x, y, z \in \boldsymbol{S}'_a$ the distributive laws hold*:

$$(x + y) \bullet z = x \bullet z + y \bullet z ,$$
$$x \bullet (y + z) = x \bullet y + x \bullet z .$$

Proof. According to Theorem 3 and its Corollary, $(\boldsymbol{J}, \circ)$ is a group and for each two elements $x, y \in \boldsymbol{S}_a$, $x = (a)\,\boldsymbol{\sigma}_\xi$, $y = (a)\,\boldsymbol{\sigma}_\eta$ we have

$$x \bullet y = (a)\,\boldsymbol{\sigma}_{\xi \circ \eta}.$$

The mapping $\boldsymbol{S}_a \to \boldsymbol{J}$, $x = (a)\,\boldsymbol{\sigma}_\xi \mapsto \xi$ is therefore an isomorphism of $(\boldsymbol{S}_a, \bullet)$ onto $(\boldsymbol{J}, \circ)$, so $(\boldsymbol{S}_a, \bullet)$ is also a group.

To prove the distributive laws in $\boldsymbol{S}'_a$ we shall use Theorem 4 and the relation (10). Let $z \in \boldsymbol{S}'_a$, $z = (a)\,\boldsymbol{\sigma}_\zeta$. Then we have

$$\begin{aligned}(x + y) \bullet z &= ((a)\,\boldsymbol{\sigma}_\xi + (a)\,\boldsymbol{\sigma}_\eta) \bullet (a)\,\boldsymbol{\sigma}_\zeta = (a)\,\boldsymbol{\sigma}_{(\xi +_0 \eta) \circ \zeta} = \\ &= (a)\,\boldsymbol{\sigma}_{\xi \circ \zeta +_0 \eta \circ \zeta} = (a)\,\boldsymbol{\sigma}_{\xi \circ \zeta} + (a)\,\boldsymbol{\sigma}_{\eta \circ \zeta} = \\ &= x \bullet z + y \bullet z\end{aligned}$$

and also

$$\begin{aligned}x \bullet (y + z) &= (a)\,\boldsymbol{\sigma}_\xi \bullet ((a)\,\boldsymbol{\sigma}_\eta + (a)\,\boldsymbol{\sigma}_\zeta) = (a)\,\boldsymbol{\sigma}_{\xi \circ (\eta +_0 \zeta)} = \\ &= (a)\,\boldsymbol{\sigma}_{\xi \circ \eta +_0 \xi \circ \zeta} = (a)\,\boldsymbol{\sigma}_{\xi \circ \eta} + (a)\,\boldsymbol{\sigma}_{\xi \circ \zeta} = \\ &= x \bullet y + x \bullet z .\end{aligned}$$

If either $x = 0$ or $y = 0$ then

$$x \bullet y = 0 \bullet y = (a)\,\boldsymbol{\sigma}_{\beta \circ \eta} = (a)\,\boldsymbol{\sigma}_\beta = 0 \quad \text{or}$$
$$x \bullet y = x \bullet 0 = (a)\,\boldsymbol{\sigma}_{\xi \circ \beta} = (a)\,\boldsymbol{\sigma}_\beta = 0 , \quad \text{respectively} . \quad \blacksquare$$

Theorem 9. *Let $\mathcal{N}$ be a Desargues net with an infinite degree k which possesses the property* (rs). *Then for each element $a \in \boldsymbol{S} \setminus \{0\}$, $(\boldsymbol{S}'_a, +, \bullet)$ is a skewfield. The skewfiled $(\boldsymbol{S}'_a, +, \bullet)$ is isomorphic with the skewfiled $(\boldsymbol{J}', +_0, \circ)$.*

Proof. The first assertion follows from Lemma 7 and Lemma 8. The elements $0 = (a)\,\boldsymbol{\sigma}_\beta$ and $a = (a)\,\boldsymbol{\sigma}_\gamma$ are the zero and the unit element, respectively, in the skewfield $(\boldsymbol{S}'_a, +, \bullet)$.

To prove the second assertion it suffices to realize that the mapping

$$\varphi\colon \boldsymbol{S}'_a \to \boldsymbol{J}' , \quad x = (a)\,\boldsymbol{\sigma}_\xi \mapsto \xi$$

is an isomorphism of the skewfield $(\boldsymbol{S}'_a, +, \bullet)$ onto the skewfield $(\boldsymbol{J}', +_0, \circ)$. For, it is a one-to-one mapping and we have

$$x + y = (a)\,\boldsymbol{\sigma}_{\xi +_0 \eta} \mapsto \xi +_0 \eta ,$$
$$x \bullet y = (a)\,\boldsymbol{\sigma}_{\xi \circ \eta} \mapsto \xi \circ \eta . \quad \blacksquare$$

Corollary. *The group* $\mathrm{Aut}(\boldsymbol{S}'_a, +, \bullet)$ *of all automorphisms of each skewfield* $(\boldsymbol{S}'_a, +, \bullet)$, $a \in \boldsymbol{S} \setminus \{0\}$, *is isomorphic to the group* $\mathrm{Aut}(\boldsymbol{J}', +_0, \circ)$ *of all automorphisms of the skewfield* $(\boldsymbol{J}', +_0, \circ)$.

Proof. For each automorphism ψ of the skewfield $(\boldsymbol{J}', +_0, \circ)$ under the isomorphism φ we have the corresponding automorphism $\varphi\psi\varphi^{-1}$ of the skewfield $(\boldsymbol{S}'_a, +, \bullet)$. Conversely, for each automorphism φ_a of the skewfield $(\boldsymbol{S}'_a, +, \bullet)$ under the isomorphism φ we have the corresponding automorphism $\varphi^{-1}\varphi_a\varphi$ of the skewfield $(\boldsymbol{J}', +_0, \circ)$. ∎

4. THE THEOREM ON ORTHOLOGIC QUADRANGLES IN NETS

By an orthogonality of lines in a net $\mathscr{N} = (\mathbf{P}, \mathscr{L}, (\mathrm{V}_\iota)_{\iota \in I})$ we understand a binary relation "$\perp$" in the set $\mathscr{L} \setminus \{\mathbf{v}\}$ of all ordinary lines of $\mathscr{N}$ satisfying the following axioms:

(o1) For each two lines $\mathbf{g}, \mathbf{h} \in \mathscr{L} \setminus \{\mathbf{v}\}$ we have

$$\mathbf{g} \perp \mathbf{h} \Rightarrow \mathbf{h} \perp \mathbf{g}.$$

(o2) For each three lines $\mathbf{g}, \mathbf{h}, \mathbf{k} \in \mathscr{L} \setminus \{\mathbf{v}\}$ we have

$$\mathbf{g} \perp \mathbf{h} \wedge \mathbf{h} \parallel \mathbf{k} \Rightarrow \mathbf{g} \perp \mathbf{k}.$$

(o3) For each line $\mathbf{g} \in \mathscr{L} \setminus \{\mathbf{v}\}$ and each point $\mathrm{B} \in \mathbf{g}$ there is one and only one line $\mathbf{h} \in \mathscr{L} \setminus \{\mathbf{v}, \mathbf{g}\}$ such that $\mathrm{B} \in \mathbf{h}$ and $\mathbf{h} \perp \mathbf{g}$.

Remark 3. a) If for two lines $\mathbf{g}, \mathbf{h} \in \mathscr{L} \setminus \{\mathbf{v}\}$ we have $\mathbf{g} \perp \mathbf{h}$ then the two lines are said to be mutually orthogonal (which is made possible by (o1)). We also say that $\mathbf{g}$ is perpendicular to $\mathbf{h}$ and vice versa.

b) The axioms (o2) and (o3) and the axioms of a net imply that through each point $\mathrm{B} \in \mathbf{P} \setminus \mathbf{v}$ there goes exactly one line $\mathbf{h}$ perpendicular to a given line $\mathbf{g} \in \mathscr{L} \setminus \{\mathbf{v}\}$.

c) According to the axioms (o3) and (o2) no line $\mathbf{g} \in \mathscr{L} \setminus \{\mathbf{v}\}$ of a net $\mathscr{N}$ with an orthogonality of lines is isotropic (i.e., we never have $\mathbf{g} \perp \mathbf{g}$).

d) In a net $\mathscr{N}$ with an orthogonality of lines, for each three distinct lines $\mathbf{g}, \mathbf{h}, \mathbf{k} \in \mathscr{L} \setminus \{\mathbf{v}\}$ we have

$$\mathbf{g} \perp \mathbf{h} \wedge \mathbf{g} \perp \mathbf{k} \Rightarrow \mathbf{h} \parallel \mathbf{k}.$$

In the usual Euclidean plane geometry the Reidemeister closure theorem on rectangles holds true. It deals with two mutually orthogonal rectangles and contains 11 parameters. This theorem serves in [6] as a starting point to build up a rectangle affine plane geometry. In the present paper we shall use it as a closure condition for the orthogonality of lines in a net. We shall call it "the theorem on orthologic quandrangles"; it will be denoted by (OQ). It reads:

(OQ) Let $\mathcal{N} = (\mathbf{P}, \mathcal{L}, (\mathsf{V}_\iota)_{\iota \in \boldsymbol{I}})$ be a net which possesses the property (Q) and has the set $\mathcal{L} \setminus \{\mathbf{v}\}$ of all ordinary lines equipped with a relation "$\perp$". Let A_i, $i \in \{1, 2, 3, 4\}$ and A'_i, $i \in \{1, 2, 3, 4\}$, $\mathsf{A}_i, \mathsf{A}'_i \in \mathbf{P} \setminus \mathbf{v}$ be the vertices of two nondegenerate quadrangles. If any five of the six relations

$$\mathsf{A}_i\mathsf{A}_k \perp \mathsf{A}'_i\mathsf{A}'_k\,, \quad i < k\,, \quad i, k \in \{1, 2, 3, 4\}$$

hold true then the remaining sixth relation holds as well.

In our investigations of properties of the coordinate algebra $\mathfrak{A}(\mathcal{N})$ of the net $\mathcal{N}$ with orthogonality of lines in which the theorem (OQ) on orthologic quadrangles holds, we shall use the orthogonal frame of this net. Instead of the property (rs) of $\mathcal{N}$ we shall assume that $\mathcal{N}$ has the following property:

(ors) $\mathcal{N}$ has an orthogonal frame $(\mathsf{O}; \alpha, \beta, \gamma)$, i.e., $\mathsf{OV}_\alpha \perp \mathsf{OV}_\beta$, and a coordinate algebra $\mathfrak{A}(\mathcal{N})$ with respect to this frame.

The singular (improper) point of the perpendicular to a line $\mathbf{g} = \mathsf{BV}_\varkappa$, where $\mathsf{B} \in \mathbf{P} \setminus \mathbf{v}$, $\varkappa \in \boldsymbol{I}$, will be denoted by $\mathsf{V}_{(\varkappa)\mathbf{p}}$. Then

$$\mathsf{BV}_{(\varkappa)\mathbf{p}} \perp \mathsf{BV}_\varkappa \quad \forall \varkappa \in \boldsymbol{I}\,, \quad \forall \mathsf{B} \in \mathbf{P} \setminus \mathbf{v}\,. \tag{11}$$

Lemma 10. *Let $\mathcal{N}$ be a net which possesses the property* (ors) *and has the set $\mathcal{L} \setminus \{\mathbf{v}\}$ equipped with a relation "$\perp$". Then the mapping $\boldsymbol{I} \to \boldsymbol{I}$, $\varkappa \mapsto (\varkappa)\,\mathbf{p}$, determined by condition* (11), *has the following properties*:

(a) *it is a one-to-one mapping of $\boldsymbol{I}$ onto itself*;

(b) *it is an involutory mapping, i.e.*, $((\varkappa)\mathbf{p})\mathbf{p} = \varkappa \quad \forall \varkappa \in \boldsymbol{I}$;

(c) $(\alpha)\,\mathbf{p} = \beta$, $(\beta)\,\mathbf{p} = \alpha$.

Proof. Properties (a), (b) and (c) follow from axioms (o1), (o2), (o3), property (ors) and condition (11). ∎

In the sequel we derive some additional nontrivial properties of the mapping $\varkappa \mapsto (\varkappa)\,\mathbf{p}$.

Lemma 11. *Let $\mathcal{N}$ be a net having the properties* (ors), (Q), (OQ) *and let the set $\mathcal{L} \setminus \{\mathbf{v}\}$ be equipped with a relation "$\perp$". Then for each two indexes $\varkappa, \lambda \in \boldsymbol{J}$ we have*

a) $(-_0\varkappa)\,\mathbf{p} = -_0(\varkappa)\,\mathbf{p}$,

b) $(\varkappa +_0 \lambda)\,\mathbf{p} = (((\varkappa)\,\mathbf{p})^{-1} +_0 ((\lambda)\,\mathbf{p})^{-1})^{-1}$,

c) $(\varkappa^{-1})\,\mathbf{p} = (\gamma)\,\mathbf{p} \circ ((\varkappa)\,\mathbf{p})^{-1} \circ (\gamma)\,\mathbf{p}$,

d) $(\varkappa \circ \lambda)\,\mathbf{p} = (\lambda)\,\mathbf{p} \circ ((\gamma)\,\mathbf{p})^{-1} \circ (\varkappa)\,\mathbf{p}$.

Proof. According to Theorem 5, (ors) and (Q) imply that $\mathcal{N}$ is a Desargues net (with an infinite degree k), such that we have $\mathsf{OV}_\alpha \perp \mathsf{OV}_\beta$. Now, according to Theorem 4, $(\boldsymbol{J}', +_0, \circ)$ is a skewfield with the unit element γ and the zero element β.

a) Since $(J', +_0)$ is a (abelian) group, for each element $\varkappa \in J' \setminus \{\beta\} = J$ there exists its (uniquely determined) inverse element $-_0\varkappa \in J$. It follows from [4] that in $\mathcal{N}$ the diagonal condition of the type (α, β) with the restriction $\mathsf{A} = \mathsf{O}$, $\mathsf{B} \neq \mathsf{A}$ (cf. [4]) holds true.

Assume that points $\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D} \in \mathbf{P} \setminus \mathbf{v}$ satisfy this closure condition, i.e., we have

$$\text{“}\mathsf{ABV}_\beta\text{”} \wedge \text{“}\mathsf{CDV}_\beta\text{”} \wedge \text{“}\mathsf{ADV}_\alpha\text{”} \wedge \text{“}\mathsf{BCV}_\alpha\text{”} \wedge \text{“}\mathsf{ACV}_\varkappa\text{”},$$

where $\mathsf{A} = \mathsf{O}$, $\mathsf{B} \neq \mathsf{A}$ (see Fig. 1). Then we have also “BDV_λ”, where $\lambda = -_0\varkappa$.

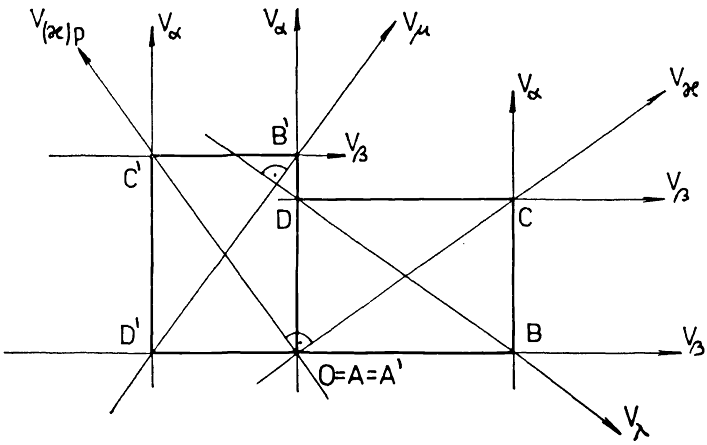

Fig. 1

Assume that points $\mathsf{A}', \mathsf{B}', \mathsf{C}', \mathsf{D}' \in \mathbf{P} \setminus \mathbf{v}$ satisfy “$\mathsf{A'B'V}_\alpha$” $\wedge$ “$\mathsf{C'D'V}_\alpha$” $\wedge$ $\wedge$ “$\mathsf{A'D'V}_\beta$” $\wedge$ “$\mathsf{B'C'V}_\beta$” $\wedge$ “$\mathsf{A'C'V}_{(\varkappa)\mathbf{p}}$”, where $\mathsf{A}' = \mathsf{O}$, $\mathsf{B}' \neq \mathsf{O}$. Since $\mathcal{N}$ satisfies the diagonal condition, we have also “$\mathsf{B'D'V}_\mu$”, $\mu \in J$ and $\mu = -_0(\varkappa)\,\mathbf{p}$.

The quandrangles ABCD and A'B'C'D' satisfy the assumptions of Theorem (OQ) and hence $\mathsf{BD} \perp \mathsf{B'D'}$ holds. This implies $\mu = (\lambda)\,\mathbf{p}$ and hence we have $(-_0\varkappa)\,\mathbf{p} = -_0(\varkappa)\,\mathbf{p}$.

b) Since $(J', +_0)$ is an (abelian) group, for each two elements $\varkappa, \lambda \in J$ there exists exactly one element $\varrho \in J$, $\varrho = \varkappa +_0 \lambda$. According to [4], this means that the net $\mathcal{N}$ satisfies the generalized diagonal condition of the type (α, β) with the restrictions $\mathsf{A} = \mathsf{O}$, $\mathsf{B} \neq \mathsf{A}$ (cf. [4]).

Assume that points $\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D} \in \mathbf{P} \setminus \mathbf{v}$ satisfy this closure condition, i.e., we have

$$\text{“}\mathsf{ABV}_\beta\text{”} \wedge \text{“}\mathsf{BCV}_\alpha\text{”} \wedge \text{“}\mathsf{ADV}_\alpha\text{”} \wedge \text{“}\mathsf{ACV}_\varkappa\text{”} \wedge \text{“}\mathsf{BDV}_\lambda\text{”},$$

where $\mathsf{A} = \mathsf{O}$, $\mathsf{B} \neq \mathsf{A}$ (see Fig. 2). Then we also have “CDV_ϱ”, where $\varrho = \varkappa +_0 \lambda$.

Assume that points $\mathsf{A}' = \mathsf{O}$, $\mathsf{B}' \neq \mathsf{A}'$, $\mathsf{C}', \mathsf{D}' \in \mathbf{P} \setminus \mathbf{v}$ satisfy “$\mathsf{A'B'V}_\alpha$” $\wedge$ $\mathsf{A'D'V}_\beta$” $\wedge$ $\wedge$ “$\mathsf{B'C'V}_\beta$” $\wedge$ “$\mathsf{A'C'V}_{(\varkappa)\mathbf{p}}$” $\wedge$ “$\mathsf{B'D'V}_{(\lambda)\mathbf{p}}$”.

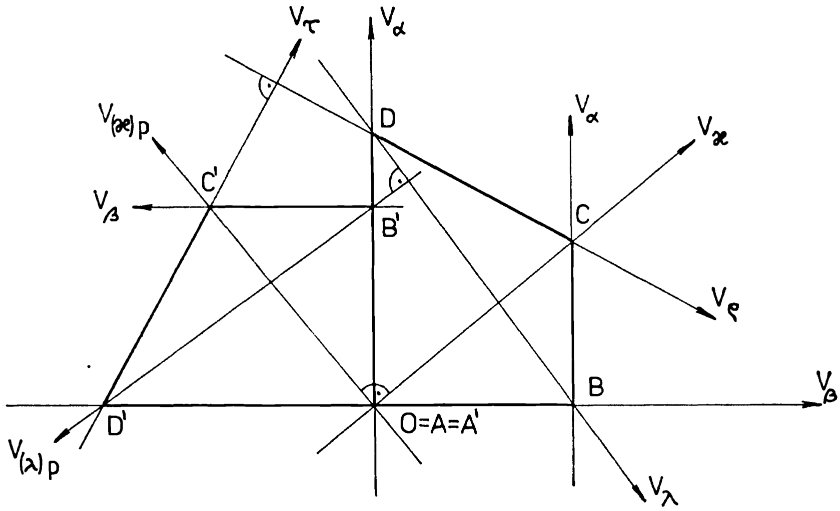

Fig. 2

It follows from (Q) that also "$\mathsf{C'D'V}_\tau$" holds true. We are going to express the index τ in terms of the indexes $(\varkappa)\,\mathbf{p}$ and $(\lambda)\,\mathbf{p}$. Since $\mathsf{B'} \in \mathsf{OV}_\alpha$ is arbitrary, put $\mathsf{B'} = (0, b')$, where $b' \in \boldsymbol{S} \setminus \{0\}$. Then $\mathsf{C'} = ((b')\,\sigma^{-1}_{(\varkappa)p}, b')$, $\mathsf{D'} = (-(b')\,\sigma^{-1}_{(\lambda)p}, 0)$ and we have

$$b' = ((b')\,\sigma^{-1}_{(\varkappa)p})\,\sigma_\tau + q \wedge 0 = (-(b')\,\sigma^{-1}_{(\lambda)p})\,\sigma_\tau + q\,.$$

Expressing q from the second equality, substituting it in to the first one and using Theorem 1 and Theorem 3 we get

$$(b')\,\sigma_\tau^{-1} = (b')\,\sigma^{-1}_{(\varkappa)p} + (b')\,\sigma^{-1}_{(\lambda)p}\,.$$

Further, we obtain

$$(b')\,\sigma_{\tau^{-1}} = (b')\,\sigma_{((\varkappa)\mathrm{p})^{-1} +_0 ((\lambda)\mathrm{p})^{-1}} \qquad \forall b' \in \boldsymbol{S} \setminus \{0\}$$

so that $\tau^{-1} = ((\varkappa)\,\mathbf{p})^{-1} +_0 ((\lambda)\,\mathbf{p})^{-1}$, i.e.

$$\tau = (((\varkappa)\,\mathbf{p})^{-1} +_0 ((\lambda)\,\mathbf{p})^{-1})^{-1}\,.$$

The quadrangles ABCD and A'B'C'D' satisfy the assumptions of (OQ), and hence CD $\perp$ C'D' holds. Consequently, $(\varrho)\,\mathbf{p} = \tau$ and

$$(\varkappa +_0 \lambda)\,\mathbf{p} = (((\varkappa)\,\mathbf{p})^{-1} +_0 ((\lambda)\,\mathbf{p})^{-1})^{-1}\,.$$

c) Since $(\boldsymbol{J}, \circ)$ is a group, for each element $\varkappa \in \boldsymbol{J}$ there exists exactly one inverse element $\lambda = \varkappa^{-1} \in \boldsymbol{J}$. According to [4] this means that $\mathcal{N}$ satisfies the Reidemeister condition of the type $(\alpha, \beta, \gamma, \varkappa, \lambda, \gamma)$ with the restriction $\mathsf{R} = \mathsf{O}$ (cf. [4]).

Assume that points $\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D} \in \mathbf{P} \setminus \mathbf{v}$ satisfy the assumptions of this closure condition, i.e., the following holds:

$$\text{“}\mathsf{ABV}_\beta\text{”} \wedge \text{“}\mathsf{CDV}_\beta\text{”} \wedge \text{“}\mathsf{ADV}_\alpha\text{”} \wedge \text{“}\mathsf{BCV}_\alpha\text{”} \wedge \text{“}\mathsf{OBDV}_\gamma\text{”} \wedge \text{“}\mathsf{OAV}_\varkappa\text{”}$$

(see Fig. 3). Then also "OCV_λ" holds, where $\lambda = \varkappa^{-1}$.

Assume that points $A', B', C', D' \in \mathbf{P} \setminus \mathbf{v}$ satisfy "$A'B'V_\alpha$" $\wedge$ "$C'D'V_\alpha$" $\wedge$ $\wedge$ "$A'D'V_\beta$" $\wedge$ "$B'C'V_\beta$" $\wedge$ "$OB'D'V_{(\gamma)\mathbf{p}}$" $\wedge$ "$OA'V_{(\varkappa)\mathbf{p}}$".

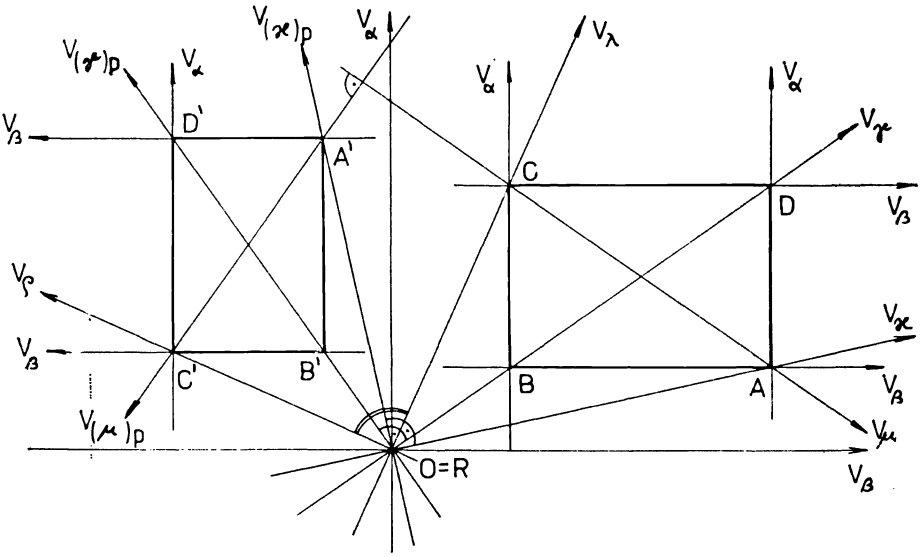

Fig. 3

The property (Q) of $\mathcal{N}$ implies that we have also "$OC'V_\varrho$". Now we express ϱ in terms of $(\gamma)\,\mathbf{p}$ and $(\varkappa)\,\mathbf{p}$. Since the point $A' \in OV_{(\varkappa)\mathbf{p}}$ is arbitrary but distinct from O and $V_{(\varkappa)\mathbf{p}}$, put $A' = (a', (a')\,\sigma_{(\varkappa)\mathbf{p}})$, where $a' \in \mathbf{S} \setminus \{0\}$. Then $B' = (a', (a')\,\sigma_{(\gamma)\mathbf{p}})$, $D' = ((a')\,\sigma_{(\varkappa)\mathbf{p}}\,\sigma^{-1}_{(\gamma)\mathbf{p}}, (a')\,\sigma_{(\varkappa)\mathbf{p}})$, $C' = ((a')\,\sigma_{(\varkappa)\mathbf{p}}\,\sigma^{-1}_{(\gamma)\mathbf{p}}, (a')\,\sigma_{(\gamma)\mathbf{p}})$ and we have

$$(a')\,\sigma_{(\gamma)\mathbf{p}} = (a')\,\sigma_{(\varkappa)\mathbf{p}}\sigma^{-1}_{(\gamma)\mathbf{p}}\sigma_\varrho \quad \forall a' \in \mathbf{S} \setminus \{0\}\,.$$

Hence $\sigma_{(\gamma)\mathbf{p}} = \sigma_{(\varkappa)\mathbf{p}}\sigma^{-1}_{(\gamma)\mathbf{p}}\sigma_\varrho$, and also

$$\sigma_\varrho = \sigma_{(\gamma)\mathbf{p}}\sigma^{-1}_{(\varkappa)\mathbf{p}}\sigma_{(\gamma)\mathbf{p}}\,, \quad \text{i.e.,} \quad \varrho = (\gamma)\,\mathbf{p} \circ ((\varkappa)\,\mathbf{p})^{-1} \circ (\gamma)\,\mathbf{p}\,.$$

The quadrangles, ABCD and A'B'C'D' satisfy the assumptions of (OQ), andhence $OC \perp OC'$, i.e., $(\lambda)\,\mathbf{p} = \varrho$. Substituting into this equality we have

$$(\varkappa^{-1})\,\mathbf{p} = (\gamma)\,\mathbf{p} \circ ((\varkappa)\,\mathbf{p})^{-1} \circ (\gamma)\,\mathbf{p}\,.$$

d) Since $(\mathbf{J}, \circ)$ is a group, for each two indexes $\varkappa, \lambda \in \mathbf{J}$ there exists exactly one index $\varrho \in \mathbf{J}$ such that $\varrho = \varkappa \circ \lambda$. According to [4] this means that $\mathcal{N}$ satisfies the Reidemeister closure condition of the type (α, β, γ) with the restriction $R = O$ (cf. [4]).

Assume that points $A, B, C, D \in \mathbf{P} \setminus \mathbf{v}$ satisfy the assumptions of this closure condition, i.e., we have

$$“\mathsf{ABV}_\beta” \wedge “\mathsf{CDV}_\beta” \wedge “\mathsf{ADV}_\alpha” \wedge “\mathsf{BCV}_\alpha” \wedge “\mathsf{OAV}_\varkappa” \wedge$$
$$\wedge “\mathsf{OBV}_\gamma” \wedge “\mathsf{OCV}_\lambda”$$

(cf. Fig. 4). Then we also have “ODV_ϱ”, where $\varrho = \varkappa \circ \lambda$.

Assume that points $\mathsf{A}', \mathsf{B}', \mathsf{C}', \mathsf{D}' \in \mathbf{P} \setminus \mathbf{v}$ satisfy

$$“\mathsf{A'B'V}_\alpha” \wedge “\mathsf{C'D'V}_\alpha” \wedge “\mathsf{A'D'V}_\beta” \wedge “\mathsf{B'C'V}_\beta” \wedge$$
$$\wedge “\mathsf{OA'V}_{(\varkappa)\mathbf{p}}” \wedge “\mathsf{OB'V}_{(\gamma)\mathbf{p}}” \wedge “\mathsf{OC'V}_{(\lambda)\mathbf{p}}” .$$

Then (Q) applied to the quadrangles OBCA, OB′C′A′ and OA′C′D′ yields the existence of the joining lines AC, A′C′ and OD′, respectively.

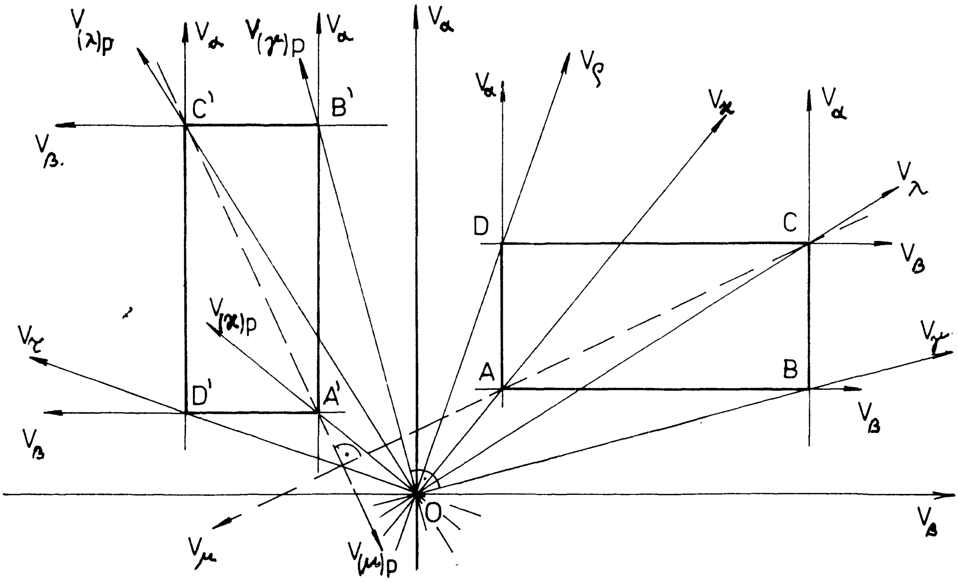

Fig. 4

If we assume “ACV_μ”, then the property (OQ) of $\mathcal{N}$ implies “$\mathsf{A'C'V}_{(\mu)\mathbf{p}}$”. Further, assume “$\mathsf{OD'V}_\tau$”, $\tau \in \boldsymbol{J}$. We express the index τ in terms of $(\gamma)\,\mathbf{p}$, $(\varkappa)\,\mathbf{p}$, $(\lambda)\,\mathbf{p}$. Since the point $\mathsf{A}' \in \mathsf{OV}_{(\varkappa)\mathbf{p}}$ is distinct from O and $\mathsf{V}_{(\varkappa)\mathbf{p}}$ but otherwise arbitrary, we can denote $\mathsf{A}' = (a', (a')\,\boldsymbol{\sigma}_{(\varkappa)\mathbf{p}})$, where $a' \in \boldsymbol{S} \setminus \{0\}$. Then $\mathsf{B}' = (a', (a')\,\boldsymbol{\sigma}_{(\gamma)\mathbf{p}})$, $\mathsf{C}' = ((a')\,\boldsymbol{\sigma}_{(\gamma)\mathbf{p}}\boldsymbol{\sigma}^{-1}_{(\lambda)\mathbf{p}}, (a')\,\boldsymbol{\sigma}_{(\gamma)\mathbf{p}})$ and $\mathsf{D}' = ((a')\,\boldsymbol{\sigma}_{(\gamma)\mathbf{p}}\boldsymbol{\sigma}^{-1}_{(\lambda)\mathbf{p}}, (a')\,\boldsymbol{\sigma}_{(\varkappa)\mathbf{p}})$, where

$$(a')\,\boldsymbol{\sigma}_{(\varkappa)\mathbf{p}} = (a')\,\boldsymbol{\sigma}_{(\gamma)\mathbf{p}}\boldsymbol{\sigma}^{-1}_{(\lambda)\mathbf{p}}\boldsymbol{\sigma}_\tau \quad \forall a' \in \boldsymbol{S} \setminus \{0\} .$$

Thus $\boldsymbol{\sigma}_{(\varkappa)\mathbf{p}} = \boldsymbol{\sigma}_{(\gamma)\mathbf{p}}\boldsymbol{\sigma}^{-1}_{(\lambda)\mathbf{p}}\boldsymbol{\sigma}_\tau$, and hence $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}_{(\lambda)\mathbf{p}\circ((\gamma)\mathbf{p})^{-1}\circ(\varkappa)\mathbf{p}}$. Consequently, $\tau = (\lambda)\,\mathbf{p} \circ ((\gamma)\,\mathbf{p})^{-1} \circ (\varkappa)\,\mathbf{p}$.

Since the quadrangles OACD and OA′C′D′ satisfy the assumptions of (OQ), we have OD ⊥ OD′. Thus $(\varrho)\,\mathbf{p} = \tau$ and, after substituting,

$$(\varkappa \circ \lambda)\,\mathbf{p} = (\lambda)\,\mathbf{p} \circ ((\gamma)\,\mathbf{p})^{-1} \circ (\varkappa)\,\mathbf{p} . \quad ■$$

Using Lemma 10 and Lemma 11, we are going to determine the algebraic equivalent of a net $\mathcal{N}$ having the properties (ors), (Q) and (OQ). In doing so, choose an element $e \in \boldsymbol{S} \setminus \{0\}$ and put

$$(0, e) = \mathsf{E}\,, \quad (0, -e) = \mathsf{D}\,, \quad (e, 0) = \mathsf{C}\,, \quad \mathsf{B} = \mathsf{OV}_\beta \sqcap \mathsf{EV}_{(\gamma)\mathbf{p}}\,,$$

where "CDV_γ" and $\mathsf{EV}_{(\gamma)\mathbf{p}} \perp \mathsf{CD}$ (see Fig. 5).

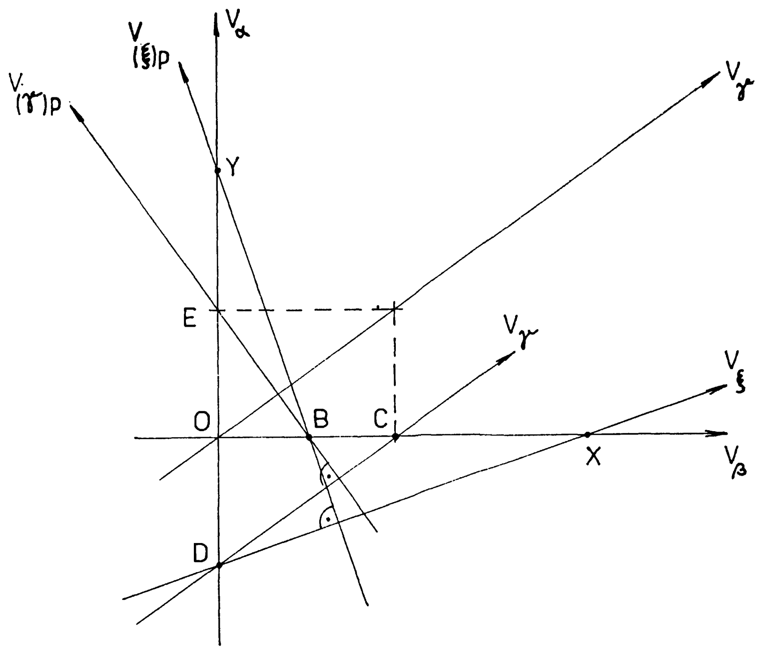

Fig. 5

It is easy to verify that we have $\mathsf{B} = (-(e)\,\sigma_{(\gamma)p}^{-1}, 0)$. Further, put $\mathsf{X} = \mathsf{OV}_\beta \sqcap \mathsf{DV}_\xi$, $\xi \in \boldsymbol{J}$ and $\mathsf{Y} = \mathsf{OV}_\alpha \sqcap \mathsf{BV}_{(\xi)\mathbf{p}}$. Then according to (11) we have $\mathsf{BV}_{(\xi)\mathbf{p}} \perp \mathsf{DV}_\xi$. If $\mathsf{X} = (x, 0)$ and $\mathsf{Y} = (0, y)$ then a straightforward calculation shows that

$$x = (e)\,\sigma_\xi^{-1}\,, \quad y = (e)\,\sigma_{(\gamma)p}^{-1}\sigma_{(\xi)\mathbf{p}}\,.$$

Under the notation from Section 3 we have $x, y \in \boldsymbol{S}'_e$. Since the mapping $\mathsf{OV}_\beta \to \mathsf{OV}_\alpha$, $\mathsf{X} \mapsto \mathsf{Y}$ is one-to-one (this follows from the axioms (o1), (o2) and (o3) of the orthogonality in $\mathcal{N}$ and from the axioms of a net), the mapping

$$\text{(12)} \qquad \varphi_e\colon \boldsymbol{S}'_e \to \boldsymbol{S}'_e\,, \quad x = (e)\,\sigma_\xi^{-1} \mapsto y = (e)\,\sigma_{(\gamma)p}^{-1}\sigma_{(\xi)\mathbf{p}}$$

is also a one-to-one mapping of the set $\boldsymbol{S}'_e$ onto itself such that

$$e = (e)\,\sigma_\gamma^{-1} \mapsto (e)\,\varphi_e = (e)\,\sigma_{(\gamma)p}^{-1}\sigma_{(\gamma)\mathbf{p}} = (e)\,\sigma_\gamma = e\,,$$
$$0 = (e)\,\sigma_\beta \mapsto (0)\,\varphi_e = 0\,.$$

The equality $y = (e)\,\sigma^{-1}_{(\gamma)p}\sigma_{(\xi)p}$ can be transformed into

$$y = (e)\,\sigma^{-1}_{((\xi)p)^{-1}\circ(\gamma)p}\,.$$

The mapping φ_e defined by (12) determines another mapping

$$\psi: J \to J\,, \quad \xi \mapsto (\xi)\,\psi = ((\xi)\,\mathbf{p})^{-1} \circ (\gamma)\,\mathbf{p}\,, \tag{13}$$

which satisfies the equalities

$$\begin{aligned}(\xi^{-1})\,\psi &= ((\xi^{-1})\,\mathbf{p})^{-1} \circ (\gamma)\,\mathbf{p} = ((\gamma)\,\mathbf{p} \circ ((\xi)\,\mathbf{p})^{-1} \circ (\gamma)\,\mathbf{p})^{-1} \circ (\gamma)\,\mathbf{p} = \\ &= (((\gamma)\,\mathbf{p})^{-1} \circ (\xi)\,\mathbf{p} \circ ((\gamma)\,\mathbf{p})^{-1}) \circ (\gamma)\,\mathbf{p} = \\ &= ((\gamma)\,\mathbf{p})^{-1} \circ (\xi)\,\mathbf{p}\,;\end{aligned}$$

observe that we have used the statement c) in Lemma 11. The mapping ψ (determined by (13)) can be extended over J' if we define

$$(\beta)\,\psi := \beta\,. \tag{13'}$$

Lemma 12. *Let $\mathcal{N}$ be a net satisfying the conditions* (ors), (Q) *and* (OQ). *Then the mapping ψ determined by the conditions* (13) *and* (13') *is an automorphism of the skewfield* $(J', +_0, \circ)$.

Proof. Observe that ψ is a one-to-one mapping of J' onto itself. Indeed, if $((\xi)\,\mathbf{p})^{-1} \circ (\gamma)\,\mathbf{p} = \eta$, $\eta \in J$ then according to Lemma 10 and Lemma 11 we have $(\xi)\,\mathbf{p} = (\gamma)\,\mathbf{p} \circ \eta^{-1}$ and

$$\begin{aligned}\xi &= ((\xi)\,\mathbf{p})\,\mathbf{p} = ((\gamma)\,\mathbf{p} \circ \eta^{-1})\,\mathbf{p} = (\eta^{-1})\,\mathbf{p} \circ ((\gamma)\,\mathbf{p})^{-1} \circ ((\gamma)\,\mathbf{p})\,\mathbf{p} = \\ &= ((\gamma)\,\mathbf{p} \circ ((\eta)\,\mathbf{p})^{-1} \circ (\gamma)\,\mathbf{p}) \circ ((\gamma)\,\mathbf{p})^{-1} \circ \gamma = (\gamma)\,\mathbf{p} \circ ((\eta)\,\mathbf{p})^{-1}\,.\end{aligned}$$

For each two elements $\xi, \eta \in J'$ we have

$$\begin{aligned}(\xi +_0 \eta)\,\psi &= ((\xi +_0 \eta)\,\mathbf{p})^{-1} \circ (\gamma)\,\mathbf{p} = (((\xi)\,\mathbf{p})^{-1} +_0 ((\eta)\,\mathbf{p})^{-1}) \circ (\gamma)\,\mathbf{p} = \\ &= ((\xi)\,\mathbf{p})^{-1} \circ (\gamma)\,\mathbf{p} +_0 ((\eta)\,\mathbf{p})^{-1} \circ (\gamma)\,\mathbf{p} = (\xi)\,\psi +_0 (\eta)\,\psi\end{aligned}$$

and also

$$\begin{aligned}(\xi \circ \eta)\,\psi &= ((\xi \circ \eta)\,\mathbf{p})^{-1} \circ (\gamma)\,\mathbf{p} = ((\eta)\,\mathbf{p} \circ ((\gamma)\,\mathbf{p})^{-1} \circ (\xi)\,\mathbf{p})^{-1} \circ (\gamma)\,\mathbf{p} = \\ &= (((\xi)\,\mathbf{p})^{-1} \circ (\gamma)\,\mathbf{p} \circ ((\eta)\,\mathbf{p})^{-1}) \circ (\gamma)\,\mathbf{p} = \\ &= (((\xi)\,\mathbf{p})^{-1} \circ (\gamma)\,\mathbf{p}) \circ (((\eta)\,\mathbf{p})^{-1} \circ (\gamma)\,\mathbf{p}) = (\xi)\,\psi \circ (\eta)\,\psi\,.\end{aligned}$$

However, according to (13') we have $(\beta)\,\psi = \beta$ and according to (13) we have $(\gamma)\,\psi = ((\gamma)\,\mathbf{p})^{-1} \circ (\gamma)\,\mathbf{p} = \gamma$. ∎

Summing up the above results we get

Theorem 13. *The algebraic equivalent of a net $\mathcal{N} = (\mathbf{P}, \mathcal{L}, (V_\iota)_{\iota \in I})$ equipped with an orthogonality for ordinary lines and having the properties* (Q) *and* (OQ) *is the geometry over the admissible algebra $\mathfrak{A}(\mathcal{N}) = (\mathbf{S}, 0, (\sigma_\iota)_{\iota \in J})$ as the coordinate* *algebra of the net $\mathcal{N}$ with respect to some orthogonal frame $(O; \alpha, \beta, \gamma)$. The algebra $\mathfrak{A}(\mathcal{N})$ satisfies the following conditions:*

(i) For each index $\iota \in \mathbf{J}$ we have $+_\iota = +_\gamma =: +$, $(\mathbf{S}, +)$ is an abelian group and $\boldsymbol{\sigma}_\iota$ is an automorphism of the group $(\mathbf{S}, +)$.

(ii) $(\mathbf{J}', +_0, \circ)$ is a skewfield, where $\mathbf{J}'$ is determined by (1) and the operations "$+_0$" and "$\circ$" are determined by (5) and (8), respectively.

The line $y = (x)\boldsymbol{\sigma}_\xi + q$, $q \in \mathbf{S}$, $\xi \in \mathbf{J}$ of $\mathcal{N}$ is orthogonal to each line $y = (x)\,\boldsymbol{\sigma}_{(\xi)\mathbf{p}} + q'$, *$q' \in \mathbf{S}$, where*

$$(\xi)\,\mathbf{p} = (\gamma)\,\mathbf{p} \circ (\xi^{-1})\,\psi\,.$$

The mapping $\xi \mapsto (\xi)\,\mathbf{p}$ is defined by (11) and ψ is an automorphism of the skewfield $(\mathbf{J}', +_0, \circ)$ satisfying the relations (13) and (13').

Proof. The assertion follows from Theorem 5 and its Corollary, Lemma 11 and Lemma 12. ■

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Súhrn

O KOLMOSTI V SIETIACH

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V článku je skúmaná kolmosť obyčajných priamok v desarguessovskej sieti, ktorá spĺňa štvoruholníkovú uzáverovú podmienku. Pritom rád aj stupeň tejto siete sú nekonečné kardinálne čísla. Odvodený je algebraický ekvivalent siete, v ktorej je splnená štvoruholníková uzáverová podmienka.

Pre kolmosť priamok v sieti sa požadujú len ,,triviálne axiómy kolmosti". Ako uzáverová podmienka kolmosti je používaná Reidemeisterova veta o ortológových štvoruholníkoch.

Hlavným výsledkom je odvodenie algebraického ekvivalentu siete, v ktorej platí štvoruholníková uzáverová veta a súčasne veta o ortológových štvoruholníkoch.

Резюме

ОБ ОРТОГОНАЛЬНОСТИ В СЕТЯХ

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В статье рассматривается ортогональность обыкновенных прямых в дезарговой сети, в которой выполняется условие замыкания четырехугольника, причем порядок и степень этой сети — бесконечные кардинальные числа. Выведен алгебраический эквивалент сети, удовретворяющей условию замыкания четырехугольника.

Для ортогональности прямых в сети требуется выполнение только ,,тривиальных аксиом ортогональности". В качестве условия замыкания ортогональности здесь используется теорема Рейдемейстера об ортологовых четырехугольниках.

Главным результатом работы является установление алгебраического эквивалента сети, в которой справедлива теорема замыкания четырехугольника одновременно с теоремой об ортологовых четырехугольниках.

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