Časopis pro pěstování matematiky

Antonín Lešanovský

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Časopis pro pěstování matematiky, Vol. 113 (1988), No. 2, 197--212

Persistent URL: `http://dml.cz/dmlcz/118343`

THICKNESS OF A FAMILY OF SETS AND UNIFORM MEASURES

ANTONÍN LEŠANOVSKÝ, Praha

(Received January 2, 1986)

Summary. Let S be a set and let w be a family of its subsets. The notion of thickness of w was defined in [4] using general molecular measures on S. The paper shows that thickness of w can be expressed in terms of only uniform measures on a modified set $\tilde{S}$. A characterization of families w with zero thickness is also given.

Keywords: thickness of a family of sets, uniform measures, molecular measures.

AMS classification: 28A10.

A measure of thickness for families of sets has been defined in [4]. The paper [4] has given a geometrical justification for its name and a review in the new light of the results of [5] and [6] dealing with a theorem on existence of certain convex combinations with applications to mathematical analysis. Particular attention was paid to an investigation of the range of this characteristic and to the relation between its value and the combinatorial structure of the family of sets in question.

These results presented in Section 1 of [4] have been proved by using measures of the simplest form which have been called uniform measures. This fact implied the problem whether, in evaluating the thickness of a family of sets, it is generally possible to consider uniform measures (on a given basic space) only. Section 2 of [4] has shown that it is not so.

Nevertheless, the uniform measures are extremely objective and easy to describe. For example, a measure λ is used to construct convex combinations $\sum \lambda(s)\, x_s$ of certain elements x_s of a linear space in order to approximate a given element. If λ is a uniform measure then this approximation is the arithmetic mean

$$\frac{1}{n}(x_1 + \ldots + x_n).$$

Thus, it has been useful to find a condition, dealing with the basic space and with the family of its substes, under which the restriction discussed above is justified.

This paper presents a way of modification of a given basic space S and a given family w of its subsets, and proves that the thicknesses of the original and modified families of sets coincide. The modified basic space $\tilde{S}$ and the modified family $\tilde{w}$ of sets fulfil the condition mentioned above. Thus, the thickness of an arbitrary family of sets can be evaluated by considering uniform measures (defined on a modified basic space) only. This fact is used in Section 4 of the present paper, where the case of families of thickness zero is investigated. Theorem 4 proves that thickness of a family w is zero if and only if there exists a sequence $X = \{x_q\}_{q=1}^{\infty} \subset \tilde{S}$ such that the intersections $\tilde{W} \cap X$ are of density zero in X uniformly for all $\tilde{W} \in \tilde{w}$, i.e.

$$\lim_{k \to \infty} \sup_{\tilde{W} \in \tilde{w}} \frac{1}{k} \operatorname{card} (\tilde{W} \cap \{x_q\}_{q=1}^{k}) = 0 .$$

This notion of density of a subset was used e.g. by Halmos in [1].

Readers interested in the connection of the subject of this paper with the inversion of the order of limit operations are referred to [3], [6], and [7]. The papers [2], [8], [9], and [10] contain applications of these ideas to game theory. There is, however, a more extensive literature on this topic.

1. DEFINITIONS

To each set S we assign a set $P(S)$ the elements of which are functions λ defined on S possessing the following properties:

(1) $$\lambda(s) \geqq 0 \quad \text{for each} \quad s \in S ,$$

(2) $$\text{the set} \quad N(\lambda) = \{s; s \in S,\ \lambda(s) > 0\} \quad \text{is finite} ,$$

(3) $$\sum_{s \in S} \lambda(s) = 1 .$$

Given a $\lambda \in P(S)$ it is possible to define a non-negative measure on the σ-algebra $\exp S$ in such a way that the value of this measure for each $M \subset S$ is

$$\sum_{s \in M} \lambda(s) .$$

Since no misunderstanding can arise we use the same symbol for $\lambda \in P(S)$ and for the corresponding measure, and we do not distinguish between λ taken as a function on S and as a measure on $\exp S$.

Let w be a nonvoid family of subsets of S, i.e. $\emptyset \neq w \subset \exp S$. We define the thickness $e(w, S)$ of the family w of subsets of the basic space S by the formula

(4) $$e(w, S) = \inf_{\lambda \in P(S)} \sup_{W \in w} \lambda(W) .$$

We observe immediately that the quantity $e(w, S)$ is non-negative and that

$$\lambda(W) = \sum_{s \in W \cap N(\lambda)} \lambda(s) \quad \text{for all} \quad \lambda \in P(S) \quad \text{and} \quad W \in w ,$$

so that

$$\operatorname{card} \{\lambda(W); W \in w\} \leqq \operatorname{card} \{ \sum_{s \in T} \lambda(s); T \subset N(\lambda)\} \leqq 2^{\operatorname{card} N(\lambda)} < \infty .$$

Thus, supremum may be replaced by maximum in (4), i.e.

$$e(w, S) = \inf_{\lambda \in P(S)} \max_{W \in w} \lambda(W) .$$

It is useful to show that $e(w, S)$ depends on both w and S. If $e(w, S) > 0$, $w \subset \subset \exp S$, $S \subset S'$ and $S \neq S'$ then there exist $s_0 \in S' - S$ and $\lambda_0 \in P(S')$ such that

$$\lambda_0(s_0) = 1 , \tag{5}$$

$$\lambda_0(s) = 0 \quad \text{for each} \quad s \in S' - \{s_0\} . \tag{6}$$

Further, the relations $s_0 \notin W$ and $\lambda_0(W) = 0$ are true for each $W \in w$. Thus, we have

$$0 \leqq e(w, S') \leqq \max_{W \in w} \lambda_0(W) = 0 ,$$

i.e.

$$e(w, S) \neq e(w, S') .$$

We have just proved

Lemma 1. *If the set* $S - \bigcup_{W \in w} W$ *is not empty then* $e(w, S) = 0$.

Definition. A measure $\lambda \in P(S)$ is said to be *uniform* if

$$\lambda(\{s\}) = \begin{cases} \dfrac{1}{\operatorname{card} N(\lambda)} & \text{if} \quad s \in N(\lambda) , \\ 0 & \text{otherwise} , \end{cases} \tag{7}$$

i.e., if it is generated by a function $\lambda \in P(S)$ such that its values at each element of its support $N(\lambda)$ are the same; due to (3) this common value must equal the reciprocal value of the cardinality of $N(\lambda)$. The set of all uniform measures defined on $\exp S$ is denoted by $U(S)$.

It is easy to see that

$$\lambda(W) = \frac{\operatorname{card} [W \cap N(\lambda)]}{\operatorname{card} N(\lambda)} \quad \text{for all} \quad \lambda \in U(S) \quad \text{and} \quad W \in w ,$$

and that

$$e(w, S) \leqq \inf_{\lambda \in U(S)} \max_{W \in w} \lambda(W) . \tag{8}$$

Section 2 of the paper [4] shows that there are a set S and a family w of its subsets such that the strict inequality holds in (8). On the other hand, it is true that for some couples (S, w) the equality

$$e(w, S) = \inf_{\lambda \in U(S)} \max_{W \in w} \lambda(W) \tag{9}$$

is valid. Such a situation is considered in the next section and a modification $(\tilde{S}, \tilde{w})$ of (S, w) described in Section 3 is based on it.

Let $Q(S)$ denote the set of such $\lambda \in P(S)$ that the values of $\lambda(s)$, for each $s \in S$, are rational numbers. It is easy to find that

$$U(S) \subset Q(S) \subset P(S) \tag{10}$$

is true.

Lemma 2. *The equality*

$$e(w, S) = \inf_{\lambda \in Q(S)} \max_{W \in w} \lambda(W) \tag{11}$$

holds for each set S and for each family w of its subsets.

Proof. The relation (10) implies

$$e(w, S) \leqq \inf_{\lambda \in Q(S)} \max_{W \in w} \lambda(W) . \tag{12}$$

On the other hand, let ε be an arbitrary positive real number. Take a $\lambda \in P(S)$ such that

$$\max_{W \in w} \lambda(W) < e(w, S) + \tfrac{1}{2}\varepsilon . \tag{13}$$

If $\lambda \in P(S) - Q(S)$ we shall find a $\mu \in Q(S)$ such that

$$\max_{W \in w} \mu(W) < e(w, S) + \varepsilon .$$

We put $m = \operatorname{card} N(\lambda)$, denote the elements of the set $N(\lambda)$ by $s_1, \ldots, s_m$, and choose a rational number r_i in the interval $[\lambda(s_i); \lambda(s_i) + \varphi/2m)$ for each $i = 1, \ldots, m - 1$, where $\varphi = \min\{\varepsilon; \lambda(s_m)\}$. Further, we put

$$\mu(s_i) = r_i \quad \text{for} \quad i = 1, \ldots, m - 1 ,$$

$$\mu(s_m) = 1 - \sum_{i=1}^{m-1} r_i ,$$

and

$$\mu(s) = 0 \quad \text{for} \quad s \in S - N(\lambda) .$$

We observe that $\mu \in P(S)$, $N(\mu) = N(\lambda)$, $\mu(s_m) \leqq \lambda(s_m)$, and that $\mu(s)$ is a rational number, for each $s \in S$. Thus, $\mu \in Q(S)$. Let W be an arbitrary element of the family w. We have

$$\mu(W) = \sum_{s \in W} \mu(s) < \sum_{\substack{i=1 \\ s_i \in W}}^{m} \left[\lambda(s_i) + \frac{\varepsilon}{2m}\right] \leqq \lambda(W) + \tfrac{1}{2}\varepsilon ,$$

so that by the relation (13)

$$\max_{W \in w} \mu(W) < \max_{W \in w} \lambda(W) + \tfrac{1}{2}\varepsilon < e(w, S) + \varepsilon .$$

Taking the infimum over $\mu \in Q(S)$ and comparing this with (12) we complete the proof of the lemma.

Theorem 1. *Given a set S and a family w of its subsets, we put*

$$w_s = \{W;\ W \in w,\ s \in W\} \quad \textit{for each} \quad s \in S\,.$$

Let the set

$$M_s = [\bigcap_{W \in w_s} W] \cap [\bigcap_{W \in w - w_s} (S - W)] \cap [\bigcup_{W \in w} W]$$

be either empty or infinite for each $s \in S$. Then the equality (9) holds.

Proof. By (8) and Lemma 2, we need to prove that for each $\lambda \in Q(S)$ there exists a $\mu \in U(S)$ such that

$$\max_{W \in w} \mu(W) \leqq \max_{W \in w} \lambda(W)\,. \tag{14}$$

Let us start with the case that there exists an $s_0 \in S$ such that the set M_{s_0} is empty. In this case, we have $s_0 \in S - \bigcup_{W \in w} W$ and put $\mu = \lambda_0$, where λ_0 was defined by (5) and (6). It is obvious that $\lambda_0 \in Q(S) \cap U(S)$. Moreover, we know the proof of Lemma 1 that $0 = \max_{W \in w} \lambda_0(W) \leqq \max_{W \in w} \lambda(W)$ for each $\lambda \in Q(S)$.

On the other hand, let the set M_s be infinite for each $s \in S$. Denote the elements of $N(\lambda)$ by $s_1, \ldots, s_m$. The values of $\lambda(s_i)$ for $i = 1, \ldots, m$ are rational numbers. We may express them as fractions

$$\lambda(s_i) = \frac{x_i}{y} \quad \text{for} \quad i = 1, \ldots, m\,,$$

where x_i for $i = 1, \ldots, m$, and y are positive integers, and

$$y = \sum_{i=1}^{m} x_i\,. \tag{15}$$

The sets M_{s_i} are assumed to be infinite so that it is possible to take x_i elements $s(i, 1), s(i, 2), \ldots, s(i, x_i)$ of the set M_{s_i}, for $i = 1, \ldots, m$, in such a way that the set

$$\mathfrak{A} = \{s(i, j);\ i = 1, \ldots, m,\ j = 1, \ldots, x_i\}$$

contains exactly y elements. Further, we put $N(\mu) = \mathfrak{A}$ and

$$\mu(s) = \frac{1}{y} \quad \text{for} \quad s \in \mathfrak{A}\,,$$

$$\mu(s) = 0 \quad \text{otherwise}\,.$$

We find that $\sum_{s \in S} \mu(s) = 1$. Thus, $\mu \in U(S)$. Finally, take an arbitrary $W_0 \in w$. Let

$$I = \{i;\ i \in \{1; \ldots; m\},\ s_i \in W_0\}\,,$$

i.e.

$$W_0 \cap N(\lambda) = \{s_i;\ i \in I\}\ .$$

We have $W_0 \in w_{s_i}$ for each $i \in I$, so that

$$s(i, j) \in W_0\ , \quad \text{if} \quad i \in I \quad \text{and} \quad j = 1, \ldots, x_i\ ,$$

and $W_0 \in w - w_{s_i}$ for each $i \in \{1; \ldots; m\} - I$, so that

$$s(i, j) \notin W_0\ , \quad \text{if} \quad i \in \{1; \ldots; m\} - I\ , \quad j = 1, \ldots, x_i\ .$$

Thus,

$$\mu(W_0) = \sum_{s \in W_0 \cap N(\mu)} \mu(s) = \sum_{i \in I} \sum_{j=1}^{x_i} \frac{1}{y} = \sum_{i \in I} \lambda(s_i) = \lambda(W_0)\ .$$

This immediately implies the equality

$$\max_{W \in w} \mu(W) = \max_{W \in w} \lambda(W)\ ,$$

so that the validity of the inequality (14) is proved.

3. GENERAL CASE

Let N be the set of all positive integers and let

$$\tilde{S} = S \times N\ , \tag{16}$$

$$\tilde{w} = \{W \times N;\ W \in w\}\ , \tag{17}$$

and

$$\tilde{w}_{(s,n)} = \{\tilde{W};\ \tilde{W} \in \tilde{w},\ (s, n) \in \tilde{W}\} \quad \text{for each} \quad (s, n) \in \tilde{S}\ .$$

Given $(s_0, n_0) \in \tilde{S}$ we have

$$\{(s_0, n);\ n \in N\} \subset \tilde{W} \quad \text{if} \quad \tilde{W} \in \tilde{w}_{(s_0,n_0)}\ ,$$

and

$$\{(s_0, n);\ n \in N\} \subset \tilde{S} - \tilde{W} \quad \text{if} \quad \tilde{W} \notin \tilde{w}_{(s_0,n_0)}\ ,$$

so that the set

$$[\bigcap_{\tilde{W} \in \tilde{w}(s_0,n_0)} \tilde{W}] \cap [\bigcap_{\tilde{W} \in \tilde{w} - \tilde{w}(s_0,n_0)} (\tilde{S} - \tilde{W})] \cap [\bigcup_{\tilde{W} \in \tilde{w}} \tilde{W}]$$

is either empty or infinite, i.e. the couple $(\tilde{S}, \tilde{w})$ fulfils the assumption of Theorem 1. Thus, we have

$$e(\tilde{w}, \tilde{S}) = \inf_{\lambda \in U(\tilde{S})} \max_{\tilde{W} \in \tilde{w}} \lambda(\tilde{W})\ . \tag{18}$$

Lemma 3. *Let S be a set and let w be a family of its subsets. Then*

$$(19) \qquad e(w, S) = e(\tilde{w}, \tilde{S}),$$

where $\tilde{S}$ and $\tilde{w}$ are defined by (16) and (17).

Proof. We shall show that for each $\lambda \in P(\tilde{S})$ there exists a $\mu \in P(S)$ such that the equality

$$(20) \qquad \mu(W) = \lambda(W \times N)$$

holds for each $W \in w$, and vice versa. Let $\lambda \in P(\tilde{S})$ be given. Then the function μ defined on S by the formula

$$\mu(s) = \sum_{n=1}^{\infty} \lambda((s, n)) \quad \text{for each} \quad s \in S$$

is an element of $P(S)$ and fulfils (20). On the other hand, if $\mu \in P(S)$ then take

$$\lambda((s, 1)) = \mu(s) \quad \text{for each} \quad s \in S,$$
$$\lambda((s, n)) = 0 \quad \text{for each} \quad s \in S \quad \text{and} \quad n \in N - \{1\}.$$

We observe that $\lambda \in P(\tilde{S})$ and that (20) is valid.

By the definition of the family $\tilde{w}$, there is a one-to-one correspondence between w and $\tilde{w}$ such that $W \in w$ corresponds to the set $W \times N \in \tilde{w}$. This fact and the equality (20) imply (19).

The following theorem is an immediate consequence of Lemma 3 and of the relation (18).

Theorem 2. *Let S be a set and w a family of its subsets. Then*

$$(21) \qquad e(w, S) = \inf_{\lambda \in U(\tilde{S})} \max_{\tilde{W} \in \tilde{w}} \lambda(\tilde{W}).$$

We conclude that, when evaluating the quantity $e(w, S)$, it is possible to consider the couple $(\tilde{S}, \tilde{w})$ instead of (S, w) and to investigate uniform measures on $\exp \tilde{S}$ only. The couple $(\tilde{S}, \tilde{w})$ is a little bit more complicated than (S, w). This is, however, more than recompensated by the fact that the values of uniform measures can be expressed in terms of cardinalities of certain finite subsets of the basic space $\tilde{S}$. To demonstrate this we present an analogue of Lemma 2.2 of the paper [4].

Theorem 3. *Let $\omega \in [0; 1]$. Then $e(w, S) = \omega$ if and only if the following two conditions are fulfilled:*

1) *for each $\varepsilon > 0$, there exists a finite nonvoid set $F_\varepsilon \subset \tilde{S}$ such that*

$$(22) \qquad \operatorname{card}(\tilde{W} \cap F_\varepsilon) < (\omega + \varepsilon) . \operatorname{card} F_\varepsilon, \quad \textit{for each} \quad \tilde{W} \in \tilde{w}.$$

2) *for each finite set $G \subset \tilde{S}$, there exists $\tilde{W} \in \tilde{w}$ such that*

$$(23) \qquad \operatorname{card}(\tilde{W} \cap G) \geqq \omega . \operatorname{card} G.$$

We observe in Theorem 3 that in the case $\omega = 0$ it is sufficient to consider the first condition only. In other words, if $e(w, S) = 0$ then Theorem 3 guarantees the existence of a finite nonvoid set $F_\varepsilon \subset \tilde{S}$, for each $\varepsilon > 0$, possessing the property

$$\operatorname{card}(\tilde{W} \cap F_\varepsilon) < \varepsilon \,.\, \operatorname{card} F_\varepsilon \quad \text{for each} \quad \tilde{W} \in \tilde{w}\,.$$

No information concerning the relation of the sets F_ε for different positive real numbers ε has been given. A heuristic idea suggests that if ε is getting smaller then the number of elements of the corresponding F_ε is increasing. (We omit the trivial case that $S - \bigcup_{W \in w} W \neq \emptyset$, of course.) It is, however, not obvious whether a set $F_{\varepsilon'}$ contains all (or many) elements of the set F_ε for $\varepsilon > \varepsilon' > 0$, or whether e.g. F_ε and $F_{\varepsilon'}$ are disjoint.

We are going to show that the sets F_ε, for ε decreasing, can be constructed by gradually adding certain elements of the basic space $\tilde{S}$. The sequence, say $\{x_q\}_{q=1}^\infty \subset \subset \tilde{S}$, arising by this procedure possesses the property that the measures $\lambda_k \in U(\tilde{S})$ determined for each $k \in N$ by their supports $N(\lambda_k) = \{x_1; \ldots; x_k\}$ fulfil

$$\lim_{k \to \infty} \max_{\tilde{W} \in \tilde{w}} \lambda_k(\tilde{W}) = 0\,. \tag{24}$$

Let us remark that, in general, it is not possible to find $\{x_q\}_{q=1}^\infty$ in such a way that the sequence

$$\{\max_{\tilde{W} \in \tilde{w}} \lambda_k(\tilde{W})\}_{k=k_0}^\infty \quad \text{is decreasing}\,, \tag{25}$$

for a suitably chosen positive integer k_0. Indeed, we shall prove that (25) implies the existence of a $k_1 \in N$ such that each $\tilde{W} \in \tilde{w}$ contains not more than k_1 elements of the sequence $\{x_q\}_{q=1}^\infty$. Further, we shall study the example considered in Section 2 of the paper [4], which possesses the property $e(w, S) = 0$. We shall show that for each sequence $\{x_q\}_{q=1}^\infty \subset \tilde{S}$ there exists a $\tilde{W} \in \tilde{w}$ containing arbitrarily many elements of $\{x_q\}_{q=1}^\infty$.

Lemma 4. *Let $\{x_q\}_{q=1}^\infty \subset \tilde{S}$. Then (25) holds with a $k_0 \in N$ (λ_k is determined by $\lambda_k \in U(\tilde{S})$ and $N(\lambda_k) = \{x_1; \ldots; x_k\}$) if and only if*

$$\sup_{\tilde{W} \in \tilde{w}} \operatorname{card}(\tilde{W} \cap \{x_q\}_{q=1}^\infty) < \infty\,. \tag{26}$$

Proof. We observe that

$$\lambda_{k+1}(\tilde{W}) = \frac{1}{k+1} \operatorname{card}(\tilde{W} \cap \{x_q\}_{q=1}^{k+1}) =$$

$$= \frac{1}{k+1}\left[\operatorname{card}(\tilde{W} \cap \{x_q\}_{q=1}^{k}) + \chi(x_{k+1}, \tilde{W})\right] = \tag{27}$$

$$= \frac{1}{k+1} [k\, \lambda_k(\tilde{W}) + \chi(x_{k+1}, \tilde{W})]$$

holds for each $k \in N$ and $\tilde{W} \in \tilde{w}$, where

$$\chi(x_{k+1}, \tilde{W}) = \begin{cases} 1, & \text{if} \quad x_{k+1} \in \tilde{W}, \\ 0 & \text{otherwise}. \end{cases}$$

Let the relation (25) be true, let $k \in N$ be such that $k \geqq k_0$ and let $\tilde{W}_k \in \tilde{w}$ fulfil

$$\lambda_k(\tilde{W}_k) = \max_{\tilde{W} \in \tilde{w}} \lambda_k(\tilde{W}).$$

If $\chi(x_{k+1}, \tilde{W}_k) = 1$, we have by (25) and (27)

$$\lambda_k(\tilde{W}_k) > \max_{\tilde{W} \in \tilde{w}} \lambda_{k+1}(\tilde{W}) \geqq \lambda_{k+1}(\tilde{W}_k) \geqq$$

$$\geqq \frac{1}{k+1} [k\, \lambda_k(\tilde{W}_k) + \lambda_k(\tilde{W}_k)] = \lambda_k(\tilde{W}_k).$$

Thus, we find that

(28) $$\chi(x_{k+1}, \tilde{W}_k) = 0.$$

Further, if $\tilde{W} \in \tilde{w}$ fulfil

$$\lambda_k(\tilde{W}) < \max_{\tilde{W} \in \tilde{w}} \lambda_k(\tilde{W})$$

we obtain

$$\operatorname{card}(\tilde{W} \cap \{x_q\}_{q=1}^k) + 1 \leqq \operatorname{card}(\tilde{W}_k \cap \{x_q\}_{q=1}^k),$$

so that by (27)

(29) $$\lambda_{k+1}(\tilde{W}) \leqq \frac{1}{k+1} [\operatorname{card}(\tilde{W}_k \cap \{x_q\}_{q=1}^k) + \chi(x_{k+1}, \tilde{W}_k)] = \lambda_{k+1}(\tilde{W}_k).$$

The relations (27), (28), and (29) imply

$$\max_{\tilde{W} \in \tilde{w}} \operatorname{card}(\tilde{W} \cap \{x_q\}_{q=1}^{k+1}) = (k+1) . \max_{\tilde{W} \in \tilde{w}} \lambda_{k+1}(\tilde{W}) =$$

$$= k . \max_{\tilde{W} \in \tilde{w}} \lambda_k(\tilde{W}) = \max_{\tilde{W} \in \tilde{w}} \operatorname{card}(\tilde{W} \cap \{x_q\}_{q=1}^k),$$

for each $k \in N$ such that $k \geqq k_0$, i.e.

(30) $$\max_{\tilde{W} \in \tilde{w}} \operatorname{card}(\tilde{W} \cap \{x_q\}_{q=1}^k) = \max_{\tilde{W} \in \tilde{w}} \operatorname{card}(\tilde{W} \cap \{x_q\}_{q=1}^{k_0}) \leqq k_0 < \infty$$

holds for each $k \in N$, $k \geqq k_0$. The assertion (26) is an immediate consequence of (30).

On the other hand, let (26) be fulfilled. We denote

$$k_1 = \sup_{\tilde{W} \in \tilde{w}} \operatorname{card}(\tilde{W} \cap \{x_q\}_{q=1}^{\infty}).$$

We find that supremum may be replaced by maximum so that there exists a set $\tilde{W}_0 \in \tilde{w}$ containing k_1 elements of the sequence $\{x_q\}_{q=1}^{\infty}$, say $x_{q(1)}, \ldots, x_{q(k_1)}$, where $q(1) < q(2) < \ldots < q(k_1)$. We put

$$k_0 = q(k_1) .$$

For each $\tilde{W} \in \tilde{w}$ and $k \in N$ such that $k \geqq k_0$ we have

$$\lambda_k(\tilde{W}) = \frac{1}{k} \operatorname{card} (\tilde{W} \cap \{x_q\}_{q=1}^{k}) \leqq \frac{k_1}{k} ,$$

and

$$\lambda_k(\tilde{W}_0) = \frac{1}{k} \operatorname{card} (\tilde{W}_0 \cap \{x_q\}_{q=1}^{k}) \geqq \frac{1}{k} \operatorname{card} \{x_{q(1)}; \ldots; x_{q(k_1)}\} = \frac{k_1}{k} ,$$

so that

$$\max_{\tilde{W} \in \tilde{w}} \lambda_k(\tilde{W}) = \frac{k_1}{k} \tag{31}$$

holds for each $k \in N$ such that $k \geqq k_0$. The equality (31) proves (25).

Example. Let S and w be given as follows:

$$S = N ,$$

$$w = \{W; W \subset S, \min W \geqq \operatorname{card} W\} . \tag{32}$$

If the sequence $\{x_q\}_{q=1}^{\infty} \subset \tilde{S}$ is such that

$$\sup_{\tilde{W} \in \tilde{w}} \operatorname{card} (\tilde{W} \cap \{x_q\}_{q=1}^{\infty})$$

is finite (say equal to $\hat{k}$) then, using the explicit expression

$$x_q = (s_q, n_q) , \quad \text{for} \quad q \in N ,$$

and (17), (32), we immediately conclude that not more than $\hat{k}$ elements of $\{x_q\}_{q=1}^{\infty}$ can be such that their first component s_q fulfils $s_q > \hat{k}$. We put

$$\tilde{W}_i = \{(s, n);\ s = i,\ n \in N\} \quad \text{for each} \quad i = 1, \ldots, \hat{k} .$$

We find that $\tilde{W}_i \in \tilde{w}$ for each $i = 1, \ldots, \hat{k}$, and

$$\operatorname{card} [(\bigcup_{i=1}^{\hat{k}} \tilde{W}_i) \cap \{x_q\}_{q=1}^{\infty}] = \infty .$$

Thus, there exists an $i_0 \in \{1; \ldots; \hat{k}\}$ such that

$$\operatorname{card} (\tilde{W}_{i_0} \cap \{x_q\}_{q=1}^{\infty}) = \infty .$$

We conclude that, in this example, the relation (25) is not true for any sequence $\{x_q\}_{q=1}^{\infty} \subset \tilde{S}$.

It remains to present a construction of a sequence $\{x_q\}_{q=1}^{\infty}$ such that (24) is valid, for any given couple (S, w).

Definition. Let A and B be finite subsets of $\tilde{S}$. We assume that the elements of A and B are numbered in a fixed way by indexes $1, 2, \ldots$, card A, and $1, 2, \ldots$, card B, respectively, i.e.

$$A = \{x_1; \ldots; x_{\operatorname{card} A}\},$$
$$B = \{y_1; \ldots; y_{\operatorname{card} B}\}.$$

Using the expression

$$x_i = (s_i, n_i),$$
$$y_i = (t_i, m_i),$$

where $s_i, t_i \in S$ and $n_i, m_i \in N$, we have

$$A = \{(s_1, n_1); \ldots; (s_{\operatorname{card} A}, n_{\operatorname{card} A})\},$$

and similarly

$$B = \{(t_1, m_1); \ldots; (t_{\operatorname{card} B}, m_{\operatorname{card} B})\}.$$

We assume that the elements of the sets A and B are numbered in such a way that the sequence $\{n_i\}_{i=1}^{\operatorname{card} A}$ and $\{m_i\}_{i=1}^{\operatorname{card} B}$ are non-decreasing. We define an *operation* $*$ by the formula

$$A * B = \{(s_1, 1); \ldots; (s_{\operatorname{card} A}, \operatorname{card} A); (t_1, \operatorname{card} A + 1); \ldots$$
$$\ldots; (t_{\operatorname{card} B}, \operatorname{card} A + \operatorname{card} B)\}.$$

Further, we put

$$A^{1*} = A * \emptyset = \{(s_1, 1); \ldots; (s_{\operatorname{card} A}, \operatorname{card} A)\},$$
$$A^{i*} = A * A^{(i-1)*} \quad \text{for each} \quad i \in N - \{1\},$$

and

$$A^{(1)} = A^{1*},$$
$$A^{(i)} = A^{i*} - A^{(i-1)*} =$$
$$= \{(s_1, (i-1) \operatorname{card} A + 1); \ldots; (s_{\operatorname{card} A}, i \operatorname{card} A)\}$$
$$\text{for each} \quad i \in N - \{1\}.$$

Lemma 5. *Let A, B, C be finite subsets of $\tilde{S}$ and let $i, j \in N$. The operation $*$ possesses the following properties:*

a) *$A * B$ is a finite subset of $\tilde{S}$;*
b) $\operatorname{card}(A * B) = \operatorname{card} A + \operatorname{card} B$;
c) $(A * B) * C = A * (B * C)$;
d) $A^{1*} \subset A * B$;
e) $(A * B)^{1*} = A * B$;

f) *if* $i \leqq \operatorname{card} A$ *and if* $B = \{(s, n); (s, n) \in A^{1*}, n < i\}$ *then* $A^{1*} = B * (A^{1*} - B)$;

g) $A^{i*} = \bigcup_{q=1}^{i} A^{(q)}$;

h) $A^{(i)} \cap A^{(j)} = \emptyset$ *if* $i \neq j$;

i) $\operatorname{card}\{(s, n); (s, n) \in A^{1*}, n = i\} = 1$.

Lemma 6. *Let* $\eta, \mu \in U(\tilde{S})$ *and let* $i \in N$. *If*

$$N(\mu) = [N(\eta)]^{i*}$$

then

$$\mu(\tilde{W}) = \eta(\tilde{W}) \quad \textit{for each} \quad \tilde{W} \in \tilde{w}\,.$$

Proof. Given a $\tilde{W} \in \tilde{w}$, we know that $\tilde{W} = W \times N$, where W is an element of w. Thus, a couple $(s, n) \in \tilde{S}$ is an element of $\tilde{W}$ regardless of the value of its second component n. The set $N(\eta)$ differs from $[N(\eta)]^{(j)}$ for $j \in N$ only by the values of the second component of the couples involved. We have

$$\operatorname{card}[\tilde{W} \cap N(\eta)] = \operatorname{card}\{\tilde{W} \cap [N(\eta)]^{(j)}\} \quad \text{for each} \quad j \in N\,.$$

Finally, by Lemma 5 we obtain

$$\operatorname{card}[\tilde{W} \cap N(\mu)] = \sum_{j=1}^{i} \operatorname{card}\{\tilde{W} \cap [N(\eta)]^{(j)}\} = i \,.\, \operatorname{card}[\tilde{W} \cap N(\eta)]\,,$$

so that

$$\mu(\tilde{W}) = \frac{1}{i \,.\, \operatorname{card} N(\eta)}\, i \,.\, \operatorname{card}[\tilde{W} \cap N(\eta)] = \eta(\tilde{W})\,.$$

Lemma 7. *Let* $\eta, \mu \in U(\tilde{S})$ *be such that*

$$\operatorname{card} N(\eta) \geqq \operatorname{card} N(\mu)\,, \tag{33}$$

and let

$$M \subset N(\mu)\,. \tag{34}$$

Then $\nu \in U(\tilde{S})$ *determined by its support*

$$N(\nu) = N(\eta) * M \tag{35}$$

fulfils

$$\max_{\tilde{W} \in \tilde{w}} \nu(\tilde{W}) \leqq \max_{\tilde{W} \in \tilde{w}} \eta(\tilde{W}) + \max_{\tilde{W} \in \tilde{w}} \mu(\tilde{W})\,.$$

If $\operatorname{card} M = \operatorname{card} N(\eta)$ *then*

$$\max_{\tilde{W} \in \tilde{w}} \nu(\tilde{W}) \leqq \tfrac{1}{2}[\max_{\tilde{W} \in \tilde{w}} \eta(\tilde{W}) + \max_{\tilde{W} \in \tilde{w}} \mu(\tilde{W})]\,. \tag{36}$$

Proof. Taking a $\tilde{W} \in \tilde{w}$ we have

$$\operatorname{card}[\tilde{W} \cap N(\nu)] = \operatorname{card}[\tilde{W} \cap N(\eta)] + \operatorname{card}[\tilde{W} \cap M] \leqq$$
$$\leqq \operatorname{card} N(\eta) \,.\, \max_{\tilde{W} \in \tilde{w}} \eta(\tilde{W}) + \operatorname{card} N(\mu) \,.\, \max_{\tilde{W} \in \tilde{w}} \mu(\tilde{W}) \,,$$

so that

$$\nu(\tilde{W}) \leqq \max_{\tilde{W} \in \tilde{w}} \eta(\tilde{W}) + \max_{\tilde{W} \in \tilde{w}} \mu(\tilde{W}) \,.$$

In the case $\operatorname{card} M = \operatorname{card} N(\eta)$, Lemma 5, (33), (34) and (35) imply $\operatorname{card} N(\nu) = 2 \,.\, \operatorname{card} N(\eta) = 2 \,.\, \operatorname{card} N(\mu)$ and (36).

Let $e(w, S) = 0$ and let the sets $F_{2^{-z}}$ for $z \in N$, the existence of which is guaranteed by Theorem 3, be given. We denote

$$E_z = F_{2^{-z}} \quad \text{for each} \quad z \in N \,,$$

and

(37) $$H_1 = [E_1]^{1*} \,,$$

(38) $$H_{z+1} = [H_z]^{\alpha_z *} * [E_{z+1}]^{\beta_z *} \,,$$

where

(39) $$\alpha_z = \operatorname{card} E_{z+1} \quad \text{and} \quad \beta_z = \operatorname{card} H_z$$

for each $z \in N$.

Lemma 8. *Let $\lambda^{(z)} \in U(\tilde{S})$ for $z \in N$ be such that $N(\lambda^{(z)}) = H_z$. Put*

(40) $$h_1 = 0.5 \,,$$

(41) $$h_z = \max_{\tilde{W} \in \tilde{w}} \lambda^{(z)}(\tilde{W}) \quad \text{for each} \quad z \in N - \{1\} \,.$$

Then

(42) $$h_z \leqq \frac{z+1}{2^{z+1}} \quad \textit{for each} \quad z \in N \,.$$

Proof. The validity of (42) for $z = 1$ is obvious. Further, we know from (36) and from Lemmas 5, 6 and 7 that

(43) $$h_{z+1} \leqq \tfrac{1}{2}(h_z + 2^{-z-1}) \quad \text{for each} \quad z \in N \,.$$

The inequality (42) for each $z \in N - \{1\}$ is an immediate consequence of (43).

We are now ready to prove the main result of this section.

Theorem 4. *Let S be a set and let w be a family of its subsets. Then $e(w, S) = 0$ if and only if there exists such a sequence $\{x_q\}_{q=1}^{\infty} \subset \tilde{S}$ that $\lambda_k \in U(\tilde{S})$ determined by their supports $N(\lambda_k) = \{x_1; \ldots; x_k\}$ for $k \in N$ fulfil the relation (24), where $\tilde{S}$ and $\tilde{w}$ were defined by (16) and (17).*

Proof. It is easy to find that (24) implies $e(\tilde{w}, \tilde{S}) = 0$, i.e. $e(w, S) = 0$ according to Lemma 3. On the other hand, let $e(w, S) = 0$. Then take the set

(44) $$H = \bigcup_{z=1}^{\infty} H_z ,$$

where H_z for $z \in N$ are defined by (37) and (38). We know from Lemma 5 that H is infinite, $H_z \subset H_{z+1}$, and that H_z is a finite subset of $\tilde{S}$ for each $z \in N$. Let x_q denote the element $(s, q) \in H$ the second component of which equals q for each $q \in N$. Its existence and uniqueness is guaranteed by Lemma 5. It remains to prove that this sequence $\{x_q\}_{q=1}^{\infty}$ possesses the required property (24). Take a $k \in N$ such that $k >$ card E_1. Then the quantity

(45) $$z(k) = \max \{z;\ z \in N,\ x_k \notin H_z\}$$

is well-defined and the support $N(\lambda_k) = [N(\lambda_k)]^{1*}$ of λ_k can be written either in the form

$$N(\lambda_k) = [H_{z(k)}]^{\gamma(k)*} * M ,$$

where $\gamma(k) \in \{1; \ldots; \text{card } E_{z(k)+1} - 1\}$ and $M \subset H_{z(k)}$, or

$$N(\lambda_k) = [H_{z(k)}]^{\alpha_{z(k)}*} * M ,$$

where $M \subset [E_{z(k)+1}]^{\beta_{z(k)}*}$. Let η_k, $\mu_k \in U(\tilde{S})$ be such that

(46) $$N(\eta_k) = [H_{z(k)}]^{\gamma(k)*} ,$$

(47) $$N(\mu_k) = H_{z(k)}$$

in the former case, and

(48) $$N(\eta_k) = [H_{z(k)}]^{\alpha_{z(k)}*} ,$$

(49) $$N(\mu_k) = [E_{z(k)+1}]^{\beta_{z(k)}*}$$

in the latter. We find by (37), (38), (39) and by Lemma 5 that

$$\text{card } H_{z(k)} \leqq \text{card } [H_{z(k)}]^{\gamma(k)*} ,$$

and

$$\text{card } [E_{z(k)+1}]^{\beta_{z(k)}*} = \alpha_{z(k)} \cdot \beta_{z(k)} = \text{card } [H_{z(k)}]^{\alpha_{z(k)}*} ,$$

so that

$$\text{card } N(\eta_k) \geqq \text{card } N(\mu_k)$$

holds in both cases. Lemmas 6, 7, and 8 imply

$$\max_{\tilde{W} \in \tilde{w}} \lambda_k(\tilde{W}) \leqq 2h_{z(k)} \leqq \frac{z(k) + 1}{2^{z(k)}}$$

provided (46) and (47) are true.

Similarly, if (48) and 49) are valid then

$$\max_{\tilde{W} \in \tilde{w}} \lambda_k(\tilde{W}) \leqq h_{z(k)} + 2^{-z(k)-1} \leqq \frac{z(k) + 2}{2^{z(k)+1}} ;$$

we conclude that the measure λ_k fulfil

$$\max_{\tilde{W} \in \tilde{w}} \lambda_k(\tilde{W}) \leqq \frac{z(k) + 1}{2^{z(k)}} \quad \text{for each} \quad k \in N\,, \quad k > \operatorname{card} E_1\,,$$

where $z(k)$ is defined by (45). We find by (45) and Lemma 5 that

$$\lim_{k \to \infty} z(k) = \infty\,,$$

i.e.

$$\lim_{k \to \infty} \max_{\tilde{W} \in \tilde{w}} \lambda_k(\tilde{W}) \leq \lim_{k \to \infty} \frac{z(k) + 1}{2^{z(k)}} = 0\,.$$

This completes the proof of the theorem.

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Souhrn

TLOUŠŤKA RODIN MNOŽIN A ROVNOMĚRNÉ MÍRY

ANTONÍN LEŠANOVSKÝ

Nechť S je množina a w rodina jejích podmnožin. Pojem tloušťky rodiny w byl definován v článku [4] pomocí obecných molekulárních měr na S. V článku je sestrojena taková množina $\tilde{S}$, že pro určení tloušťky rodiny w se stačí omezit jen na rovnoměrné molekulární míry na $\tilde{S}$. Této skutečnosti je využito pro charakterizaci rodin w s nulovou tloušťkou.

Резюме

ТОЛЩИНА СЕМЕЙСТВА МНОЖЕСТВ

ANTONÍN LEŠANOVSKÝ

Пусть S-множество и w-семейство его подмножеств. Понятие толщины семейства w было введено в [4] с помощью общих молекулярных мер на S. В статье строится такое множество $\tilde{S}$, что для определения толщины семейства w можно ограничиться равномерными молекулярными мерами на $\tilde{S}$. Этот факт используется для характеризации семейств w с нулевой толщиной.

Author's address: Matematický ústav ČSAV, Žitná 25, 115 67 Praha 1.