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Pavel Kostyrko

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ON A LOCAL FORM OF LOBACHEVSKI'S FUNCTIONAL EQUATION

PAVEL KOSTYRKO, Bratislava

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Summary. A function $f: (A, B) \to R$ (R — the real line, $(A, B) \subset R$) is said to be locally Lobachevskian if for each $x \in (A, B)$ there exists $\delta(x) > 0$ such that

$$f(x + h) f(x - h) = f(x)^2$$

holds for each h, $0 < h < \delta(x)$. In the paper a full description of the family of all locally Lobachevskian functions is given.

Keyword: Lobachevski's functional equation.

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1. INTRODUCTION, DEFINITIONS AND RESULTS

In the present paper we shall deal with real functions which are defined on a real open interval (A, B), $-\infty \leqq A < B \leqq +\infty$. In the general theory of functional equations, Lobachevski's functional equation

$$\text{(L)} \qquad f(x + h) f(x - h) = f(x)^2 ,$$

is well known (see e.g. [A]). Similarly to the paper [K], where Jensen's functional equation in its local form is investigated, we can deal with a local form of Lobachevski's functional equation. Note that the following local property is introduced analogously to [R].

Definition 1. *A function $f: (A, B) \to R$ (R — the real line) is said to be locally Lobachevskian (lL) at $x \in (A, B)$ if there exists $\delta(x) > 0$ such that* (L) *holds for each h, $0 < h < \delta(x)$. We say that f is locally Lobachevskian if it is lL at x for each $x \in (A, B)$. Let LL stand for the family of all locally Lobachevskian functions.*

Obviously, each Lobachevskian function, i.e. a solution of (L), belongs to LL. Recall that each Lobachevskian function f is of the form $f(x) = ce^{a(x)}$, where $a: R \to R$ is an additive function and c is a real constant. There are functions in LL which are not Lobachevskian functions. Such functions are e.g. the functions $g: R \to R$ and $h: R \to R$ defined in the following way: $g(x) = -1$ for $x \in (-\infty, 0)$, $g(x) = 0$ for $x \in [0, 1]$ and $g(x) = 2^x$ for $x \in (1, \infty)$; $h(x) = -1$ for $x \in Z$ (Z — the set of integers) and $h(x) = 3^{-2k} 3^x$ for $x \in (2k - 1, 2k) \cup (2k, 2k + 1)$, $k \in Z$. In what follows a full description of the family LL will be given in terms of Lobachevskian functions.

Definition 2. *A set $N \subset (A, B)$ is said to be a semi-symmetric $(ss-)$ set if*
(i) *N is closed;*
(ii) *for each $x \in N$ there exists $\delta_x > 0$ such that for each h, $0 < h < \delta_x$, $x + h \in N$ or $x - h \in N$.*

Definition 3. ([K]) *A set $M \subset (A, B)$ is said to be an s-set if*
(i) *M is closed and countable;*
(ii) *for each $x \in M$ there exists $\delta_x > 0$ such that for each h, $0 < h < \delta_x$, $x + h \in M$ if and only if $x - h \in M$.*

Theorem 1. *Let $f \in LL$. Then $N_f = \{x \in (A, B): f(x) = 0\}$ is an ss-set and for each interval (a, b) contiguous to N_f there exists an interval $(u, v) \subset (a, b)$ such that the restriction $f|\,(u, v)$ is a Lobachevskian function.*

Theorem 2. *Let $f: (A, B) \to R$. Then the following statements are equivalent:*
(a) *$f \in LL$;*
(b) *there exists an ss-set N such that $N = N_f = \{x \in (A, B): f(x) = 0\}$; for each interval (a, b) contiguous to N there exists a Lobachevskian function $g: (a, b) \to$ $\to R$, an s-set $M \subset (a, b)$ with the collection $\{J_n\}$ of contiguous intervals of M in (a, b), and a real sequence $\{a_n\}$ such that $f|\, J_n = a_n g|\, J_n$ holds for each n, and f is lL at each $x \in M$.*

2. PROOFS

In the following proofs we shall use modifications of ideas used in [T] and [K].

Lemma. *Let $f \in LL$, $f(x) \neq 0$ for each $x \in (U, V)$, $A \leqq U < V \leqq B$. Then there exists an interval $(u, v) \subset (U, V)$ such that for each subinterval $I = [x - 2h, x + 2h]$ of (u, v) we have $f(x + 2h) f(x - 2h)^{-1} = f(x + h)^2 f(x - h)^{-2}$.*

Proof. Let $\delta(x)$ be introduced by Definition 1 and write $E_n = \{x \in (U, V): \delta(x) >$ $> n^{-1}\}$, $n = 1, 2, \ldots$. Then, since the sets E_n cover (U, V), according to the Baire Category Theorem there must exist an interval $(u, v) \subset (U, V)$ and n such that E_n is dense in (u, v). Without loss of generality we may assume that $v - u < n^{-1}$. Let $I \subset (u, v)$. The notation is simplified if we assume that $I = [-2h, 2h]$, E_n is dense in $(-2h, 2h)$ and $4h < n^{-1}$. If so, choose a negative x' in E_n such that $0 <$ $< x' - (-h/2) < \delta(h)/2$, and a positive x'' in E_n such that $0 < x'' - h/2 < \delta(-h)/2$. This means that $0 < 2x' + h < \delta(h)$ and $0 < 2x'' - h < \delta(-h)$. Clearly we can also achieve that $x'' + 2x' < 0$ and $2x'' + x' > 0$.

Let us define the following intervals:

$$I_1 = [-2h, 2x' + 2h] = [x' - (x' + 2h), x' + (x' + 2h)],$$
$$I_2 = [-2x', 2x' + 2h] = [h - (2x' + h), h + (2x' + h)],$$
$$I_3 = [2x'' + 2x', -2x'] = [x'' + (x'' + 2x'), x'' - (x'' + 2x')],$$
$$I_4 = [-2x'', 2x' + 2x''] = [x' - (2x'' + x'), x' + (2x'' + x')],$$
$$I_5 = [-2x'', 2x'' - 2h] = [-h - (2x'' - h), -h + (2x'' - h)], \quad \text{and}$$
$$I_6 = [2x'' - 2h, 2h] = [x'' - (2h - x''), x'' + (2h - x'')].$$

For $f \in LL$, $f(x) \neq 0$ for each $x \in (U, V)$ and $J = [r, s] \subset (U, V)$, put $F(J) = f(r) f(s) f((r + s)/2)^{-2}$. It is straightforward to check that each I_i $(i = 1, 2, \ldots, 6)$ is a subinterval of I, that $F(I_i) = 1$ for each I_i and that for f we have

$$f(-2h) f(2h)^{-1} f(h)^2 f(-h)^{-2} = \prod_{i=1}^{6} F(I_i)^{(-1)^{i+1}} = 1.$$

It remains to relabel and the proof is complete.

Proof of Theorem 1. The fact that $N_f = \{x \in (A, B): f(x) = 0\}$ is an ss-set is obvious. Let (a, b) be an interval contiguous to N_f and let $(u, v) \subset (a, b)$ be an interval of the type whose existence is guarranteed by Lemma. Choose $x \in (u, v)$ and $h > 0$ such that $u < x - h < x + h < v$. By induction we can verify that

$$(*) \qquad f(x + h) f(x - h) f(x)^{-2} = [f(x + h\, 2^{-n}) f(x - h\, 2^{-n}) f(x)^{-2}]^{4^n}$$

holds for each $n = 0, 1, \ldots$. Clearly, for $n = 0$ $(*)$ is fulfilled. Using Lemma we can write

$$\begin{aligned}
f(x + h\, 2^{-n}) f(x - h\, 2^{-n}) f(x)^{-2} &= \\
= f(x + h\, 2^{-n}) f(x)^{-1} [f(x) f(x - h\, 2^{-n})^{-1}]^{-1} &= \\
= f(x + \tfrac{3}{4}h\, 2^{-n})^2 f(x + \tfrac{1}{4}h\, 2^{-n})^{-2} [f(x - \tfrac{1}{4}h\, 2^{-n})^2 f(x - \tfrac{3}{4}h\, 2^{-n})^{-2}]^{-1} &= \\
= f(x + \tfrac{3}{4}h\, 2^{-n})^2 f(x - \tfrac{1}{4}h\, 2^{-n})^{-2} [f(x + \tfrac{1}{4}h\, 2^{-n}) f(x - \tfrac{3}{4}h\, 2^{-n})^{-1}]^{-2} &= \\
= [f(x + \tfrac{1}{2}h\, 2^{-n}) f(x)^{-1}]^4 [f(x) f(x - \tfrac{1}{2}h\, 2^{-n})^{-1}]^{-4} &= \\
= [f(x + h\, 2^{-n-1}) f(x - h\, 2^{-n-1}) f(x)^{-2}]^4 &.
\end{aligned}$$

This proves $(*)$ for each $n = 0, 1, \ldots$. If we choose m such that $0 < h\, 2^{-m} < \delta(x)$ then

$$f(x + h) f(x - h) f(x)^{-2} = [f(x + h\, 2^{-m}) f(x - h\, 2^{-m}) f(x)^{-2}]^{4^m} = 1.$$

Proof of Theorem 2. (a) implies (b): According to Theorem 1 the set N_f is an ss-set, hence it is closed. Let (a, b) be a contiguous interval to N_f. If $M \subset (a, b)$ is an s-set, then $(a, b) - M = \bigcup_n J_n$, where $\{J_n\}_n$ is a countable collection of mutually disjoint contiguous open intervals of M. Further, we shall say that the restriction

$f|(c, d)$, $(c, d) \subset (a, b)$, has an acceptable form if there is a Lobachevskian function $g: (c, d) \to R$, an s-set $M \subset (c, d)$ with the collection of contiguous intervals $\{J_n\}_n$, and a real sequence $\{a_n\}_n$ such that $f| J_n = a_n g| J_n$ holds for each n. Let (u, v) be an interval such that $f|(u, v)$ is a Lobachevskian function (Theorem 1). Choose an arbitrary $z \in (u, v)$ and put

$$y = \sup \{w: f|(z, w) \text{ has an acceptable form}\} .$$

We will prove that $y = b$. On the contrary, suppose $y < b$. Obviously $y \geqq v$. Let $\delta(y)$ have the meaning from Definition 1. We can suppose without loss of generality that $z \leqq y - \delta(y)$. Since $f_1 = f|(y - \delta(y), y)$ has an acceptable form, the same is true for $f_2 = f|(y, y + \delta(y))$. Indeed, it follows from the hypothesis $f \in LL$ that $f_2(y + h) = f(y)^2 f_1(y - h)^{-1}$ holds for each h, $0 < h < \delta(y)$. Hence $f|(z, y + \delta(y))$ has an acceptable form — a contradiction. Analogously it can be verified that $f|(a, z)$ has an acceptable form and consequently, also $f|(a, b)$ has an acceptable form.

The fact that (b) implies (a) is an easy consequence of the structure of the ss-set N and of the s-set M $(= M(a, b))$.

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Súhrn

O LOKÁLNOM TVARE LOBAČEVSKÉHO FUNKCIONÁLNEJ ROVNICE

Pavel Kostyrko

Funkcia $f: (A, B) \to R$ (R — reálna priamka, $(A, B) \subset R$) sa nazýva lokálne Lobačevského ak pre každé $x \in (A, B)$ existuje $\delta(x) > 0$ tak, že

$$f(x + h) f(x - h) = f(x)^2$$

platí pre každé h, $0 < h < \delta(x)$. V článku sa podáva úplný opis systému všetkých lokálne Lobačevského funkcií.

Резюме

О ЛОКАЛЬНОМ ВИДЕ ФУНКЦИОНАЛЬНОГО УРАВНЕНИЯ ЛОБАЧЕВСКОГО

Pavel Kostyrko

Отображение $f: (A, B) \to R$ (R — вещественная прямая, $(A, B) \subset R$) является локально отображением Лобачевского, если для каждого $x \in (A, B)$ существует $\delta(x) > 0$ так, что

$$f(x + h) f(x - h) = f(x)^2$$

имеет место для всякого h, $0 < h < \delta(x)$. В работе дана полная характеристика всех таких отображений.

Author's address: Katedra algebry a teórie čísel MFF UK, 842 15 Bratislava.