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REFLECTION AND THE NEUMANN PROBLEM ON DOUBLY
CONNECTED REGIONS

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Summary. This paper is a continuation of the paper "Reflection and the Dirichlet problem on doubly connected regions". Analogously to that paper it is shown that using the reflection function the system of two integral equations corresponding the Neumann problem can be reduced to a single integral equation.

Keywords: Laplace's equation, Neumann problem, integral equations.

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The Dirichlet problem for doubly connected regions bounded by two Jordan curves in the plane has been solved in [6]. The "exterior curve" was supposed to be analytic and such that it has a "global reflection function"; for the existence and properties of the reflection function see [16]. The "interior curve" was non-smooth in general but of finite length and with bounded cyclic variation. It was shown that the use of the reflection function makes it possible to reduce the system of two integral equations in solving the Dirichlet problem to a single integral equation considered on the "interior curve". The original idea is due to J. M. Sloss [17], who considered only the case of smooth boundary curves. In this sense, the results in [6] are a mere generalization of the results in [17]. Only the Dirichlet problem was studied in [17]. The aim of the present paper is to show that, in a similar way, the reflection function can be used also in solving the Neumann problem. We shall show that, as in the case of the Dirichlet problem, by using the reflection function the system of two integral equations corresponding to the Neumann problem can be reduced to a single integral equation considered on the "interior curve".

1. PRELIMINARY REMARKS AND NOTATION

In this part we state one simple assertion concerning the reflection function. Further, we introduce the necessary notation and recall some assertions we shall need in the sequel. We keep the notation used in [6] (the present paper is a continuation of [6]), nevertheless we recall briefly some points.

Just as in [6] we shall deal with the real plane R^2 which we shall identify with the complex plane C .

Throughout the paper L stands for an analytic Jordan curve with a parametrization $\Phi(\theta) = \Phi_1(\theta) + i \Phi_2(\theta)$, $\theta \in \langle 0, 2\pi \rangle$, of the form

$$(1.1) \quad \Phi_1(\theta) = x(\theta) = \sum_{k=0}^n (a_k \cos k\theta + b_k \sin k\theta),$$

$$(1.2) \quad \Phi_2(\theta) = y(\theta) = \sum_{k=0}^m (\alpha_k \cos k\theta + \beta_k \sin k\theta),$$

where $n \geq m$,

$$\begin{aligned} &(\Phi'_1(\theta))^2 + (\Phi'_2(\theta))^2 \neq 0, \\ &(a_n, b_n) \neq (0, 0) \neq (\alpha_m, \beta_m). \end{aligned}$$

Moreover, if L is not a circle then in the case $m = n$ we suppose that either

$$\alpha_n^2 + \beta_n^2 \neq a_n^2 + b_n^2$$

or

$$\alpha_n a_n + \beta_n b_n \neq 0.$$

Denote further

$$R = \text{Int } L.$$

It is shown in [16] that there are finitely many points $e_1, \dots, e_r \in R^2$ (the so called critical points with respect to L) and a neighbourhood R_0 of L such that, if L_i is a Jordan arc lying in R and joining all the points e_1, \dots, e_r , then there is a function g with the following properties:

- g is defined and analytic on $(R - \{e_1, \dots, e_r\}) \cup R_0$;
- g is single-valued on $(R - L_i) \cup R_0$;
- $g'(z) \neq 0$ for $z \in (R - L_i) \cup R_0$;
- $\bar{g}(z) = z$ for $z \in L$;
- $\bar{g}(R - L_i) \cap (R - L_i) = \emptyset$;
- g can be uniquely extended onto

$$R_g = (R - L_i) \cup L \cup \bar{g}(R - L_i)$$

to be holomorphic there;

$$\bar{g}(\bar{g}(z)) = z \quad \text{for } z \in R_g.$$

The function (mapping) \bar{g} is called the reflection function (mapping) with respect to L . It can be seen that \bar{g} is one-to-one on R_g , $\bar{g}(R_g) = R_g$, $\bar{g}(R_g \cap R) = R_g \cap \text{Ext } L$, $\bar{g}(R_g \cap \text{Ext } L) = R_g \cap R$.

For $r = 1, 2$ let \mathcal{H}_r stand for the (normalized) r -dimensional Hausdorff measure on R^2 .

We shall need the following auxiliary assertion (cf. [5], Lemma 1.1).

1.1. Lemma. *Let M be an open set such that either $M \subset R - L_i$ or $M \subset \bar{g}(R - L_i)$. Let (real) functions φ, h be defined on $\bar{g}(M)$ and have continuous*

first partial derivatives there. Then the integral

$$\int_M \text{grad}(\varphi * \bar{g}) \text{grad}(h * \bar{g}) d\mathcal{H}_2$$

exists if and only if the integral

$$\int_{\bar{g}(M)} \text{grad} \varphi \text{grad} h d\mathcal{H}_2$$

does. If these integrals exist then

$$(1.3) \quad \int_M \text{grad}(\varphi * \bar{g}) \text{grad}(h * \bar{g}) d\mathcal{H}_2 = \int_{\bar{g}(M)} \text{grad} \varphi \text{grad} h d\mathcal{H}_2.$$

1.2. Remark. The assertion of Lemma 1.1 will be used in various forms. Let us consider, for instance, the following situation. Let $M \subset R - L_i$ be open,

$$S = M \cup \bar{g}(M)$$

and let φ, h be functions defined and continuously differentiable on S . Then Lemma 1.1 yields (the equalities are valid also in the sense of the existence of the integrals considered)

$$\int_M \text{grad}(\varphi * \bar{g}) \text{grad}(h * \bar{g}) d\mathcal{H}_2 = \int_{\bar{g}(M)} \text{grad} \varphi \text{grad} h d\mathcal{H}_2$$

and at the same time

$$\int_M \text{grad} \varphi \text{grad} h d\mathcal{H}_2 = \int_{\bar{g}(M)} \text{grad}(\varphi * \bar{g}) \text{grad}(h * \bar{g}) d\mathcal{H}_2.$$

Hence

$$(1.4) \quad \int_S \text{grad}(\varphi * \bar{g}) \text{grad}(h * \bar{g}) d\mathcal{H}_2 = \int_S \text{grad} \varphi \text{grad} h d\mathcal{H}_2.$$

In a similar way we can obtain, for example, the equalities

$$(1.5) \quad \int_S \text{grad}(\varphi * \bar{g}) \text{grad} h d\mathcal{H}_2 = \int_S \text{grad} \varphi \text{grad}(h * \bar{g}) d\mathcal{H}_2,$$

$$(1.6) \quad \begin{aligned} \int_S \text{grad}(\varphi * \bar{g}) \text{grad}(h + h * \bar{g}) d\mathcal{H}_2 &= \\ &= \int_S \text{grad} \varphi \text{grad}(h + h * \bar{g}) d\mathcal{H}_2 \end{aligned}$$

and so on.

1.3. Remark. Let R_0 be an open neighbourhood of L , $R_0 \cap L_i = \emptyset$, and let h be a harmonic function on

$$S^+ = R_0 \cap R$$

such that h and its first partial derivatives are continuously extendable from S^+ to $S^+ \cup L$. Further, let φ be a continuously differentiable function with compact support in R^2 ,

$$\text{spt } \varphi \subset S^+ \cup L \cup \bar{g}(S^+)$$

($\text{spt } \varphi$ stands for the support of φ). As h is harmonic, we have

$$\text{div} [\varphi \text{ grad } h] = \text{grad } \varphi \text{ grad } h.$$

Let n_e denote the exterior normal to S^+ on L and n_i the exterior normal to $\bar{g}(S^+)$ on L ; of course, we have $n_i = -n_e$. Since

$$\text{spt } \varphi \cap \partial S^+ = \text{spt } \varphi \cap \partial(\bar{g}(S^+)) \subset L,$$

we obtain from the Gauss-Green theorem and Lemma 1.1

$$\begin{aligned} \int_L \varphi n_e \text{ grad } h \, d\mathcal{H}_1 &= \int_{S^+} \text{grad } \varphi \text{ grad } h \, d\mathcal{H}_2 = \\ &= \int_{\bar{g}(S^+)} \text{grad} (\varphi * \bar{g}) \text{ grad} (h * \bar{g}) \, d\mathcal{H}_2 = \int_L (\varphi * \bar{g}) n_i \text{ grad} (h * \bar{g}) \, d\mathcal{H}_1. \end{aligned}$$

However, for $z \in L$ we have $(\varphi * \bar{g})(z) = \varphi(z)$ and thus

$$\int_L \varphi n_e \text{ grad } h \, d\mathcal{H}_1 = \int_L \varphi n_i \text{ grad} (h * \bar{g}) \, d\mathcal{H}_1.$$

Since the last equality is valid for each continuously differentiable function φ with compact support in $S^+ \cup L \cup \bar{g}(S^+)$ (which is a neighbourhood of L), we see now that

$$(1.7) \quad \frac{\partial h}{\partial n_e} = \frac{\partial(h * \bar{g})}{\partial n_i} = - \frac{\partial(h * \bar{g})}{\partial n_e}$$

on L (cf. [6], Lemma 1.2).

1.4. Remark. Let h be a harmonic function on a ‘‘symmetric’’ neighbourhood U of L , that is, on such a neighbourhood U that $\bar{g}(z) \in U$ for any $z \in U$. Suppose that $h(\bar{g}(z)) = h(z)$ for $z \in U$ and let n be a normal to L . Then for $\zeta \in L$ the identity

$$\frac{\partial h}{\partial n}(\zeta) = 0$$

holds since by (1.7) we have

$$\frac{\partial}{\partial n} (h * \bar{g}) = - \frac{\partial h}{\partial n}$$

on L and at the same time (by the assumption $h * \bar{g} = h$)

$$\frac{\partial}{\partial n} (h * \bar{g}) = \frac{\partial h}{\partial n}.$$

1.5. Further notation. Let $\langle a, b \rangle$ be a compact interval in R^1 , $\psi: \langle a, b \rangle \rightarrow R^2$ a simple closed path of finite length; we write

$$K = \psi(\langle a, b \rangle).$$

For $z \in R^2$ let $\mathfrak{g}_z = \mathfrak{g}_z^K$ be a single-valued continuous branch of $\arg[\psi - z]$ on $\langle a, b \rangle - \psi^{-1}(z)$. For $0 < r \leq +\infty$ let $\gamma_{z,r}$ be the family of all components of the set

$$\{t \in \langle a, b \rangle; 0 < |\psi(t) - z| < r\},$$

and for $\alpha \in R^1$ let $n_r^\psi(\alpha, z)$ be the number of points in

$$\{t \in \langle a, b \rangle; \psi(t) - z = |\psi(t) - z| e^{i\alpha}, 0 < |\psi(t) - z| < r\}$$

(finite or $+\infty$). It is known (see for example [11]) that $n_r^\psi(\alpha, z)$ as a function of the variable $\alpha \in R^1$ is Lebesgue measurable, and if we put

$$(1.8) \quad v_r^\psi(z) = \int_0^{2\pi} n_r^\psi(\alpha, z) d\alpha$$

then

$$v_r^\psi(z) = \sum_{I \in \gamma_{z,r}} \text{var} [\mathfrak{g}_z; I].$$

For $r = +\infty$ we write $v_{+\infty}^\psi(z) = v^\psi(z)$ and this term is called the cyclic variation of the curve K (or of the path ψ) at the point z .

If $M \subset R^2$ is compact, then, as usual, $\mathcal{C}(M)$ stands for the space of all (real) continuous functions on M endowed with the supremum norm. $\mathcal{C}'(M)$ will denote the space of all signed (finite) Borel measures on R^2 with support contained in M . For $\mu \in \mathcal{C}'(M)$ we put

$$\|\mu\| = |\mu|(M),$$

where $|\mu|$ is the total variation of μ . $\mathcal{C}'(M)$ endowed with this norm is the dual space of $\mathcal{C}(M)$.

Let $z \in R^2$ be such that $v^\psi(z) < \infty$. Then for $f \in \mathcal{C}(K)$ the value of the double layer potential $W_K(z, f)$ is defined by

$$(1.9) \quad W_K(z, f) = \frac{1}{\pi} \sum_{I \in \gamma_{z, \infty}} \int_I f(\psi(t)) d\mathfrak{g}_z(t).$$

Under the assumption $\text{var} [\psi; \langle a, b \rangle] < \infty$ we have $v^\psi(z) < \infty$ for each $z \in R^2 - K$ and

$$W_K(z, f) = \frac{1}{\pi} \text{Im} \int_{\psi} \frac{f(\zeta)}{\zeta - z} d\zeta.$$

Put further $\iota = \iota_K = 1$ if ψ is positively oriented and $\iota = \iota_K = -1$ in the opposite case. It is known (see [13], [11], [12]) that if

$$(1.10) \quad \sup_{z \in K} v^\psi(z) < \infty$$

then for each $f \in \mathcal{C}(K)$, $\zeta \in K$ there exist finite limits

$$(1.11) \quad W_K^i(\zeta, f) = \lim_{\substack{z \rightarrow \zeta \\ z \in \text{Int } K}} W_K(z, f),$$

$$(1.12) \quad W_K^e(\zeta, f) = \lim_{\substack{z \rightarrow \zeta \\ z \in \text{Ext } K}} W_K(z, f)$$

and

$$W_K^i(\zeta, f) - \iota f(\zeta) = W_K^e(\zeta, f) + \iota f(\zeta).$$

If (1.10) is fulfilled then for $f \in \mathcal{C}(K)$, $\zeta \in K$ put

$$(1.13) \quad \bar{W}_K f(\zeta) = W_K^i(\zeta, f) - \iota f(\zeta) = W_K^e(\zeta, f) + \iota f(\zeta).$$

Then $\bar{W}_K f \in \mathcal{C}(K)$ for each $f \in \mathcal{C}(K)$ and \bar{W}_K ($\bar{W}_K: f \mapsto \bar{W}_K f$) is a bounded linear operator acting on $\mathcal{C}(K)$.

Let \mathcal{X} stand for the set of all compact (linear) operators acting on $\mathcal{C}(K)$. Given a linear continuous operator $A: \mathcal{C}(K) \rightarrow \mathcal{C}(K)$ denote

$$\omega A = \inf_{D \in \mathcal{X}} \|A - D\|.$$

The reciprocal value of ωA is called the Fredholm radius of A . It is known that (see [13], [12])

$$(1.14) \quad \omega \bar{W}_K = \frac{1}{\pi} \limsup_{r \rightarrow 0^+} v_r^\psi(\zeta).$$

Further, let \mathcal{D} denote the space of all infinitely differentiable (real) functions with compact supports in R^2 . Given a Borel set $M \subset R^2$ the perimeter $P(M)$ of M is defined by

$$P(M) = \sup_w \int_M \text{div } w \, d\mathcal{H}_2,$$

where $w = (w_1, w_2)$ ranges over all vector-valued functions with components $w_1, w_2 \in \mathcal{D}$ such that $w_1^2 + w_2^2 \leq 1$. It is known that if either $M = \text{Int } K$ or $M = \text{Ext } K$ then

$$P(M) = \text{var} [\psi; \langle a, b \rangle] = \mathcal{H}_1(K).$$

Suppose M is open, and for $z \in R^2$ let $\mathbf{n}^M(z)$ denote the exterior normal in Federer's sense of M at z . We shall need the following form of the divergence theorem (the Gauss-Green theorem):

Suppose that $P(M) < \infty$ and let $w_1, w_2 \in \mathcal{D}$, $w = (w_1, w_2)$. Then

$$\int_{\partial M} w(\zeta) \mathbf{n}^M(\zeta) d\mathcal{H}_1(\zeta) = \int_M \operatorname{div} w(z) d\mathcal{H}_2(z).$$

For $z \in R^2$ define a function h_z on R^2 such that $h_z(z) = +\infty$ and

$$h_z(\zeta) = \frac{1}{\pi} \ln \frac{1}{|\zeta - z|}$$

for $\zeta \in R^2 - \{z\}$. Recall another expression for the double layer potential (cf. [6], (2.39)). Suppose that $\operatorname{var} [\psi; \langle a, b \rangle] < \infty$. Then for $z \in R^2 - K$, $f \in \mathcal{C}(K)$ such that $f = \varphi^f|_K$, where $\varphi^f \in \mathcal{D}$, $z \notin \operatorname{spt} \varphi^f$ we have

$$\begin{aligned} (1.15) \quad W_K(z, f) &= -\iota \int_{\operatorname{Int} K} \operatorname{grad} \varphi^f \operatorname{grad} h_z d\mathcal{H}_2 = \\ &= -\iota \int_K f(\zeta) \mathbf{n}^K(\zeta) \operatorname{grad} h_z(\zeta) d\mathcal{H}_1(\zeta) \end{aligned}$$

if \mathbf{n}^K denotes the exterior normal in Federer's sense of $\operatorname{Int} K$.

Let $M \subset R^2$ be open and let h be a harmonic function on M such that for any bounded open set $G \subset R^2$

$$(1.16) \quad \int_{M \cap G} |\operatorname{grad} h| d\mathcal{H}_2 < \infty.$$

Then the (generalized or weak) normal derivative $N_M h$ of h with respect to M is defined as a functional (a distribution) on \mathcal{D} (see for example [13] or [9]) by

$$(1.17) \quad \langle \varphi, N_M h \rangle = \int_M \operatorname{grad} \varphi \operatorname{grad} h d\mathcal{H}_2, \quad \varphi \in \mathcal{D}.$$

It is known that the support of $N_M h$ is contained in ∂M (the boundary of M).

If μ is a signed (finite) Borel measure with compact support in R^2 then the logarithmic potential U_μ is defined by

$$(1.18) \quad U_\mu(z) = \int_{R^2} h_z(\zeta) d_\mu(\zeta)$$

for all such $z \in R^2$ for which the integral on the right-hand side exists. U_μ is defined at least on $R^2 - \operatorname{spt} \mu$ and is harmonic there.

Let either $M = \operatorname{Int} K$ or $M = \operatorname{Ext} K$. It is known that for any $\mu \in \mathcal{C}'(K)$ the potential $h = U_\mu$ satisfies the condition (1.16) and thus the weak normal derivative $N_M U_\mu$ of U_μ is defined by (1.17). The support of $N_M U_\mu$ is contained in K . Then also the following assertion is valid (see [13] or [9]):

Let either $M = \operatorname{Int} K$ or $M = \operatorname{Ext} K$. Then the distribution $N_M U$ can be represented by a charge $\nu_\mu \in \mathcal{C}'(K)$ for each $\mu \in \mathcal{C}'(K)$ in the sense that

$$\langle \varphi, N_M U_\mu \rangle = \int_K \varphi \, d\nu_\mu$$

for each $\varphi \in \mathcal{D}$ if and only if

$$(1.19) \quad \sup_{\zeta \in K} v^\psi(\zeta) < \infty .$$

If (1.19) is fulfilled then for each $\mu \in \mathcal{C}'(K)$ the charge $\nu_\mu \in \mathcal{C}'(K)$ is uniquely determined and

$$(1.20) \quad \|\nu_\mu\| \leq \left(2 + \frac{1}{\pi} \sup_{\zeta \in K} v^\psi(\zeta) \right) \|\mu\| .$$

If the condition (1.19) is fulfilled then we identify the functional $N_M U_\mu$ with the charge ν_μ . $N_M U$ is then a bounded linear operator acting on $\mathcal{C}'(K)$:

$$N_M U: \mu \mapsto N_M U_\mu = \nu_\mu, \quad N_M U: \mathcal{C}'(K) \rightarrow \mathcal{C}'(K) .$$

The following assertion also holds (see [13] or [9], [12]):

Suppose that (1.19) is fulfilled. Then the operator $N_M U$ is adjoint to $(I - \iota \bar{W})$ (I denotes the identity operator on $\mathcal{C}(K)$) in the case $M = \text{Int } K$ and adjoint to $(I + \iota \bar{W})$ in the case $M = \text{Ext } K$.

2. OPERATOR N_K AND THE NEUMANN PROBLEM

Throughout this part we suppose that L is an analytic Jordan curve having a global reflection function and K a Jordan curve such that

$$K \subset \text{Int } L .$$

We shall always suppose that K, L are related as follows. Let e_1, \dots, e_r be all the critical points with respect to L lying in $\text{Int } L$. If

$$G = \text{Int } K$$

(we keep this notation in the sequel) we suppose that

$$\{e_1, \dots, e_r\} \subset G .$$

Further, let L_i be an arc which joins all the points e_1, \dots, e_r ,

$$L_i \subset G ,$$

and \bar{g} the corresponding reflection function with respect to L . Then \bar{g} is defined and one-to-one on

$$R_g = (\text{Int } L - L_i) \cup L \cup \bar{g}(\text{Int } L - L_i)$$

and g is holomorphic there. If we denote

$$(2.1) \quad \hat{K} = \bar{g}(K)$$

then \hat{K} is also a Jordan curve and it follows from the properties of the reflection function \bar{g} (and the above assumptions) that

$$K \cup L \subset \text{Int } \hat{K}.$$

Further, denote

$$(2.2) \quad S^+ = \text{Int } L \cap \text{Ext } K, \quad S^- = \bar{g}(S^+), \quad S = S^+ \cup L \cup S^-.$$

Then we have

$$\begin{aligned} S^- &= \text{Ext } L \cap \text{Int } \hat{K}, \quad S = \text{Ext } K \cap \text{Int } \hat{K}, \\ \partial S^+ &= K \cup L, \quad \partial S^- = L \cup \hat{K}, \quad \partial S = K \cup \hat{K}. \end{aligned}$$

We will often denote

$$E = \text{Ext } K.$$

Let ψ be a parametrization of the curve K defined on an interval $\langle a, b \rangle$, $\hat{\psi}$ a parametrization of \hat{K} defined on the same interval $\langle a, b \rangle$. Throughout this part we suppose that K with the parametrization ψ is negatively oriented while \hat{K} with the parametrization $\hat{\psi}$ is positively oriented (the orientation plays a role in the definition of the double layer potential).

Since g is holomorphic it is easy to see that if

$$(2.3) \quad \text{var } [\psi; \langle a, b \rangle] < \infty$$

then also

$$(2.4) \quad \text{var } [\hat{\psi}; \langle a, b \rangle] < \infty.$$

Note that there is a connection even between the cyclic variations v^ψ and $v^{\hat{\psi}}$ of K and \hat{K} , respectively (see [5]).

We shall solve the Neumann problem on the region S^+ . The solution will be found in the form

$$U_\mu + U_\mu * \bar{g},$$

where μ is a suitable measure supported by $\partial S^+ = K \cup L$.

First we shall investigate the normal derivative of $(U_\mu + U_\mu * \bar{g})$ with respect to S^+ , where either $\mu \in \mathcal{E}'(K)$ or $\mu \in \mathcal{E}'(L)$. If $\mu \in \mathcal{E}'(K)$ then $|\text{grad } U_\mu|$ is locally integrable on R^2 and the function $U_\mu * \bar{g}$ is harmonic on $R_g - \hat{K}$ and thus its partial derivatives are bounded on S^+ . Now it is seen that the weak normal derivative of $(U_\mu + U_\mu * \bar{g})$ with respect to S^+ can be defined as a functional $N_{S^+}(U_\mu + U_\mu * \bar{g})$ on \mathcal{D} by (cf. (1.17))

$$(2.5) \quad \langle \varphi, N_{S^+}(U_\mu + U_\mu * \bar{g}) \rangle = \int_{S^+} \text{grad } \varphi \text{ grad } (U_\mu + U_\mu * \bar{g}) \, d\mathcal{H}_2,$$

$\varphi \in \mathcal{D}$. In accordance with the preliminary remarks, the support of $N_{S^+}(U_\mu + U_\mu * \bar{g})$ is contained in $\partial S^+ = K \cup L$. Let us notice that if $\mu \in \mathcal{E}'(K)$ then even the support of $N_{S^+}(U_\mu + U_\mu * \bar{g})$ is contained in K . Indeed, let $\varphi \in \mathcal{D}$ be such that $\text{spt } \varphi \cap K = \emptyset$

and let \mathbf{n}^+ stand for the exterior normal with respect to S^+ on L , \mathbf{n}^- for the interior normal with respect to S^+ on L (clearly $\mathbf{n}^- = -\mathbf{n}^+$ and \mathbf{n}^- is at the same time the exterior normal with respect to S^- on L). In virtue of the fact that U_μ is harmonic on $R^2 - K$ and the assumption $\text{spt } \varphi \cap K = \emptyset$ the divergence theorem yields

$$\int_{S^+} \text{grad } \varphi \text{ grad } U_\mu \, d\mathcal{H}_2 = \int_L \varphi \mathbf{n}^+ \text{ grad } U_\mu \, d\mathcal{H}_1.$$

Since $(\varphi * \bar{g})(\zeta) = \varphi(\zeta)$ for $\zeta \in L$, we obtain from Lemma 1.1 (again using the divergence theorem)

$$\begin{aligned} \int_{S^+} \text{grad } \varphi \text{ grad } (U_\mu * \bar{g}) \, d\mathcal{H}_2 &= \int_{S^-} \text{grad } (\varphi * \bar{g}) \text{ grad } U_\mu \, d\mathcal{H}_2 = \\ &= \int_L \varphi * \bar{g} \mathbf{n}^- \text{ grad } U_\mu \, d\mathcal{H}_1 = - \int_L \varphi \mathbf{n}^+ \text{ grad } U_\mu \, d\mathcal{H}_1. \end{aligned}$$

Now we obtain immediately from (2.5) that $\langle \varphi, N_{S^+}(U_\mu + U_\mu * \bar{g}) \rangle = 0$, which means that the support of $N_{S^+}(U_\mu + U_\mu * \bar{g})$ is contained in K .

Note that in fact, we have just proved that normal the derivative of $(U_\mu + U_\mu * \bar{g})$ vanishes on L . But this is clear with regard to Remark 1.4, the fact that $h = (U_\mu + U_\mu * \bar{g})$ is harmonic on S , and $h * \bar{g} = h$.

2.1. Lemma. *Suppose the condition (2.3) is fulfilled. Then the functional $N_{S^+}(U_\mu + U_\mu * \bar{g})$ can be represented by a charge $\nu_\mu \in \mathcal{C}'(K)$ for each $\mu \in \mathcal{C}'(K)$, in the sense that*

$$(2.6) \quad \langle \varphi, N_{S^+}(U_\mu + U_\mu * \bar{g}) \rangle = \int_K \varphi \, d\nu_\mu$$

for any $\varphi \in \mathcal{D}$ if and only if

$$(2.7) \quad \sup_{\zeta \in K} v^\psi(\zeta) < \infty.$$

If (2.7) is fulfilled then the operator \bar{N}_K ,

$$\bar{N}_K: \mu \mapsto \nu_\mu, \quad \bar{N}_K: \mathcal{C}'(K) \rightarrow \mathcal{C}'(K),$$

is a bounded linear operator.

Proof. Denote

$$(2.8) \quad \mathcal{D}_L = \{\varphi \in \mathcal{D}; \text{spt } \varphi \subset R_g, \text{spt } \varphi \cap L = \emptyset\}.$$

Since $\text{spt } N_{S^+}(U_\mu + U_\mu * \bar{g}) \subset K$ for $\mu \in \mathcal{C}'(K)$, we can restrict our considerations to functions $\varphi \in \mathcal{D}_L$. Let $\mu \in \mathcal{C}'(K)$, $\varphi \in \mathcal{D}_L$. Then (assuming (2.3) only)

$$\begin{aligned} \langle \varphi, N_{S^+}(U_\mu * \bar{g}) \rangle &= \int_{S^+} \text{grad } \varphi \text{ grad } (U_\mu * \bar{g}) \, d\mathcal{H}_2 = \\ &= \int_{S^-} \text{grad } (\varphi * \bar{g}) \text{ grad } U_\mu \, d\mathcal{H}_2 = \int_K \varphi * \bar{g} \mathbf{n}^K \text{ grad } U_\mu \, d\mathcal{H}_1, \end{aligned}$$

where $\mathbf{n}^{\hat{K}}$ denotes the exterior normal in Federer's sense with respect to S^- on \hat{K} .

As $\text{dist}(K, \hat{K}) > 0$, it is easily seen that there is a constant $c \in \mathbb{R}^1$ such that (for each $\mu \in \mathcal{C}'(K)$)

$$\sup_{\zeta \in \hat{K}} |\text{grad } U_\mu(\zeta)| \leq c \|\mu\|.$$

If $\varphi \in \mathcal{D}_L$, then certainly $\|\varphi\| = \|\varphi * \bar{g}\|$. Now we see that

$$|\langle \varphi, N_{S^+}(U_\mu * \bar{g}) \rangle| \leq \|\varphi\| c \|\mu\| \mathcal{H}_1(\hat{K}),$$

which means that $N_{S^+}(U_\mu * \bar{g})$ can be always (only under the assumption (2.3)) represented by a charge $v_\mu^1 \in \mathcal{C}'(K)$ in the sense that

$$(2.9) \quad \langle \varphi, N_{S^+}(U_\mu * \bar{g}) \rangle = \int_K \varphi \, dv_\mu^1$$

for each $\varphi \in \mathcal{D}_L$; moreover we have

$$(2.10) \quad \|v_\mu^1\| \leq c \mathcal{H}_1(\hat{K}) \|\mu\|.$$

(Note that if $\hat{v}_\mu^1 = \mathbf{n}^{\hat{K}} \text{grad } U_\mu \mathcal{H}_1|_{\hat{K}}$ then, in the sense of the notation used in [6], $v_\mu^1 \approx \hat{v}_\mu^1$.) Further, it follows from (2.10) that the operator \hat{N}_K ,

$$(2.11) \quad \hat{N}_K: \mu \mapsto v_\mu^1, \quad \hat{N}_K: \mathcal{C}'(K) \rightarrow \mathcal{C}'(K),$$

is a bounded (linear) operator (we shall keep this notation in the sequel).

Now it suffices to note that

$$\begin{aligned} \langle \varphi, N_{S^+}(U_\mu + U_\mu * \bar{g}) \rangle &= \langle \varphi, N_{S^+}U_\mu \rangle + \langle \varphi, N_{S^+}(U_\mu * \bar{g}) \rangle, \\ \langle \varphi, N_{S^+}U_\mu \rangle &= \int_{S^+} \text{grad } \varphi \text{ grad } U_\mu \, d\mathcal{H}_2 = \\ &= \int_E \text{grad } \varphi \text{ grad } U_\mu \, d\mathcal{H}_2 = \langle \varphi, N_EU_\mu \rangle \end{aligned}$$

for $\varphi \in \mathcal{D}_L$ ($E = \text{Ext } K$), and (cf. preliminary remarks) N_EU_μ can be represented by a charge from $\mathcal{C}'(K)$ for each $\mu \in \mathcal{C}'(K)$ if and only if (2.7) is fulfilled. Further, if (2.7) is supposed, the operator N_EU ,

$$N_EU: \mu \mapsto v_\mu^E,$$

where $v_\mu^E \in \mathcal{C}'(K)$ is such that

$$\langle \varphi, N_EU_\mu \rangle = \int_K \varphi \, dv_\mu^E, \quad \varphi \in \mathcal{D},$$

is a bounded linear operator mapping $\mathcal{C}'(K)$ into itself.

In what follows we shall always suppose that the condition (2.7) is fulfilled (with the exception of Lemma 2.4 below) and \bar{N}_K will stand for the operator defined in Lemma 2.1. We shall investigate the equation

$$(2.12) \quad \bar{N}_K \mu = v,$$

where $v \in \mathcal{C}'(K)$ is a given charge, $\mu \in \mathcal{C}'(K)$ unknown. Note that if $\mu \in \mathcal{C}'(K)$ is a solution of (2.12) then, in accordance with the above, the function $(U_\mu + U_\mu * \bar{g})$ can be regarded as a (generalized) solution of the Neumann problem on S^+ with the boundary condition v prescribed on K and with the zero boundary condition on L .

2.2. Lemma. *Suppose that*

$$(2.13) \quad \frac{1}{\pi} \limsup_{r \rightarrow 0^+} v_r^\mu(\zeta) < 1$$

and let $\mu \in \mathcal{C}'(K)$ be such that

$$(2.14) \quad \bar{N}_K \mu = 0.$$

Then the function $(U_\mu + U_\mu * \bar{g})$ is constant on $S = S^+ \cup L \cup S^-$; in particular, U_μ is constant on L .

Proof. It suffices to show that $(U_\mu + U_\mu * \bar{g})$ is constant on S^+ since this function is harmonic on S . If $(U_\mu + U_\mu * \bar{g}) = c$ on S then (in view of $\bar{g}(\zeta) = \zeta$) $U_\mu(\zeta) = \frac{1}{2}c$ for $\zeta \in L$, that is, U_μ is constant on L .

Let $\mu \in \mathcal{C}'(K)$ be a charge for which (2.14) holds. The function $U_\mu * \bar{g}$ is harmonic on $R_g - \hat{K}$. There is a Jordan curve L_1 of the class \mathcal{C}^2 such that $L_1 \subset \text{Int } K$ and if

$$S_{L_1}^+ = \text{Ext } L_1 \cap \text{Int } L$$

then $\text{cl}(S_{L_1}^+) \subset R_g$ (and $\text{cl}(S_{L_1}^+) \subset R_g - \hat{K}$, of course), that is, $U_\mu * \bar{g}$ is harmonic on a neighbourhood of $\text{cl}(S_{L_1}^+)$. Since L is analytic, L_1 of the class \mathcal{C}^2 , it follows from the classical theory concerning the solution of the Neumann problem by means of integral equations that there are $\mu_{L_1} \in \mathcal{C}'(L_1)$, $\mu_L \in \mathcal{C}'(L)$ (and μ_{L_1}, μ_L are even absolutely continuous with respect to \mathcal{H}_1) such that

$$U_{\mu_{L_1}}(z) + U_{\mu_L}(z) = (U_\mu * \bar{g})(z)$$

for $z \in S_{L_1}^+$. By a classical result on the balayage of measures (see [14], pp. 258, 260, Theorem 4.2, Corollary 2) there is a $\mu_K \in \mathcal{C}'(K)$ such that $U_{\mu_{L_1}}(z) = U_{\mu_K}(z)$ for $z \in \text{Ext } K$. In particular, if $z \in S^+$ then

$$U_{\mu_K}(z) + U_{\mu_L}(z) = (U_\mu * \bar{g})(z).$$

Put

$$\mu_0 = \mu + \mu_K + \mu_L.$$

Then $\mu_0 \in \mathcal{C}'(B)$, where $B = \partial S^+ = K \cup L$, and for $z \in S^+$ we have

$$U_{\mu_0}(z) = U_\mu(z) + U_{\mu_K}(z) + U_{\mu_L}(z) = U_\mu(z) + (U_\mu * \bar{g})(z).$$

Thus (as (2.14) is fulfilled)

$$\langle \varphi, N_{S^+} U_{\mu_0} \rangle = \langle \varphi, N_{S^+} (U_\mu + U_\mu * \bar{g}) \rangle = 0$$

for $\varphi \in \mathcal{D}$, that is $N_{S^+} U_{\mu_0} = 0$. Since L is analytic and the condition (2.13) is fulfilled

for K it follows from [9], Lemma 5.8 that U_{μ_0} and thus $(U_\mu + U_\mu * \bar{g})$ is constant on S^+ .

2.3. Lemma. *Suppose the condition (2.13) is fulfilled. Then in $\mathcal{C}'(K)$ there is at most one linearly independent charge μ such that U_μ is constant on L .*

Proof. Let us distinguish two cases: either there is a non-trivial charge $\mu \in \mathcal{C}'(K)$ such that U_μ vanishes on L , or no such charge exists.

a) Suppose that there is no $\mu \in \mathcal{C}'(K)$, $\mu \neq 0$, such that $U_\mu|_L = 0$. Let $\mu_0 \in \mathcal{C}'(K)$ be such that $U_{\mu_0}|_L = 1$ and suppose that there is $\mu \in \mathcal{C}'(K)$ such that $U_\mu|_L = c$, where $c \in \mathbb{R}^1$ is a constant. Putting $\mu_1 = \mu - c\mu_0$ we have $U_{\mu_1}|_L = U_\mu|_L - cU_{\mu_0}|_L = 0$ and by the assumption $\mu_1 = 0$, that is $\mu = c\mu_0$. Note that in this case we did not use the assumption (2.13).

b) Suppose that there is a $\mu_0 \in \mathcal{C}'(K)$, $\mu_0 \neq 0$, such that $U_{\mu_0}|_L = 0$.

First let us show that if $\mu \in \mathcal{C}'(K)$ is such that $U_\mu|_L = 0$ and $\mu(K) = 0$ then necessarily $\mu = 0$. If μ is a charge with a compact support in \mathbb{R}^2 such that $\mu(\mathbb{R}^2) = 0$, then

$$(2.15) \quad \lim_{|z| \rightarrow +\infty} U_\mu(z) = 0.$$

Since $\text{spt } \mu \subset K$, U_μ is harmonic on $\mathbb{R}^2 - K \supset \text{Ext } L$ and continuous on $\overline{\text{Ext } L}$. If $U_\mu|_L = 0$ and (2.15) holds then by the maximum principle U_μ vanishes on $\text{Ext } L$ and thus (as U_μ is harmonic on $\text{Ext } K$) also vanishes on $\text{Ext } K$. Now we see that U_μ is a solution of the Neumann problem on $\text{Ext } K$ with the zero boundary condition. Since the condition (2.13) is fulfilled it follows from [13], Theorem 14.7 that $\mu = 0$.

Thus we see that $\mu_0(K) \neq 0$ and we can suppose that $\mu_0(K) = 1$.

Now let $\mu \in \mathcal{C}'(K)$ be such that $U_\mu|_L = 0$. Let us show that then μ is a multiple of μ_0 . If we put $\mu_1 = \mu - c\mu_0$, where $c = \mu(K)$, then $\mu_1(K) = 0$ and $U_{\mu_1}|_L = 0$. By the preceding case $\mu_1 = 0$, that is $\mu = c\mu_0$.

Finally, we shall show that in this case there is no $\mu \in \mathcal{C}'(K)$ such that $U_\mu|_L = c$, where $c \in \mathbb{R}^1 \setminus \{0\}$ is a constant. Suppose that $U_\mu|_L = c \neq 0$ and put $\mu_1 = \mu - k\mu_0$, where $k = \mu(K)$. Then $\mu_1(K) = 0$ and thus

$$\lim_{|z| \rightarrow +\infty} U_{\mu_1}(z) = 0.$$

Further, we have

$$U_{\mu_1}|_L = U_\mu|_L - kU_{\mu_0}|_L = c \neq 0.$$

Thus U_{μ_1} is a bounded harmonic function on $\text{Ext } L$ such that

$$\lim_{\substack{z \rightarrow y \\ z \in \text{Ext } L}} U_{\mu_1}(z) = c$$

for all $y \in L$. By the maximum principle for harmonic functions in the plane (see [13], Theorem 14.2) we have $U_{\mu_1}(z) = c$ for $z \in \text{Ext } L$ — a contradiction.

Now we shall look for the form of the operator adjoint to \bar{N}_K . We have seen in the proof of Lemma 2.1 that if the condition (2.7) is fulfilled then the operator \bar{N}_K can be written in the form ($E = \text{Ext } K$)

$$(2.16) \quad \bar{N}_K = N_E U + \hat{N}_K.$$

The operator adjoint to $N_E U$ is equal to $(I - \bar{W}_K)$ (cf. 1.5; recall that K is supposed to be negatively oriented). It suffices to find the operator adjoint to \hat{N}_K .

For $\varphi \in \mathcal{D}$ we denote $W_K(z, \varphi) = W_K(z, \varphi|_K)$. Recall that \hat{K} is positively oriented and that for $z \in R^2 - K$, $\varphi \in \mathcal{D}$, $z \notin \text{spt } \varphi$ we have (cf. (1.15))

$$(2.17) \quad W_K(z, \varphi) = - \int_{\text{Int } K} \text{grad } \varphi \text{ grad } h_z \, d\mathcal{H}_2.$$

For $f \in \mathcal{C}(K)$ put $\hat{f} = f * \bar{g}$ and write $f \approx \hat{f}$. Then, of course, $\hat{f} \in \mathcal{C}(\hat{K})$ and the given relation is an isometric isomorphism of the spaces $\mathcal{C}(K)$, $\mathcal{C}(\hat{K})$. Define the operator \tilde{W} on $\mathcal{C}(K)$ by putting

$$(2.18) \quad \tilde{W}f = W_K(\cdot, \hat{f})|_K$$

for $f \in \mathcal{C}(K)$. If $\text{var } [\psi; \langle a, b \rangle] < \infty$, then (as we have noted) $\text{var } [\hat{\psi}; \langle a, b \rangle] < \infty$ and certainly $\tilde{W}f \in \mathcal{C}(K)$ for any $f \in \mathcal{C}(K)$. This means that \tilde{W} can be considered as an operator on $\mathcal{C}(K)$,

$$\tilde{W}: f \mapsto \tilde{W}f, \quad \tilde{W}: \mathcal{C}(K) \rightarrow \mathcal{C}(K).$$

Note that \tilde{W} is linear.

2.4. Lemma. *Suppose the condition (2.3) is fulfilled. Then the operator \tilde{W} is compact and the operators \tilde{W} , $-\hat{N}_K$ are adjoint to each other*

Proof. The fact that \tilde{W} is compact can be proved in the same way as Lemma 2.2 in [6]; we omit the details.

Let us show that \tilde{W} , $-\hat{N}_K$ are adjoint to each other.

Let $\varphi \in \mathcal{D}_L$ and suppose, in addition, that

$$\text{spt } \varphi \cap \text{Ext } L = \emptyset.$$

The function $\hat{\varphi} = \varphi * \bar{g}$ is defined on R_g and $\text{spt } \hat{\varphi} \subset R_g$. If we define $\hat{\varphi}$ on R^2 such that $\hat{\varphi}$ vanishes outside R_g , then, of course, $\hat{\varphi} \in \mathcal{D}_L$ and, in addition,

$$\text{spt } \hat{\varphi} \cap \text{Int } L = \emptyset.$$

By the definition of \hat{N}_K and by (2.17) we now obtain for $\mu \in \mathcal{C}'(K)$

$$\begin{aligned} \langle \varphi, \hat{N}_K \mu \rangle &= \langle \varphi, N_{S^+}(U_\mu * \bar{g}) \rangle = \\ &= \int_{S^+} \text{grad } \varphi \text{ grad } (U_\mu * \bar{g}) \, d\mathcal{H}_2 = \int_{S^-} \text{grad } \hat{\varphi} \text{ grad } U_\mu \, d\mathcal{H}_2 = \end{aligned}$$

$$\begin{aligned}
&= \int_{\text{Int } R} \text{grad } \phi(\zeta) \int_K \text{grad}_\zeta h_\zeta(z) d\mu(z) d\mathcal{H}_2(\zeta) = \\
&= \int_K \int_{\text{Int } R} \text{grad } \phi(\zeta) \text{grad}_\zeta h_\zeta(z) d\mathcal{H}_2(\zeta) d\mu(z) = \\
&= - \int_K W_R(z, \phi) d\mu(z) = \langle -\tilde{W}(\phi|_K), \mu \rangle.
\end{aligned}$$

Thus we see that for each $\mu \in \mathcal{C}'(K)$ and each $f \in \mathcal{C}(K)$ such that $f = \phi|_K$, where $\phi \in \mathcal{D}_L$, $\text{spt } \phi \cap \text{Ext } L = \emptyset$, the equality

$$(2.19) \quad \langle f, -\hat{N}_K \mu \rangle = \langle \tilde{W}f, \mu \rangle$$

is valid. Now let $f \in \mathcal{C}(K)$ be arbitrary. Since $K \subset \text{Int } L \cap R_g$, there are functions $\phi_n \in \mathcal{D}_L$, $\text{spt } \phi_n \cap \text{Ext } L = \emptyset$ such that $f_n = \phi_n|_K \rightarrow f$ uniformly on K . Since $\hat{N}_K \mu \in \mathcal{C}'(K)$, then

$$\langle f_n, \hat{N}_K \mu \rangle \rightarrow \langle f, \hat{N}_K \mu \rangle.$$

Since \tilde{W} is compact (and thus also continuous), we have $\tilde{W}f_n \rightarrow \tilde{W}f$ uniformly on K and hence

$$\langle \tilde{W}f_n, \mu \rangle \rightarrow \langle \tilde{W}f, \mu \rangle.$$

Now we see that (2.19) is valid for any $f \in \mathcal{C}(K)$ and any $\mu \in \mathcal{C}'(K)$, which means that the operators $-\hat{N}_K, \tilde{W}$ are adjoint to each other as required.

As \tilde{W} is compact and the Fredholm radius of \overline{W}_K is known (see (1.14)), the following assertion is valid (ωA for $A: \mathcal{C}(K) \rightarrow \mathcal{C}(K)$ has the same meaning as in 1.5):

2.5. Lemma. *Suppose that the condition (2.7) is fulfilled. Then the operators $(I - (\overline{W}_K + \tilde{W}))$ and \overline{N}_K are adjoint to each other and*

$$(2.20) \quad \omega(\overline{W}_K + \tilde{W}) = \frac{1}{\pi} \lim_{r \rightarrow 0^+} \sup_{\zeta \in K} v_r^\psi(\zeta).$$

2.6. Remark. Recall that for $f \in \mathcal{C}(K)$, $\zeta \in K$ we have (K is negatively oriented)

$$(I - \overline{W}_K)f(\zeta) = -W_K^i(\zeta, f) = - \lim_{\substack{z \rightarrow \zeta \\ z \in G}} W_K(z, f)$$

($G = \text{Int } K$). Since for $f \in \mathcal{C}(K)$, $\zeta \in K$ we have (by the definition)

$$\tilde{W}f(\zeta) = W_K(\zeta, \hat{f}),$$

where $\hat{f} = f * \bar{g} \in \mathcal{C}(\hat{K})$ and $W_K(\cdot, \hat{f})$ is continuous on $R^2 - \hat{K} \supset K$, we can write for $f \in \mathcal{C}(K)$, $\zeta \in K$

$$(2.21) \quad (I - (\overline{W}_K + \tilde{W}))f(\zeta) = - \lim_{\substack{z \rightarrow \zeta \\ z \in G}} (W_K(z, f) + W_K(z, \hat{f})).$$

2.7. Lemma. *Suppose that the condition (2.13) is fulfilled. Then the space of all solutions of the homogeneous equation*

$$(2.22) \quad \bar{N}_K \mu = 0$$

(considered on $\mathcal{C}'(K)$) and also the space of all solutions of the homogeneous equation

$$(2.23) \quad (I - (\bar{W}_K + \tilde{W}))f = 0$$

(considered on $\mathcal{C}(K)$) are of dimension one. The solutions of (2.23) are then just all the constant functions on K .

Proof. Let $f \in \mathcal{C}(K)$ be constant, $f = c$. Then, since K is negatively oriented, we have for $z \in G$

$$W_K(z, f) = \frac{1}{\pi} \int_{\langle a, b \rangle} c \, d\vartheta_z = -2c.$$

Since \hat{K} is positively oriented and, of course, $\hat{f} = f * \bar{g} = c$ on \hat{K} , we analogously obtain that $W_{\hat{K}}(z, \hat{f}) = 2c$ for $z \in \text{Int } \hat{K}$. For $z \in G$ we thus have

$$W_K(z, f) + W_{\hat{K}}(z, \hat{f}) = 0$$

and (2.21) yields

$$(I - (\bar{W}_K + \tilde{W}))f = 0.$$

Hence, the constant functions on K are solutions of (2.23) and we see that the space of all solutions of (2.23) has dimension at least one.

On the other hand, it follows immediately from Lemmas 2.2, 2.3 that under the condition (2.13) the dimension of the space of all solutions of (2.22) is at most one. However, under the condition (2.13) (according to Lemma 2.5) the Fredholm alternatives are valid for the operators $\bar{N}_K, (I - (\bar{W}_K + \tilde{W}))$ and the spaces of all solutions of (2.22), (2.23) have the same dimension, whence the assertion follows.

2.8. Remark. If the condition (2.13) is fulfilled then by Lemma 2.7 there is a $\mu_0 \in \mathcal{C}'(K)$, $\mu_0 \neq 0$ such that $\bar{N}_K \mu_0 = 0$. Then, according to Lemma 2.2, $(U_{\mu_0} + U_{\mu_0} * \bar{g})$ is constant on $S = S^+ \cup L \cup S^-$, in particular, U_{μ_0} is constant on L . Let $\mu \in \mathcal{C}'(K)$ be such that U_μ is constant on L . Then μ is a multiple of μ_0 by Lemma 2.3 and thus $\bar{N}_K \mu = 0$ and, consequently, $(U_\mu + U_\mu * \bar{g})$ is constant on S . Thus we see that, under the assumption (2.13), $\mu \in \mathcal{C}'(K)$ is a solution of the homogeneous equation $\bar{N}_K \mu = 0$ if and only if U_μ is constant on L . In particular:

Let (2.13) be fulfilled, $\mu \in \mathcal{C}'(K)$. If U_μ is constant on L then $(U_\mu + U_\mu * \bar{g})$ is constant on S .

Note that one could now easily prove the following assertion (we omit the proof):

Let h be a harmonic function on a connected neighbourhood of L , let h be constant on L and let the normal derivative of h vanish on L . Then h is a constant function.

Further, let us consider the special case where L is a circle of the form

$$L = \{z \in \mathbb{R}^2; |z - z_0| = r\}$$

($r > 0$). Then the reflection function \bar{g} is of the form

$$\bar{g}(z) = z_0 + \frac{r^2}{z - z_0}$$

and the unique critical point with respect to L is z_0 . Note that h_{z_0} is constant on L . Let K be a Jordan curve such that $K \subset \text{Int } L$, $z_0 \in \text{Int } K$. Since

$$h_{z_0} = U_{\varepsilon_{z_0}},$$

where ε_{z_0} is the Dirac measure concentrated at z_0 , there is a $\mu \in \mathcal{C}'(K)$ such that

$$h_{z_0}(z) = U_\mu(z)$$

for $z \in \text{Ext } K$ ([14], Theorem 4.2, Corollary 2). In particular, U_μ is constant on L . We see that in this case the solution of $\bar{N}_K \mu = 0$ are just multiples of the balayaged Dirac measure ε_{z_0} on K (on $\mathbb{R}^2 - G$). Moreover, the answer to the question whether there is a non-trivial charge $\mu \in \mathcal{C}'(K)$ such that $U_\mu|_L = 0$ does not in this case depend at all on the curve K , but on the radius r . Namely, if $r = 1$, then $h_{z_0}|_L = 0$ and if $r \neq 1$ then $h_{z_0}|_L$ is a non-zero constant.

Also in the general case, when L is not a circle, one can easily show that the condition whether there is a non-trivial charge $\mu \in \mathcal{C}'(K)$ with $U_\mu|_L = 0$ is the property of the curve L only, not of K . It is not clear at first sight if in the general case the solutions of $\bar{N}_K \mu = 0$ can be described analogously to the case of the circle.

2.9. Lemma. *Suppose that the condition (2.13) is fulfilled and let $v \in \mathcal{C}'(K)$. Then the equation*

$$(2.24) \quad \bar{N}_K \mu = v$$

admits a solution $\mu \in \mathcal{C}'(K)$ if and only if $v(K) = 0$. If $v(K) = 0$, $\mu_0 \in \mathcal{C}'(K)$ is a non-trivial solution of the homogeneous equation (2.22) and $\mu \in \mathcal{C}'(K)$ is a solution of (2.24), then all solutions of (2.24) can be written in the form $\mu + c\mu_0$, where $c \in \mathbb{R}^1$.

Proof. It suffices to notice that under the condition (2.13), for (2.24) and for the equation

$$(I - (\bar{W}_K + \tilde{W}))f = g$$

the Fredholm alternatives are valid. The assertion follows from Lemma 2.7.

Now let $\mu \in \mathcal{C}'(L)$. We shall investigate the normal derivative of the function $(U_\mu + U_\mu * \bar{g})$ with respect to S^+ . This derivative is defined, of course, in the same way as above, that is, for $\varphi \in \mathcal{D}$ we put

$$\langle \varphi, N_{S^+}(U_\mu + U_\mu * \bar{g}) \rangle = \int_{S^+} \text{grad } \varphi \text{ grad } (U_\mu + U_\mu * \bar{g}) \, d\mathcal{H}_2.$$

This definition is correct since $\text{grad } U_\mu$ is locally integrable in R^2 , and by Lemma 1.1

$$\int_{S^+} \text{grad } \varphi \text{ grad } (U_\mu * \bar{g}) \, d\mathcal{H}_2 = \int_{S^-} \text{grad } (\varphi * \bar{g}) \text{ grad } U_\mu \, d\mathcal{H}_2,$$

where the equality is valid also in the sense of the existence of the given integrals; the integral on the right-hand side converges, of course.

We also know that the support of the distribution $N_{S^+}(U_\mu + U_\mu * \bar{g})$ is contained in

$$B = \partial S^+ = K \cup L.$$

Now let $\varphi \in \mathcal{D}$ be such that $\text{spt } \varphi \cap K = \emptyset$; we can suppose, in addition, that $\text{spt } \varphi \cap \emptyset$. Further let us suppose, for a while, that $\varphi * \bar{g} = \varphi$ on R_θ and $\text{spt } \varphi \cap \text{Ext } \hat{K} = \emptyset$ (by the above we have $\text{spt } \varphi \cap \hat{K} = \emptyset$). Then (if $\mu \in \mathcal{C}'(L)$)

$$\begin{aligned} \langle \varphi, N_{S^+}(U_\mu + U_\mu * \bar{g}) \rangle &= \\ &= \int_{S^+} \text{grad } \varphi \text{ grad } U_\mu \, d\mathcal{H}_2 + \int_{S^+} \text{grad } \varphi \text{ grad } (U_\mu * \bar{g}) \, d\mathcal{H}_2 = \\ &= \int_{S^+} \text{grad } \varphi \text{ grad } U_\mu \, d\mathcal{H}_2 + \int_{S^-} \text{grad } (\varphi * \bar{g}) \text{ grad } U_\mu \, d\mathcal{H}_2 = \\ &= \int_{\text{Int } L} \text{grad } \varphi \text{ grad } U_\mu \, d\mathcal{H}_2 + \int_{\text{Ext } L} \text{grad } \varphi \text{ grad } U_\mu \, d\mathcal{H}_2 = \\ &= 2 \int_L \varphi \, d\mu = 2\langle \varphi, \mu \rangle \end{aligned}$$

(see for example [13], Theorem 13.34). If $\varphi \in \mathcal{D}$ then for the function $\tilde{\varphi} = \frac{1}{2}(\varphi + \varphi * \bar{g})$ we have $\tilde{\varphi}|_L = \varphi|_L$ and $\tilde{\varphi} * \bar{g} = \tilde{\varphi}$ on R_θ . Now it is seen that the equality (still we have $\mu \in \mathcal{C}'(L)$)

$$(2.25) \quad \langle \varphi, N_{S^+}(U_\mu + U_\mu * \bar{g}) \rangle = 2\langle \varphi, \mu \rangle$$

is valid for any $\varphi \in \mathcal{D}$ with $\text{spt } \varphi \cap K = \emptyset$.

Suppose further that $\text{var } [\psi; \langle a, b \rangle] < \infty$ and let \mathbf{n}^K denote the exterior normal in Federer's sense with respect to S^+ on K . Let $\varphi \in \mathcal{D}$, $\text{spt } \varphi \cap L = \emptyset$. Then ($\mu \in \mathcal{C}'(L)$)

$$\begin{aligned} \langle \varphi, N_{S^+}(U_\mu + U_\mu * \bar{g}) \rangle &= \int_{S^+} \text{grad } \varphi \text{ grad } (U_\mu + U_\mu * \bar{g}) \, d\mathcal{H}_2 = \\ &= \int_K \varphi \mathbf{n}^K \text{ grad } (U_\mu + U_\mu * \bar{g}) \, d\mathcal{H}_1. \end{aligned}$$

Denoting

$$(2.26) \quad \mathbf{v}_\mu^K = \mathbf{n}^K \text{ grad } (U_\mu + U_\mu * \bar{g}) \, \mathcal{H}_1|_K$$

we can write

$$(2.27) \quad \langle \varphi, N_{S^+}(U_\mu + U_\mu * \bar{g}) \rangle = \langle \varphi, \mathbf{v}_\mu^K \rangle$$

for any $\varphi \in \mathcal{D}$ with $\text{spt } \varphi \cap L = \emptyset$. Since each $\varphi \in \mathcal{D}$ can be written in the form $\varphi = \varphi_1 + \varphi_2$, where $\varphi_1, \varphi_2 \in \mathcal{D}$, $\text{spt } \varphi_1 \cap K = \emptyset$, $\text{spt } \varphi_2 \cap L = \emptyset$, it follows from (2.25), (2.27) that for any $\mu \in \mathcal{C}'(L)$ the distribution $N_{S^+}(U_\mu + U_\mu * \bar{g})$ can be represented by a charge from $\mathcal{C}'(B)$ (here we suppose only that $\text{var} [\psi; \langle a, b \rangle] < \infty$). Denoting this charge also by $N_{S^+}(U_\mu + U_\mu * \bar{g})$, we have in addition

$$(2.28) \quad N_{S^+}(U_\mu + U_\mu * \bar{g})|_L = 2\mu,$$

$$(2.29) \quad N_{S^+}(U_\mu + U_\mu * \bar{g})|_K = v_\mu^K,$$

where v_μ^K is defined by (2.26). If $\varphi \in \mathcal{D}$ is such that $\varphi = 1$ on $\overline{S^+}$, then, of course,

$$\langle \varphi, N_{S^+}(U_\mu + U_\mu * \bar{g}) \rangle = 0.$$

This together with (2.28) implies

$$(2.30) \quad N_{S^+}(U_\mu + U_\mu * \bar{g})(K) = -2\mu(L).$$

In what follows let $v_K \in \mathcal{C}'(K)$, $v_L \in \mathcal{C}'(L)$. We shall look for a solution of the Neumann problem on S^+ with the boundary condition v_K on K and the boundary condition v_L on L , that is, for such a function h on S^+ for which

$$\langle \varphi, N_{S^+}h \rangle = \langle \varphi, v_K \rangle + \langle \varphi, v_L \rangle$$

for any $\varphi \in \mathcal{D}$. We shall find the solution in the form

$$(2.31) \quad h = (U_{\mu_K} + U_{\mu_K} * \bar{g}) + (U_{\mu_L} + U_{\mu_L} * \bar{g}),$$

where $\mu_K \in \mathcal{C}'(K)$, $\mu_L \in \mathcal{C}'(L)$ are suitable charges. Since for any $\mu_K \in \mathcal{C}'(K)$ the normal derivative of $(U_{\mu_K} + U_{\mu_K} * \bar{g})$ vanishes on L , then by (2.28) the function h of the form (2.31) satisfies the Neumann condition v_L on L if and only if $\mu_L = \frac{1}{2}v_L$. Thus we shall find the solution in the form

$$(2.32) \quad h = (U_{\mu_K} + U_{\mu_K} * \bar{g}) + \frac{1}{2}(U_{v_L} + U_{v_L} * \bar{g}).$$

The function h of the form (2.32) satisfies the Neumann condition v_K on K if and only if

$$(2.33) \quad \bar{N}_K \mu_K = v_K - \frac{1}{2}N_{S^+}(U_{v_L} + U_{v_L} * \bar{g})|_K.$$

By Lemma 2.9 the equation (2.33) has a solution $\mu_K \in \mathcal{C}'(K)$ if and only if ((2.13) is supposed)

$$v_K(K) - \frac{1}{2}N_{S^+}(U_{v_L} + U_{v_L} * \bar{g})(K) = 0.$$

According to (2.30) this condition can be written in the form

$$(2.34) \quad v_K(K) + v_L(L) = 0.$$

If (2.34) is fulfilled then all solutions of (2.33) are of the form $\mu_K = \mu_1 + c\mu_0$, where μ_1 is a fixed solution of (2.33), μ_0 a non-trivial solution of the equation $\bar{N}_K \mu = 0$. The function $(U_{\mu_0} + U_{\mu_0} * \bar{g})$ is constant on S^+ (constant even on S)

by Lemma 2.2. The solution of the form (2.32) (of the Neumann problem) is thus determined uniquely up to an additive constant. If there is a non-trivial charge $v \in \mathcal{C}'(K)$ such that $U_v|_L = 0$ then $(U_{\mu_0} + U_{\mu_0} * \bar{g}) = 0$ on S^+ for any $\mu_0 \in \mathcal{C}'(K)$ such that $\bar{N}_K \mu_0 = 0$ (see the proof of Lemma 2.3). Hence in this case the solution of the form (2.32) is even uniquely determined (apart from the fact that u_K is not unique).

We have just proved the following assertion.

2.10. Theorem. *Suppose the condition (2.13) is fulfilled and let $v_K \in \mathcal{C}'(K)$, $v_L \in \mathcal{C}'(L)$. Then the Neumann problem on S^+ with the boundary conditions v_K on K and v_L on L has a solution of the form (2.32) if and only if (2.34) is valid. If the condition (2.34) is fulfilled then $\mu_K \in \mathcal{C}'(K)$ in (2.32) is determined by the equation (2.33) and the solution of the form (2.32) is determined uniquely up to an additive constant. If, in addition, there is a $v \in \mathcal{C}'(K)$, $v \neq 0$ such that $U_v|_L = 0$, then the solution of the form (2.32) is determined uniquely.*

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Souhrn

REFLEXE A NEUMANNOVA ÚLOHA
NA DVOJNÁSOBNĚ SOUVISLÝCH OBLASTECH

EVA DONTOVÁ

Tento článek je pokračováním článku „Reflexe a Dirichletova úloha na dvojnásobně souvislých oblastech“. Podobně jako v předchozím článku se zde ukazuje, že užitím reflexní funkce lze soustavu dvou integrálních rovnic odpovídajících Neumannově úloze redukovat na jedinou integrální rovnici.

Резюме

РЕФЛЕКСИЯ И ЗАДАЧА НЕЙМАНА ДЛЯ ДВУСВЯЗНЫХ ОБЛАСТЕЙ

EVA DONTOVÁ

Статья является продолжением статьи „Рефлексия и задача Дирихле для двусвязных областей“. Аналогично предыдущей статье показывается, что при помощи рефлексной функции соответствующая задаче Неймана система двух интегральных уравнений превращается в одно интегральное уравнение.

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