Časopis pro pěstování matematiky

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Časopis pro pěstování matematiky, Vol. 112 (1987), No. 3, 308--311

Persistent URL: http://dml.cz/dmlcz/118312

112 (1987) ČASOPIS PRO PĚSTOVÁNÍ MATEMATIKY No. 3, 308—311

THE CHROMATIC NUMBER OF EXTENDED ODD GRAPHS IS FOUR

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(Received March 23, 1985)

Summary. The result is obtained using isomorphism between the extended odd graphs (defined by Mulder in [2]) and hypercubes of even dimensions with diagonals.

Keywords: chromatic number, cube-like graphs, extended odd graph, graph, halfcube, n-dimensional cube, n-dimensional cube with diagonals.

The *extended odd graphs* were introduced by Mulder [2] as follows: for $k \geqq 2$, the extended odd graph E_k has $\{A \subseteq \{1, \ldots, 2k - 1\};\ |A| \leqq k - 1\}$ as its vertex set, and two vertices A and B are joined by an edge whenever $|A \triangle B| = 1$ or $|A \triangle B| = 2k - 2$. The small extended odd graphs are the complete graph $K_4(E_2)$ and the Greenwood-Gleason graph (E_3).

Mulder showed that the graph E_k is regular of degree $2k - 1$, is distance-transitive, and the smallest odd circuit in E_k has the length $2k - 1$.

The aim of the present note is to prove

Theorem. *For* $k \geqq 2$, $\chi(E_k) = 4$.

Here $\chi(G)$ denotes as usual the chromatic number of G. We shall use $V(G)$ and $E(G)$ to denote the vertex set and the edge set of G, respectively. When dealing with colourings of G, we shall mean the well-known regular colourings, i.e. mappings of $V(G)$ into integers which assign different values to vertices u, v whenever they are adjacent.

In order to prove the theorem we shall establish an isomorphism between the extended odd graphs and graphs arising from the n-dimensional cubes by adding certain new edges. As usual, we denote the graph of the *n-dimensional cube* $(n \geqq 1)$ by Q_n; then $V(Q_n) = \{A \subseteq \{1, \ldots, n\}\}$ and for $A, B \in V(Q_n)$, $(A, B) \in E(Q_n)$ iff $|A \triangle B| = 1$. If $A \in V(Q_n)$, then $A' = \{1, \ldots, n\} - A$ will be called the *opposite vertex* to A in Q_n. Let $n \geqq 2$; the *n-dimensional cube with diagonals* Q_n^d arises from Q_n by adding 2^{n-1} new edges — called *diagonals* — each of which joins a pair of opposite vertices in Q_n. Thus $V(Q_n^d) = V(Q_n)$ and for $A, B \in V(Q_n^d)$, $(A, B) \in E(Q_n^d)$ iff $|A \triangle B| = 1$ or $|A \triangle B| = n$. Cubes with diagonals are a particular case of Lovász' *cube-like graphs* (cf. Harary [1]). The small cubes with diagonals are $K_4(Q_2^d)$ and $K_{4,4}(Q_3^d)$. Clearly, Q_{2k+1}^d is bipartite for $k \geqq 1$ (in fact, Q_{2k+1}^d is isomorphic to the so called *halfcube* $\frac{1}{2}Q_{2k+2}^d$, see [2]).

Further, Q^d_{2k-2} is isomorphic to E_k, $k \geqq 2$. (It is easy to verify that a mapping $f: V(E_k) \to V(Q^d_{2k-2})$, $f(A) = A$ if $2k - 1 \notin A$, and $f(A) = \{1, \ldots, 2k - 1\} - A$ if $2k - 1 \in A$, is an isomorphism.) Hence, we have to prove

(*) *for* $k \geqq 1$, $\chi(Q^d_{2k}) = 4$.

Since Q^d_{2k} contains odd circuits, $\chi(Q^d_{2k}) > 2$. On the other hand, it is not difficult to show that Q^d_{2k} is 4-colourable. In order to do it choose $i \in \{1, \ldots, 2k\}$ and put $V^+ = \{A \in V(Q^d_{2k});\ i \in A\}$, $V^- = \{A \in V(Q^d_{2k});\ i \notin A\}$. Then $V(Q^d_{2k}) = V^+ \cup V^-$, $V^+ \cap V^- = \emptyset$. Further, the subgraphs of Q^d_{2k} induced by V^+ and V^- are isomorphic to Q_{2k-1}, hence bipartite. Thus Q^d_{2k} can be coloured by 4 colours (one uses the colours 1, 2 for vertices in V^+ and the colours 3, 4 for those in V^-). Hence, to prove the theorem it is sufficient to show that

(**) *for* $k \geqq 1$, $\chi(Q^d_{2k}) > 3$.

Let c be a colouring of Q_n ($n \geqq 2$). We say that c fulfils the *condition of opposite vertices* – and write $O(Q_n, c)$ – if there are $A, A', B, B' \in V(Q_n)$ such that $(A, B) \in E(Q_n)$, A' is opposite to A, B' is opposite to B (hence also $(A', B') \in E(Q_n)$), and $c(A) = c(B')$, $c(A') = c(B)$. For example, if c is a 2-colouring of Q_n, then $O(Q_n, c)$ holds iff n is odd.

Proposition 1. *Let* $n \geqq 3$; *if there is a* 3-*colouring* c *of* Q_n *for which* $O(Q_n, c)$ *does not hold, then there is a* 3-*colouring of* Q^d_{n-1}.

Proof. Notice first that Q^d_{n-1} ($n \geqq 3$) is isomorphic to the graph G_n defined in the following way: $V(G_n) = \{(A, A');\ A, A' \in V(Q_n),\ A' \text{ is opposite to } A\}$; (A, A') and (B, B') are adjacent in G_n whenever $(A, B) \in E(Q_n)$ or $(A, B') \in E(Q_n)$ (cf. [2], p. 122).

Let c be a 3-colouring of Q_n, and assume $O(Q_n, c)$ does not hold. Define a mapping $\bar{c}: V(G_n) \to \{1, 2, 3\}$ as follows:

$$\begin{aligned} \bar{c}((A, A')) = 1 \quad &\text{if} \quad \{c(A), c(A')\} = \{1, 2\} \quad \text{or} \quad c(A) = c(A') = 1\,, \\ 2 \quad &\text{if} \quad \{c(A), c(A')\} = \{2, 3\} \quad \text{or} \quad c(A) = c(A') = 2\,, \\ 3 \quad &\text{if} \quad \{c(A), c(A')\} = \{1, 3\} \quad \text{or} \quad c(A) = c(A') = 3\,. \end{aligned}$$

We are going to show that $\bar{c}$ is a colouring of G_n. Suppose on the contrary that for some $(A, A'), (B, B')$ from $V(G_n)$ which are adjacent in G_n, $\bar{c}((A, A')) = \bar{c}((B, B'))$. Without loss of generality, let $\bar{c}((A, A')) = 1$, $c(A) = 1$ and $(A, B) \in E(Q_n)$. Since $(A, B) \in E(Q_n)$, we have $(A', B') \in E(Q_n)$ as well. Either $c(B) = 1$ or $c(B') = 1$, hence $c(A') \neq 1$, therefore $c(A') = 2$. This yields $c(B) = 2$, $c(B') = 1$, which means $O(Q_n, c)$ and the contradiction proves the proposition.

Proposition 2. *Let* $n \geqq 1$, *suppose that for every* 3-*colouring* c *of* Q_n, $O(Q_n, c)$ *holds. Then there is no* 3-*colouring of* Q^d_{n+1}.

Proof. Assume the contrary, let $\bar{c}$ be a 3-colouring of Q^d_{n+1}. In a similar manner as above when proving $\chi(Q^d_{2k}) \leqq 4$, choose $i \in \{1, \ldots, n+1\}$ and put $V^+ = \{A \in V(Q^d_{n+1});\ i \in A\}$, $V^- = \{A \in V(Q^d_{n+1});\ i \notin A\}$. The subgraphs induced in Q^d_{n+1} by V^+ and V^- are isomorphic to Q_n; denote them by Q^+_n and Q^-_n, respectively. Let $\bar{c}^+$ be the colouring of V^+ induced by $\bar{c}$ on V^+. We assume that $O(Q^+_n, c)$ for any colouring c of Q^+_n; hence there exist $A, B, A', B' \in V(Q^+_n)$ such that $(A, B) \in E(Q^+_n)$, A' is opposite to A in Q^+_n, B' is opposite to B in Q^+_n, and $\bar{c}^+(A) = \bar{c}^+(B') \neq \bar{c}^+(B) = \bar{c}^+(A')$. Denote by A'' and B'' the vertex opposite to A and B, respectively, in Q_{n+1}. Consider the subgraph of Q^d_{n+1} induced by $\{A, A', A'', B, B', B''\}$. A'' is adjacent to both A and A', B'' is adjacent to both B and B'. Consequently, $\bar{c}(A'') = \bar{c}(B'')$ which contradicts $(A'', B'') \in E(Q^d_{n+1})$.

Proposition 3. *For $n \geqq 2$, if $\chi(Q^d_n) > 3$, then $\chi(Q^d_{n+2}) > 3$.*

Proof. Use Propositions 2 and 1. From $\chi(Q^d_{n+2}) \leqq 3$ it would follow that there is a 3-colouring c of Q_{n+1} such that $O(Q_{n+1}, c)$ does not hold, hence $\chi(Q^d_n) \leqq 3$.

Proof of Theorem. Since Q^d_2 is K_4 and therefore $\chi(Q^d_2) = 4$, Proposition 3 proves $(**)$ from which the theorem follows.

Remark: Proposition 1 and $(**)$ immediately imply that for every 3-colouring c of Q_{2k+1} $(k \geqq 1)$, $O(Q_{2k+1}, c)$ holds.

Acknowledgement. The author wishes to express her thank to J. Matoušek for formulating the problem and to Dr. I. Havel for his help in preparing the manuscript.

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Souhrn

CHROMATICKÉ ČÍSLO ROZŠÍŘENÝCH LICHÝCH GRAFŮ JE ČTYŘI

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Dokazuje se (pomocí tzv. krychlí s diagonálami), že chromatické číslo rozšířených lichých grafů definovaných Mulderem v [2] je 4.

Резюме

ХРОМАТИЧЕСКОЕ ЧИСЛО РАСШИРЕННЫХ НЕЧЕТНЫХ ГРАФОВ РАВНО ЧЕТЫРЕМ

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Доказывается (с помощью т.н. кубов с диагоналями), что хроматическое число расширенных нечетных графов, определенных в [2], равно 4.

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