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AN APPLICATION OF l-CONDITION IN THE THEORY OF STOCHASTIC DIFFERENTIAL EQUATIONS

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Summary. Stability of solutions of stochastic differential equations in the space of probability measures (distributions) is investigated. A modification of A. Lasota's "l-condition" is used to show (under suitable assumptions) asymptotical stability in the total variation topology and in the time-homogeneous case also the existence of an unique invariant measure.

Keywords: stochastic differential equations, stability, invariant measure.

AMS Classification: 60H10.

INTRODUCTION

In this paper we deal with some ergodic properties of solutions of stochastic differential equations. We slightly improve some earlier results of R. Z. Khasminskii contained in [4] (cf. Remark 3.8) and of A. M. Il'iin and R. Z. Khasminskii contained in [5] (cf. Remark 3.6). The method of proofs used here is based on the "l-condition" for a family of Markov operators which was introduced by A. Lasota in [1].

The paper is divided into three sections. Section 1 contains the basic definitions and notations. In Section 2 we give theorems (2.1 and 2.2) which are modifications of the "l-condition" to a space of measures. Their proofs are similar to the original proof in [1] and we give them mainly for the convenience of the reader. Section 3 contains the main results of the paper (Theorem 3.1, Corollaries 3.4 and 3.7) – the applications of Theorems 2.1 and 2.2 in stochastic differential equations problems.

Fundamental statements from the theory of stochastic differential equations used in this paper can be found for instance in [6].

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1. DEFINITIONS AND NOTATIONS

Let $(X, \mathscr{B})$ be a measurable space and denote by by $\mathscr{L}$, $\mathscr{K}$, $\mathscr{P}$, respectively, the sets of finite real, finite nonnegative and probabilistic measures defined on $\mathscr{B}$.

Definition 1.1. A linear mapping $S: \mathscr{L} \to \mathscr{L}$ will be called a *Markov operator*, if it satisfies the conditions

$$S(\mathscr{K}) \subset \mathscr{K} \tag{1.1}$$

and

$$S\,v(X) = v(X) \quad \text{for} \quad v \in \mathscr{K}\,. \tag{1.2}$$

In this paper we deal with a two-parameter family of Markov operators $\{S_{s,t}\}$, $s \in \langle 0, \infty)$, $t \in \langle s, \infty)$, satisfying

$$S_{u,t} \circ S_{s,u} = S_{s,t} \tag{1.3}$$

for all $s \leqq u \leqq t$, and with a one-parameter family of Markov operators $\{S_t\}$, $t \in \langle 0, \infty)$, satisfying

$$S_{t_1} \circ S_{t_2} = S_{t_1 + t_2} \tag{1.4}$$

for all $t_1 \geqq 0$, $t_2 \geqq 0$ (which is the "time homogeneous" case).

For $v \in \mathscr{L}$ we denote by v^+, v^-, var v, respectively, the positive, negative and total variation of the measure v. We set $\|v\| = \text{var}\, v(X)$ and $R_+ = \langle 0, \infty)$.

Definition 1.2. Let a family of Markov operators $\{S_{s,t}\}$ satisfying (1.3) be given. A system $\{\mu\}_{s \in R_+}$, $\mu_s \in \mathscr{K}$, will be called a *nontrivial system of lower measures with respect to* $\{S_{s,t}\}$, if

$$\|(S_{s,t} - \mu_s)^-\| \to 0\,, \quad t \to \infty\,,$$

is fulfilled for all $v \in \mathscr{P}$, and $\mu_s(X) \geqq \alpha$ holds for some $\alpha > 0$ independent of $s \geqq 0$.

Definition 1.3. Let a family of Markov operators $\{S_t\}$ satisfying (1.4) be given. A measure $\mu \in \mathscr{K}$ will be called a *lower measure with respect to* $\{S_t\}$, if

$$\|(S_t v - \mu)^-\| \to 0\,, \quad t \to \infty\,,$$

holds for all $v \in \mathscr{P}$. If, moreover, $\mu(X) > 0$ holds, then the lower measure μ will be called nontrivial.

A measure $\mu^* \in \mathscr{L}$ will be called *invariant* (with respect to $\{S_t\}$), if $S_t \mu^* = \mu^*$ holds for every $t \geqq 0$.

2. THE l-CONDITION IN THE SPACE OF MEASURES

In this section we give the following two theorems which slightly modify the l-condition (cf. [1]):

Theorem 2.1. *Let a family of Markov operators $\{S_{s,t}\}$ satisfying (1.3) be given. Assume that there exists a nontrivial system $\{\mu_s\}_{s \in R_+}$ of lower measures with respect to $\{S_{s,t}\}$. Then*

$$\|S_{s,t}\nu_1 - S_{s,t}\nu_2\| \to 0\,, \quad t \to \infty\,,$$

holds for every $\nu_1 \in \mathscr{P}$, $\nu_2 \in \mathscr{P}$, $s \geqq 0$.

Theorem 2.2. *Let a family of Markov operators* $\{S_t\}$ *satisfying* (1.4) *be given. Then there exists an invariant measure* $\mu^* \in \mathscr{P}$ *and*

$$\|S_t\nu - \mu^*\| \to 0\,, \quad t \to \infty\,, \tag{2.1}$$

holds for every $\nu \in \mathscr{P}$, *if and only if there exists a nontrival lower measure* μ *with respect to* $\{S_t\}$.

Before proving the theorems we give (without proof) a simple lemma:

Lemma 2.3. *Let* S *be a Markov operator. Then*

$$\|S\nu\| \leqq \|\nu\| \tag{2.2}$$

and

$$\|(S\nu)^-\| \leqq S\,\nu^-(X) \tag{2.3}$$

holds for every $\nu \in \mathscr{L}$.

Proof of Theorem 2.1. Let $\nu_1 \in \mathscr{P}$, $\nu_2 \in \mathscr{P}$, $s \geqq 0$ and write

$$\nu = \nu_1 - \nu_2\,, \quad \eta = \nu^+(X) = \nu^-(X) = \tfrac{1}{2}\|\nu\|\,.$$

For every $t \geqq \lambda \geqq s \geqq 0$ we have

$$\|S_{s,t}\nu\| \leqq \|S_{s,\lambda}\nu\| \tag{2.4}$$

which follows from (1.3) and (2.2). Thus, if $\eta = 0$, the proof is complete. Suppose $\eta > 0$. For $t \geqq s$ we have

$$\|S_{s,t}\nu\| = \eta\left\|\left(S_{s,t}\frac{1}{\eta}\,\nu^+ - \mu_s\right) - \left(S_{s,t}\frac{1}{\eta}\,\nu^- - \mu_s\right)\right\|. \tag{2.5}$$

Noting that $(1/\eta)\,\nu^+ \in \mathscr{P}$, $(1/\eta)\,\nu^- \in \mathscr{P}$, we get

$$\left\|\left(S_{s,t_1}\frac{1}{\eta}\,\nu^+ - \mu_s\right)^-\right\| \leqq \tfrac{1}{4}\,\mu_s(X)\,,$$

$$\left\|\left(S_{s,t_1}\frac{1}{\eta}\,\nu^- - \mu_s\right)^-\right\| \leqq \tfrac{1}{4}\mu_s(X)$$

for some $t_1 > s$. This implies that

$$\left\|S_{s,t_1}\frac{1}{\eta}\,\nu^+ - \mu_s\right\| \leqq S_{s,t_1}\frac{1}{\eta}\,\nu^+(X) - \mu_s(X) + 2\left\|\left(S_{s,t_1}\frac{1}{\eta}\,\nu^+ - \mu_s\right)^-\right\| \leqq$$
$$\leqq 1 - \mu_s(X) + \tfrac{1}{2}\,\mu_s(X) = 1 - \tfrac{1}{2}\,\mu_s(X)\,.$$

Similarly, we get

$$\left\| S_{s,t_1} \frac{1}{\eta} v^- - \mu_s \right\| \leqq 1 - \tfrac{1}{2} \mu_s(X) ,$$

and so, by (2.5),

$$\|S_{s,t_1} v\| \leqq \eta(2 - \mu_s(X)) \leqq \left(1 - \frac{\alpha}{2}\right) \|v\| .$$

Repeating the same arguments for $S_{s,t_1}v$ instead of v we find $t_2 > t_1$ such that

$$\|S_{s,t_2} v\| = \|S_{t_1,t_2}(S_{s,t_1} v)\| \leqq (1 - \tfrac{1}{2} \mu_{t_1}(X)) \|S_{s,t_1} v\| \leqq \left(1 - \frac{\alpha}{2}\right)^2 \|v\| ,$$

and by induction we find an increasing sequence t_n such that

$$\|S_{s,t_n} v\| \leqq \left(1 - \frac{\alpha}{2}\right)^n \|v\|$$

holds. Thus we have $\|S_{s,t_n} v\| \to 0$ for $n \to \infty$ which together with (2.4) completes the proof.

Proof of Theorem 2.2. If there exists an invariant measure $\mu^* \in \mathscr{P}$ satisfying (2.1), then it is clearly a nontrivial lower measure with respect to $\{S_t\}$.

Conversely, Theorem 2.1 implies that it suffices to prove the existence of an invariant measure $\mu^* \in \mathscr{P}$. Denote by $\mathscr{S}$ the set of all lower measures with respect to $\{S_t\}$. We define the partial ordering $\leqq$ on $\mathscr{S}$ by a natural way,

$$v_1 \leqq v_2 \Leftrightarrow v_1(A) \leqq v_2(A) \quad \text{for every} \quad A \in \mathscr{B} .$$

We will show that in $\mathscr{S}$ there exists the greatest element with respect to the ordering $\leqq$. For this purpose we show:

α) $(\mathscr{S}, \leqq)$ is a directed set, i.e., for every couple $(\mu_1, \mu_2) \in \mathscr{S} \times \mathscr{S}$ we find $\mu_3 \in \mathscr{S}$ such that $\mu_1 \leqq \mu_3$, $\mu_2 \leqq \mu_3$. Indeed, we define

$$\mu_3(A) = \tfrac{1}{2}\{\operatorname{var}(\mu_1 - \mu_2)(A) + \mu_1(A) + \mu_2(A)\} , \quad A \in \mathscr{B} ,$$

and we have

$$\mu_3(A) \geqq \tfrac{1}{2}\{|\mu_1(A) - \mu_2(A)| + \mu_1(A) + \mu_2(A)\} = \max(\mu_1(A), \mu_2(A)) .$$

Furthermore, for every $v \in \mathscr{P}$

$$\|(S_t v - \mu_3)^-\| \leqq \|(S_t v - \mu_1)^-\| + \|(S_t v - \mu_2)^-\|$$

holds, and thus, $\mu_3 \in \mathscr{S}$.

β) In $\mathscr{S}$ there exists a maximal element with respect to $\leqq$. Let $\mathscr{R} \subset \mathscr{S}$ be a nonempty chain. Take a nondecreasing sequence $\mu_n \in \mathscr{R}$ such that

$$\mu_n(X) \nearrow s = \sup\{v(X), v \in \mathscr{R}\}$$

holds (clearly, $s \leqq 1$). Set

$$\bar{\mu}(A) = \lim \mu_n(A) , \quad A \in \mathscr{B} .$$

It can be easily seen that $\bar{\mu}$ is a measure on $\mathscr{B}$, $\mu \in \mathscr{K}$ and $\nu \leqq \bar{\mu}$ for every $\nu \in \mathscr{R}$. For every $\psi \in \mathscr{P}$ we have

$$\|(S_t\psi - \bar{\mu})^-\| \leqq \|(S_t\psi - \mu_n)^-\| + \|\bar{\mu} - \mu_n\| = \|(S_t\psi - \mu_n)^-\| + \bar{\mu}(X) - \mu_n(X).$$

It follows that $\bar{\mu} \in \mathscr{S}$ and we get $\beta)$ by Zorn's lemma. Now, $\alpha)$ and $\beta)$ imply that in $\mathscr{S}$ there exists the greatest element which we denote by $\hat{\mu}$. It is obvious that $\hat{\mu}(X) > 0$. From (2.3) and (2.2) we get for every $\nu \in \mathscr{P}$, $t_0 > 0$, $t > t_0$,

$$\|(S_t\nu - S_{t_0}\hat{\mu})^-\| = \|[S_{t_0}(S_{t-t_0}\nu - \hat{\mu})]^-\| \leqq$$
$$\leqq S_{t_0}[(S_{t-t_0}\nu - \hat{\mu})^-](X) \leqq \|(S_{t-t_0}\nu - \hat{\mu})^-\|.$$

Thus, $S_{t_0}\hat{\mu} \in \mathscr{S}$ holds for every $t_0 > 0$ which implies $S_{t_0}\hat{\mu} \leqq \hat{\mu}$. Hence, taking into account (1.2), we have $S_{t_0}\hat{\mu} = \hat{\mu}$ and so, the measure

$$\mu^* = \frac{1}{\hat{\mu}(X)}\,\hat{\mu}$$

is the invariant measure belonging to $\mathscr{P}$. The proof is complete.

3. APPLICATIONS OF THEOREMS 2.1 AND 2.2

In this section we consider the n-dimensional differential equation

(3.1) $$\mathrm{d}\zeta_t = b(t, \zeta_t)\,\mathrm{d}t + \sigma(t, \zeta_t)\,\mathrm{d}w_t,$$

where w_t is an l-dimensional Wiener process, $b = (b_i)$ is an n-dimensional vector and $\sigma = (\sigma_{ij})$ is a matrix $n \times l$, b and σ are defined on $\mathbb{R}_+ \times \mathbb{R}_n$. For $Q \subset \mathbb{R}_+ \times \mathbb{R}_n$ we denote by $C_{1,2}(Q)$ the set of functions defined on Q whose first time derivatives and first and second space derivatives exist and are continuous on Q. Furthermore, we denote by L the infinitesimal operator connected with (3.1), i.e., for $V \in C_{1,2}(Q)$ we have

$$LV(t, x) = \frac{\partial V}{\partial t} + \sum_i b_i(t, x)\frac{\partial V}{\partial x_i} + \tfrac{1}{2}\sum_{i,j} a_{ij}(t, x)\frac{\partial^2 V}{\partial x_i\,\partial x_j}, \quad (t, x) \in Q,$$

where $(a_{ij}(t, x))_{i,j=1,2,\ldots,n} = \sigma(t, x)\,\sigma^{\mathrm{T}}(t, x)$ (M^{T} stands for the transposed matrix to M). We assume that b and σ are continuous, locally Lipschitz continuous in x and for some functions $W \in C_{1,2}(\mathbb{R}_+ \times \mathbb{R}_n)$, $W \geqq 0$, and β: $\langle 0, \infty) \to \langle 0, \infty)$, β non-decreasing and continuously differentiable, the following holds:

(3.2a) $$LW(t, x) \leqq c\beta(W(t, x)) \quad \text{for some} \quad c > 0 \quad \text{and all} \quad (t, x),$$

(3.2b) $$\lim_{|x| \to \infty}\ \inf_{0 \leqq t \leqq T} W(t, x) = \infty \quad \text{for every} \quad T > 0,$$

(3.2c) $$\int_0^\infty \frac{\mathrm{d}u}{1 + \beta(u)} = \infty.$$

In particular, for $\beta(x) = x$ and $W(t, x) = |x|^2 + 1$ the conditions (3.2a)–(3.2c) have the following well-known simple form

(3.2d) $\quad |b(t, x)| + |\sigma(t, x)| \leqq K(1 + |x|)$ for some $K > 0$ and all (t, x).

The conditions (3.2a)–(3.2c) guarantee the existence and uniqueness of the strong solution of (3.1) (cf. [2]), which is a Markov process whose transition function we denote by $P(s, x, t, A)$, $0 \leqq s \leqq t < \infty$, $x \in \mathbb{R}_n$, A- a Borel set. In this section we set $X = \mathbb{R}_n$, $\mathscr{B}$ is the σ-algebra of Borel sets on X and the family $\{S_{s,t}\}$ is given by the equality

$$S_{s,t}\, \nu(A) = \int_X P(s, x, t, A)\, \nu(\mathrm{d}x)\,, \quad \nu \in \mathscr{L}\,, \quad A \in \mathscr{B}\,. \tag{3.3}$$

It is obvious that $\{S_{s,t}\}$ is a family of Markov operators satisfying (1.3). We will impose the following "nondegeneracy condition" (3.4):

(3.4) Let there exist a bounded nonempty region U such that

$$\sum_{i,j} a_{ij}(t, x)\, \lambda_i \lambda_j \geqq \alpha |\lambda|^2 \tag{3.4a}$$

holds for some $\alpha > 0$ and all $\lambda = (\lambda_i) \in \mathbb{R}_n$, $(t, x) \in \mathbb{R}_+ \times U$,

$$|b_i(t_1, x_1)| + |a_{ij}(t_1, x_1)| \leqq K \tag{3.4b}$$

and

$$|b_i(t_1, x_1) - b_i(t_2, x_2)| + |a_{ij}(t_1, x_1) - a_{ij}(t_2, x_2)| \leqq \leqq K(|x_1 - x_2|^\beta + |t_1 - t_2|^{\beta/2}) \tag{3.4c}$$

holds for some $K > 0$, $\beta > 0$ and all $t_1, t_2 \in \mathbb{R}_+$, $x_1, x_2 \in U$.

Theorem 3.1. *Assume that* (3.4) *is fulfilled and that there exists a nonempty region* W, $\overline{W} \subset U$, *and* $\lambda > 0$ *such that*

$$P(s, x, t, W) \geqq \lambda \tag{3.5}$$

holds for all $x \in X$, $s \geqq 0$ *and* $t \geqq t_0(s, x)$. *Then*

$$\|S_{s,t}\nu_1 - S_{s,t}\nu_2\| \to 0\,, \quad t \to \infty$$

holds for all measures $\nu_1 \in \mathscr{P}$, $\nu_2 \in \mathscr{P}$.

First we prove two auxiliary lemmas. With no loss of generality we can assume that the region U has a C_2 boundary.

Lemma 3.2. *Let* (3.4) *hold. Denoting by* $G(x, t, y, \tau)$, $t \geqq \tau$, *Green's function of the problem*

$$\frac{\partial u}{\partial t} = \sum_i b_i(t, x) \frac{\partial u}{\partial x_i} + \tfrac{1}{2} \sum_{i,j} a_{ij}(t, x) \frac{\partial^2 u}{\partial x_i \partial x_j}\,, \quad (t, x) \in \mathbb{R}_+ \times U\,, \tag{3.6}$$

we have for every set W, $\overline{W} \subset U$,

$$\gamma(y) = \inf_{(x,\tau)\in \overline{W} \times R_+} G(x, \tau + 1, y, \tau) > 0$$

for $y \in \overline{W}$.

Proof. Supposing that the assertion is false we find $y \in \overline{W}$ and sequences $\tau_k \in R_+$, $x_k \in \overline{W}$, $x_k \to x_0$, such that

$$G(x_k, \tau_k + 1, y, \tau_k) \to 0 .$$

From Schauder's interior estimate for the solution of (3.6) we get for some $\hat{K} > 0$,

$$\left|\frac{\partial G}{\partial x_i}(x_0, \tau_k + 1, y, \tau_k)\right| \leqq \hat{K} \quad \text{for} \quad i = 1, 2, \ldots, n, \quad k \in N,$$

and thus,

$$G(x_0, \tau_k + 1, y, \tau_k) \to 0 .$$

Denoting by

$$a^k_{ij}(t, x) = a_{ij}(t + \tau_k, x), \quad b^k_i(t, x) = b_i(t + \tau_k, x), \quad t \in R_+ ,$$

we get by Arzela's theorem

$$a^{k_l}_{ij} \rightrightarrows \hat{a}_{ij}, \quad b^{k_l}_i \rightrightarrows \hat{b}_i \quad \text{in} \quad \langle 0, 1 \rangle \times \overline{U}$$

for appropriate subsequences of coefficients. From [3] (theorem 3.15) it follows that $G(x_0, \tau_{k_l} + 1, y, \tau_{k_l}) \to \Gamma(x_0, 1, y, 0)$, where Γ is Green's function of the problem

$$\frac{\partial u}{\partial t} = \sum_i \hat{b}_i(t, x) \frac{\partial u}{\partial x_i} + \tfrac{1}{2} \sum_{i,j} \hat{a}_{ij}(t, x) \frac{\partial^2 u}{\partial x_i \, \partial x_j}, \quad (t, x) \in R_+ \times U . \tag{3.7}$$

Hence, we get $\Gamma(x_0, 1, y, 0) = 0$ which cannot hold in view of the strong maximum principle for the equation (3.7).

Lemma 3.3. *Let* (3.4) *hold. Denoting by* $\tau(s)$ *the exit time* (*after* s) *from* U *and*

$$w(t, z, s) = \mathbf{P}_{s,z}[\zeta_t \in H, \ \tau(s) > t], \quad t \geqq s, \quad z \in U,$$

where $H \subset \overline{W} \subset U$, H *is open, and* ζ_t *is the solution of* (3.1), *we have*

$$w(t, z, s) = \int_H G(z, t, y, s) \, \mathrm{d}y .$$

Proof. Let φ_k be a sequence of continuous functions, supp $\varphi_k \subset U$, $\varphi_k \searrow \chi_H$ (χ is the characteristic function). Denote by w_k the solution of the problem

$$\frac{\partial w_k}{\partial t}(t, z, s) = \sum_i b_i(t, z) \frac{\partial w_k}{\partial z_i}(t, z, s) + \tfrac{1}{2} \sum_{i,j} a_{ij}(t, z) \frac{\partial^2 w_k}{\partial z_i \, \partial z_j} \tag{3.8}$$

$$w_k(s, z, s) = \varphi_k(z), \quad w_k(t, z, s)|_{\langle s,\infty) \times \partial U} = 0 .$$

We have

$$(3.9) \qquad w_k(t, z, s) = \int_U G(z, t, y, s)\, \varphi_k(y)\, dy \searrow \int_H G(z, t, y, s)\, dy\,.$$

Furthermore, by a standard application of Itô's formula we get for $s \leqq T < t$, $z \in U$,

$$\mathbf{E}_{T,z} w_k(t + s - (\tau(T) \wedge t),\ \zeta_{\tau(T)\wedge t}, s) = w_k(t + s - T, z, s)\,.$$

Hence, we have $(T = s)$

$$w_k(t, z, s) = \mathbf{E}_{s,z} w_k(s, \zeta_t, s)\, \chi_{[\tau(s) > t]} =$$

$$= \mathbf{E}_{s,z} \varphi_k(\zeta_t)\, \chi_{[\tau(s)>t]} \searrow \mathbf{E}_{s,z} \chi_H(\zeta_t)\, \chi_{[\tau(s)>t]} = w(t, z, s)\,,$$

which, together with (3.9), completes the proof.

Proof of Theorem 3.1. Let $s \geqq 0$ and $v \in \mathscr{P}$. From (3.5) it follows that $S_{s,t}\, v(W) \geqq \geqq \frac{1}{2}\lambda$ holds for $t \geqq t_0(s, v)$. Thus, we get by Lemma 3.3 for any $H \subset W$, H open, and $t \geqq t_0$,

$$(3.10) \qquad S_{s,t+1}\, v(H) = S_{t,t+1}(S_{s,t} v)(H) =$$

$$= \int_X P(t, x, t + 1, H)\, S_{s,t}\, v(dx) \geqq \int_W P(t, x, t + 1, H)\, S_{s,t}\, v(dx) \geqq$$

$$\geqq \tfrac{1}{2}\lambda \inf_{x \in W} P(t, x, t + 1, H) \geqq$$

$$\geqq \tfrac{1}{2}\lambda \inf_{x \in W} \mathbf{P}_{t,x}[\zeta_{t+1} \in H,\ \tau(t) > t + 1] = \tfrac{1}{2}\lambda \inf_{x \in W} \int_H G(x, t + 1, y, t)\, dy\,.$$

We define the measure $\mu \in \mathscr{K}$,

$$\mu(A) = \tfrac{1}{2}\lambda \int_{A \cap W} \gamma(y)\, dy\,, \quad A \in \mathscr{B}$$

($\gamma(y)$ was introduced in Lemma 3.2), and from (3.10) we get

$$S_{s,t+1}\, v(H) \geqq \tfrac{1}{2}\lambda \int_H \gamma(y)\, dy = \mu(H)\,, \quad t \geqq t_0\,.$$

It follows that

$$\|(S_{s,t} v - \mu)^-\| = 0 \quad \text{for} \quad t \geqq t_0 + 1$$

and Lemma 3.2 implies $\mu(X) > 0$. Hence, the system $\mu_s = \mu$, $s \geqq 0$, is a nontrivial system of lower measures with respect to $\{S_{s,t}\}$ and our assertion follows from Theorem 2.1.

For $R > 0$ we set $U_R = \{x \in \mathbb{R}_n, |x| < R\}$.

Corollary 3.4. *Let* (3.4) *be fulfilled with* $U = U_{R_0}$ *for some* $R_0 > 0$ *and let there exist a function* $V \in C_{1,2}(\mathbb{R}_+ \times \mathbb{R}_n)$, $V \geqq 0$, *satisfying*

(3.11) $LV \leqq -\alpha V + \beta$ in $\mathbb{R}_+ \times \mathbb{R}_n$ *for some* $\alpha > 0$, $\beta > 0$,
and

(3.12) $$V_{R_1} = \inf_{R_+(R_n \setminus U_{R_1})} V > \frac{\beta}{\alpha} \quad \textit{for some} \quad 0 < R_1 < R_0 .$$

Then the assertion of Theorem 3.1 *is valid.*

Proof. Let $s \geqq 0$ and denote by h_k the exit time (after s) from U_k. From (3.11) and Itô's formula it follows that

$$\mathbf{E}_{s,x}V(h_k \wedge \theta, \zeta_{h_k h_k \wedge \theta}) \leqq V(s, x) + \beta \mathbf{E}_{s,x}[(h_k \wedge \theta) - s]$$

holds for $\theta \geqq s$, $x \in \mathbb{R}_n$. Hence, we get $(k \to \infty)$

(3.13) $$\mathbf{E}_{s,x}V(\theta, \zeta_\theta) \leqq V(s, x) + \beta(\theta - s) < \infty .$$

Now, let $t > \theta \geqq s$ and denote by τ_k the exit time (after θ) from U_k. From (3.11) and (3.13) by Itô's formula we get

$$\mathbf{E}_{s,x}V(\tau_k \wedge t, \zeta_{\tau_k \wedge t}) \leqq$$
$$\leqq \mathbf{E}_{s,x} V(\theta, \zeta_\theta) - \alpha \mathbf{E}_{s,x} \int_\theta^{\tau_k \wedge t} V(u, \zeta_u)\, \mathrm{d}u + \mathbf{E}_{s,x}\beta[(\tau_k \wedge t) - \theta] .$$

Taking $k \to \infty$ and using Fatou's lemma (on the left-hand side) and Lebesgue's monotone convergence theorem (on the right-hand side of the inequality) we get

(3.14) $$\mathbf{E}_{s,x}V(t, \zeta_t) \leqq \mathbf{E}_{s,x}V(\theta, \zeta_\theta) - \alpha \int_\theta^t \mathbf{E}_{s,x}V(u, \zeta_u)\, \mathrm{d}u + \beta(t - \theta) .$$

For $t \geqq s$ we set $\psi_t = \mathbf{E}_{s,x}V(t, \zeta_t)$ and denote by η_t the solution of the equation $\dot{\eta}_t = -\alpha \eta_t + \beta$, $t \geqq s$, $\eta_s = \psi_s$. We show that $\psi_t \leqq \eta_t$ for $t \geqq s$. Suppose, on the contrary, that $\psi_{t_1} > \eta_{t_1}$ holds for some $t_1 > s$ and set $\theta = \sup_{\lambda < t_1} \{\psi_\lambda \leqq \eta_\lambda\}$. From (3.14) it follows that

$$\liminf_{t \to t'-} \psi_t \geqq \psi_{t'}$$

holds for $t' > s$ and thus we have $\psi_\theta < \eta_\theta$ and $\theta < t_1$. Furthermore, (3.14) yields

$$|\psi_t - \eta_t| = \psi_t - \eta_t \leqq \psi_\theta - \eta_\theta - \alpha \int_\theta^t (\psi_\lambda - \eta_\lambda)\, \mathrm{d}\lambda \leqq$$
$$\leqq -\alpha \int_\theta^t (\psi_\lambda - \eta_\lambda)\, \mathrm{d}\lambda \leqq \alpha \int_\theta^t |\psi_\lambda - \eta_\lambda|\, \mathrm{d}\lambda$$

for all $t \in \langle \theta, t_1 \rangle$. By Gronwall's lemma we get $\psi_{t_1} = \eta_{t_1}$ which is a contradiction. Hence, we have for $t \geqq s$

(3.15) $$\mathbf{E}_{s,x}V(t, \zeta_t) = \psi_t \leqq \eta_t = (V(s, x) - \beta/\alpha)\, \mathrm{e}^{-\alpha(t-s)} + \beta/\alpha .$$

Now, take $\lambda_0 > 0$ such that $V_{R_1}(1 - \lambda_0) > \beta/\alpha$ holds. We show that the assumptions of Theorem 3.1 are fulfilled with $U = U_{R_0}$, $W = U_{R_1}$ and $\lambda = \frac{1}{2}\lambda_0$. From (3.15) it follows that

$$\mathsf{E}_{s,x}V(t, \zeta_t) \leqq \beta/\alpha + \tfrac{1}{2}\lambda_0 V_{R_1}$$

holds for $t \geqq t_2(s, x)$. Hence, we have

$$P(s, x, t, U_{R_1}) = 1 - P(s, x, t, \mathbb{R}_n \setminus U_{R_1}) \geqq 1 - \frac{\mathsf{E}_{s,x}V(t, \zeta_t)}{V_{R_1}} \geqq$$

$$\geqq 1 - \frac{\beta/\alpha + \frac{1}{2}\lambda_0 V_{R_1}}{V_{R_1}} \geqq \lambda_0 - \tfrac{1}{2}\lambda_0 = \tfrac{1}{2}\lambda_0 ,$$

which completes the proof.

Example 3.5. Assume that

$$(3.16) \qquad 2\sum_i x_i b_i(t, x) + \sum_i a_{ii}(t, x) \leqq -\alpha|x|^2 + \beta$$

holds for some $\alpha > 0$, $\beta > 0$, and the assumption (3.4) is fulfilled with $U = U_{R_0}$, where $R_0 > \sqrt{(\beta/\alpha)}$. Then the assertion of Theorem 3.1 is valid. The proof we get from Corollary 3.4 setting $V(t, x) = |x|^2$.

Remark 3.6. Using a partial result from [5] (lemma 3.3) we can show that the assumption (3.5) is satisfied if instead of (3.16) the following weaker condition is fulfilled:

$$(3.17) \qquad 2\sum_i x_i b_i(t, x) + \sum_i a_{ii}(t, x) \leqq -\eta \quad \text{for some} \quad \beta > 0 \quad \text{and all}$$
$$(t, x) \in \mathbb{R}_+ \times (\mathbb{R}_n \setminus U_{R_0}) .$$

This result slightly improves the one in [5] (Theorem 5′), where it is shown that

$$\|S_{s,t}\delta_x - S_{s,t}\delta_y\| \to 0 , \quad t \to \infty ,$$

for all $s \geqq 0$, $x, y \in \mathbb{R}_n$ (where δ_z is the Dirac measure at the point z), provided (3.17) holds and, furthermore, some boundedness of the fundamental solutions of the equation

$$\frac{\partial u}{\partial t} - \sum_i \frac{\partial}{\partial x_i}(b_i u) + \tfrac{1}{2}\sum_{i,j}\frac{\partial^2}{\partial x_i\, \partial x_j}(a_{ij}u) = 0$$

is required (a condition which is fulfilled, for instance, whenever the coefficients b_i, a_{ij} are bounded in $\mathbb{R}_+ \times \mathbb{R}_n$ together with their derivatives to the third order and the matrix a_{ij} is positive definite uniformly in $\mathbb{R}_+ \times \mathbb{R}_n$).

Now, let us consider the autonomous equation

$$(3.18) \qquad \mathrm{d}\zeta_t = b(\zeta_t)\,\mathrm{d}t + \sigma(\zeta_t)\,\mathrm{d}w_t ,$$

whose coefficients are locally Lipschitz continuous and satisfy (3.2a)–(3.2c). Thus, there exists a (unique) solution of (3.18) which is a homogeneous Markov process whose transition function we denote by $P(t, x, A)$. For $\nu \in \mathscr{L}$, $t \geqq 0$ we set

$$S_t \nu(A) = \int_X P(t, x, A)\, \nu(\mathrm{d}x)\,, \quad A \in \mathscr{B}\,.$$

It is obvious that $\{S_t\}_{t \geqq 0}$ is a family of Markov operators satisfying (1.4). We have the following

Corollary 3.7. *Let there exist a nonempty region* $U \subset R_n$ *with a* C_2 *boundary such that*

(3.19a) $\sum_{i,j} a_{ij}(x)\, \alpha_i \alpha_j \geqq \beta |\alpha|^2$ *for some* $\beta > 0$ *and all* $\alpha = (\alpha_i) \in R_n$, $x \in U$,

(3.19b) $\sup_{x \in K} \mathbf{E}_x \tau < \infty$, *where* τ *is the first hitting time of the set* $\bar{U}$ *and* $K \subset R_n$ *is an arbitrary compact set.*

Then there exists an invariant measure $\mu^* \in \mathscr{P}$ *and for every* $\nu \in \mathscr{P}$ *we have*

$$\|S_t \nu - \mu^*\| \to 0\,, \quad t \to \infty\,. \tag{3.20}$$

Proof. The conditions (3.19a), (3.19b) imply (3.5) (cf. [4], Lemma 4.6.5) and hence, the assumptions of Theorem 3.1 are fulfilled. As we have shown in the proof of Theorem 3.1, there exists a nontrivial lower measure with respect to $\{S_t\}$ (the measure μ). Thus, we can apply Theorem 2.2.

Remark 3.8. The assertion of Corollary 3.7 improves a little the result contained in [4] (Theorems 4.4.1 and 4.7.1) which claims that if (3.19a), (3.19b) are fulfilled, then there exists an invariant measure $\mu^* \in \mathscr{P}$ and for every $\nu \in \mathscr{P}$, $S_t \nu$ converges weakly to μ^* for $t \to \infty$.

The assertion (3.20) follows directly from [4] provided the coefficients b_i, a_{ij} are bounded in R_n together with their first and second derivatives and the matrix (a_{ij}) is positive definite uniformly in R_n. In this case, the measure $S_t \nu$ has the density $u(t, x)$ which is a solution of the equation

$$\frac{\partial u}{\partial t} = - \sum_i \frac{\partial}{\partial x_i} (b_i u) + \tfrac{1}{2} \sum_{i,j} \frac{\partial^2}{\partial x_i\, \partial x_j} (a_{ij} u)\,,$$

and therefore, it is locally Lipschitzian in x uniformly with respect to t (for t sufficiently large). Thus, the weak convergence of $S_t \nu$'s implies the "strong" convergence (3.20).

Remark 3.9. In the time-homogeneous case the assertion of Theorem 3.1 can be also obtained as a consequence of a statement in [7] (for a more general result see also [8]).

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Souhrn

APLIKACE *l*-PODMÍNKY V TEORII STOCHASTICKÝCH DIFERENCIÁLNÍCH ROVNIC

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Práce pojednává o ergodických vlastnostech řešení stochastických diferenciálních rovnic. Je nalezena dosti obecná postačující podmínka pro asymptotickou stabilitu řešení v prostoru pravděpodobnostních měr.

Резюме

ПРИМЕНЕНИЕ *l*-УСЛОВИЯ В ТЕОРИИ СТОХАСТИЧЕСКИХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ

BOHDAN MASLOWSKI

Исследуется устойчивость решений стохастических дифференциальных уравнений в пространстве вероятностных мер (распределений). С помощью модификации „*l*-условия" А. Ласоты показана (при соответствующих предположениях) асимптотическая устойчивость в топологии полной вариации, в однородном случае тоже существование и единственность инвариантной меры.

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