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ON LOCAL SPECTRAL RADIUS

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Summary. For a bounded linear operator there is defined a local spectral radius and it is proved that the local spectral radius is equal to the spectral radius on a set with the 1st category complement. The connection to the local spectral theory is also discussed.

Keywords: local spectral radius, spectral radius, local spectrum, spectrum.

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Let A be a linear bounded operator in a complex Banach space X. Then $r(A)$, the spectral radius of A, may be defined as the least number r such that the series $\sum_{n=0}^{\infty} \lambda^{-(n+1)} A^n$ is convergent for all λ outside the closed r-circle at 0. Now fix any x in X. The local spectral radius of A at x may be defined as the least number r such that the series $\sum_{n=0}^{\infty} \lambda^{-(n+1)} A^n x$ is convergent for all λ outside the closed r-circle at 0, i.e. $\limsup_{n\to\infty} \|A^n x\|^{1/n}$. This leads to

Definition. Let X be a (real or complex) normed linear space, $A: X \to X$ a linear bounded operator and $x \in X$. Define

$$r(A, x) = \limsup_{n\to\infty} \|A^n x\|^{1/n}$$

and call it the *local spectral radius* of A at x.

One sees at once that $0 \leqq r(A, x) \leqq r(A)$ for any x in X (where $r(A)$ is defined by $r(A) = \lim_{n\to\infty} \|A^n\|^{1/n} = \inf_{n\geqq 1} \|A^n\|^{1/n}$) and $r(A, x)$ depends only on the norm of $\operatorname{sp}\{A^n x: n \geqq 0\}$. Our main result asserts that $r(A, x) = r(A)$ for all x from a 2nd category subset of the Banach space X. On the other hand, the limit $\lim_{n\to\infty} \|A^n x\|^{1/n}$ does not exist generally and it may happen that $L(A, x)$, the set of limits of all convergent subsequences of the sequence $\{\|A^n x\|^{1/n}\}_{n=1}^{\infty}$, is the whole segment $[0, r(A)]$ for x from a dense subset of X.

In what follows, X will be a normed linear space and $A: X \to X$ a linear bounded operator.

Lemma 1.
(1) $r(aA, bx) = |a|\, r(A, x)$ *for all x in X, $b \neq 0$ and a a scalar.*

(2) $r(A, x + y) \leqq \max\{r(A, x), r(A, y)\}$ *for all* x, y *in* X.
(3) *If* $r(A, x) \neq r(A, y)$, *then* $r(A, x + y) = \max\{r(A, x), r(A, y)\}$.
(4) $r(A, x) = 0$ *iff* $r(A, x + y) = r(A, y)$ *for all* y *in* X.
(5) $r(A, A^k x) = r(A, x)$ *for all* x *in* X *and all nonnegative integers* k.
(6) $r(A^k, x) = r(A, x)^k$ *for all* x *in* X *and all positive integers* k.
(7) *If* B *is a linear bounded operator in* X *and* $BA = AB$, *then* $r(A + B, x) \leqq \leqq r(A, x) + r(B)$ *for all* x *in* X.
(8) *If* B *is as in* (7), *then* $r(AB, x) \leqq r(A, x)\, r(B)$ *for all* x *in* X.

Proof. (1) is trivial.

(2) Let $e > 0$ be arbitrary. Take m such that $\|A^n x\| \leqq (r(A, x) + e)^n$ and $\|A^n y\| \leqq \leqq (r(A, y) + e)^n$ for all $n \geqq m$. Then

$$\left\|A^n\left(\frac{x + y}{2}\right)\right\| \leqq \tfrac{1}{2}((r(A, x) + e)^n + (r(A, y) + e)^n) \leqq$$
$$\leqq \max\{(r(A, x) + e)^n, (r(A, y) + e)^n\} =$$
$$= (\max\{r(A, x), r(A, y)\} + e)^n \quad \text{for all} \quad n \geqq m\,.$$

Using (1) we obtain $r(A, x + y) = r(A, (x + y)/2) \leqq \max\{r(A, x), r(A, y)\} + e$ for each $e > 0$. Hence the result.

(3) Assume $r(A, y) < r(A, x)$ and take any $e \in (0, \frac{1}{2}(r(A, x) - r(A, y)))$. There are $m > 0$ and an increasing sequence of positive integers $\{n_k\}$ such that $\|A^{n_k} x\|^{1/n_k} \to \to r(A, x)$ and $\|A^n y\|^{1/n} \leqq r(A, y) + e$ for all $n \geqq m$. Let k_0 be such that $n_k \geqq m$ and $\|A^{n_k} x\|^{1/n_k} \geqq r(A, x) - e$ for all $k \geqq k_0$. Then we have, for $k \geqq k_0$,

$$\|A^{n_k}(x + y)\|^{1/n_k} \geqq (\|A^{n_k} x\| - \|A^{n_k} y\|)^{1/n_k} \geqq \|A^{n_k} x\|^{1/n_k} (1 - d^{n_k})^{1/n_k}\,,$$

where $d = (r(A, y) + e)/(r(A, x) - e) \in (0, 1)$. This implies $\liminf_{k \to \infty} \|A^{n_k}(x + y)\|^{1/n_k} \geqq \geqq r(A, x)$ and hence $r(A, x + y) \geqq r(A, x)$. Using (2) we obtain the result. (Let us point out that we have proved, in fact, a stronger result: if $\|A^{n_k} x\|^{1/n_k} \to r(A, x) > > r(A, y)$, then $\|A^{n_k}(x + y)\|^{1/n_k} \to r(A, x) = r(A, x + y)$.)

(4) follows easily from (2) and (3), and (5) is trivial.

(6) Clearly, $r(A^k, x) \leqq r(A, x)^k$. For any integer n let $m(n)$ be the integral part of n/k, and $r(n) = n - k\, m(n)$. Set $M = \max\{\|A^s\| : 0 \leqq s \leqq k - 1\}$. Then, for all $n \geqq k$,

$$\|A^n x\|^{1/n} \leqq \|A^{r(n)}\|^{1/n} \|A^{km(n)} x\|^{1/n} \leqq$$
$$\leqq M^{1/n} (\|(A^k)^{m(n)} x\|^{1/m(n)})^{(1/k) \cdot (km(n)/n)}\,.$$

As $\lim_{m \to \infty} k\, m(n)/n = 1$ and $\limsup_{n \to \infty} \|(A^k)^{m(n)} x\|^{1/m(n)} = \limsup_{n \to \infty} \|(A^k)^m x\|^{1/m} = r(A^k, x)$, we have $r(A, x) \leqq \lim_{n \to \infty} M^{1/n} r(A^k, x)^{1/k} \leqq r(A^k, x)^{1/k}$.

(7) Let $e > 0$ be given and take an m such that

$$\|A^n x\| \leqq (r(A, x) + e)^n \quad \text{and} \quad \|B^n\| \leqq (r(B) + e)^n \quad \text{for all} \quad n \geqq m\,.$$

Take any $n \geqq 2m$. Then

$$(A + B)^n x = \sum_{k=0}^{m-1} \binom{n}{k} B^{n-k} A^k x + \sum_{k=m}^{n-m} \binom{n}{k} B^{n-k} A^k x + \sum_{k=n-m+1}^{n} \binom{n}{k} B^{n-k} A^k x$$

and hence

$$\|(A + B)^n x\| \leqq \sum_{k=0}^{m-1} \binom{n}{k} \|A^k x\| (r(B) + e)^{n-k} +$$

$$+ \sum_{k=m}^{n-m} \binom{n}{k} (r(B) + e)^{n-k} (r(A, x) + e)^k + \sum_{k=n-m+1}^{n} \binom{n}{k} \|B^{n-k}\| (r(A, x) + e)^k =$$

$$= (r(B) + r(A, x) + 2e)^n + \sum_{k=0}^{m-1} \binom{n}{k} (\|A^k x\| - (r(A, x) + e)^k) (r(B) + e)^{n-k} +$$

$$+ \sum_{k=n-m+1}^{n} \binom{n}{k} (\|B^{n-k}\| - (r(B) + e)^{n-k}) (r(A, x) + e)^k \leqq$$

$$\leqq (r(B) + r(A, x) + 2e)^n \left(1 + \sum_{k=0}^{m-1} \binom{n}{k} \frac{\|A^k x\| + (r(A, x) + e)^k}{(r(B) + e)^k} s^n + \right.$$

$$\left. + \sum_{k=n-m+1}^{n} \binom{n}{k} \frac{\|B^{n-k}\| + (r(B) + e)^{n-k}}{(r(A, x) + e)^{n-k}} s^n \right) \leqq$$

$$\leqq (r(B) + r(A, x) + 2e)^n (1 + c_m(n) s^n),$$

where $c_m(n)$ is a polynomial in n of order $m - 1$ and $s = (\max\{r(B), r(A, x)\} + e)/(r(B) + r(A, x) + 2e)$. As $s \in (0, 1)$, we have $r(A + B, x) \leqq \lim_{n\to\infty} (r(B) + r(A, x) + 2e)(1 + c_m(n) s^n)^{1/n} = r(B) + r(A, x) + 2e$. This gives our assertion because $e > 0$ was arbitrary.

(8) is trivial.

Lemma 2. *Let N be a subset of X. Then*
(1) $\sup\{r(A, x): x \in N\} = \sup\{r(A, x): x \in \operatorname{sp}(N)\}$;
(2) *if X is complete and $X = \operatorname{sp}(N)$, then*

$$r(A) = \max\{r(A, x): x \in N\}$$

(so that $r(A)$ is equal to $r(A, x)$ for at least one x in X; we shall see later that $r(A) = r(A, x)$ for "almost" all x in X).

Proof. (1) Let $M = \operatorname{sp}(N)$. Clearly, $\sup\{r(A, x): x \in N\} \leqq \sup\{r(A, x): x \in M\}$. Let x in M be given. Then $x = \sum_{i=1}^{n} t_i x_i$ with x_i in N. By (1) and (2) of Lemma 1, $r(A, x) \leqq \max_i r(A, t_i x_i) \leqq \max_i r(A, x_i) \leqq \sup\{r(A, y): y \in N\}$.

(2) First we show that

(§) $$r(A) = \max\{r(A, x): x \in X\}.$$

By (1) in Lemma 1 we may assume that $r(A) = 1$ (the case $r(A) = 0$ being trivial).

Assume (§) is false. Then for each x in X we can take some $r(x) \in (r(A, x), 1)$. For each x in X there exists $n(x)$ such that

$$\|A^n x\| \leqq r(x)^n \quad \text{for all} \quad n \geqq n(x).$$

Fix any λ with $|\lambda| = 1$. We have

$$\sum_{n=n(x)}^{\infty} \|(\lambda^{-1}A)^n x\| = \sum_{n=n(x)}^{\infty} \|A^n x\| \leqq \sum_{n=n(x)}^{\infty} r(x)^n < \infty$$

and hence

$$\sum_{n=0}^{\infty} \|(\lambda^{-1}A)^n x\| < \infty \quad \text{for each } x \text{ in } X.$$

For any m define a linear bounded operator T_m in X by

$$T_m(x) = \lambda^{-1} \sum_{n=0}^{m} (\lambda^{-1}A)^n x.$$

We have shown above that

$$\sup\{\|T_m(x)\| : m \geqq 0\} < \infty \quad \text{for each } x \text{ in } X.$$

By the Banach-Steinhaus Theorem we conclude that the operator $T: X \to X$ (well-) defined by $T(x) = \lim_{m\to\infty} T_m(x)$ is a linear bounded operator. We will show that $(\lambda - A)\, T = T(\lambda - A) = I$. Let x in X be given. Then

$$(\lambda - A)\, T_m(x) = T_m(\lambda - A)\, x = x - (\lambda^{-1}A)^{m+1} x \tag{§§}$$

for each m. But $\|(\lambda^{-1}A)^{m+1} x\| \leqq r(x)^{m+1}$ for all $m \geqq n(x)$, so that, taking limit in (§§), we obtain $(\lambda - A)\, Tx = T(\lambda - A)\, x = x$. Thus $(\lambda - A)\, T = T(\lambda - A) = I$. This implies that each λ with $|\lambda| = 1$ is in the resolvent set of A, which contradicts the fact that $1 = r(A) = \max |\sigma(A)|$. Hence (§) holds.

As $X = \text{sp}(N)$, we have by (§) and (1)

$$r(A) = \max\{r(A, x) : x \in X\} = \sup\{r(A, x) : x \in N\}.$$

To show that "sup" on the right hand side can be replaced by "max", it is sufficient to show that for each y in X there exists some x in N with $r(A, y) \leqq r(A, x)$. Let y in X be given. Then $y = \sum_{i=1}^{n} t_i x_i$ with x_i in N. We have shown in the proof of (1) that $r(A, y) \leqq \max_i r(A, x_i)$. Hence for at least one x_i we have $r(A, y) \leqq r(A, x_i)$.

Corollary 1. *Let N be a subset of X. Then*

$$\sup\{r(A, x) : x \in N\} = \sup\{r(A, x) : x \in M\},$$

where $M = \text{sp}\{A^k x : x \in N,\ k \geqq 0\}$.

Proof follows from (5) of Lemma 1 and (1) in Lemma 2.

Corollary 2. *Let N be a finite subset of X and let M be defined as in Corollary 1. Then*

$$\max\{r(A, x) : x \in N\} = \max\{r(A, x) : x \in M\}.$$

Corollary 3. *Let X be complete, N a subset of X, and let M be defined as in Corollary 1. If $X = M$, then*

$$r(A) = \max\{r(A, x): x \in N\}.$$

Proof follows from (2) in Lemma 2 and Corollary 1.

Lemma 3. *Let $X(<r) = \{x \in X: r(A, x) < r\}$ $(r > 0)$ and $X(\leqq r) = \{x \in X: r(A, x) \leqq r\}$ $(r \geqq 0)$. Then*

(1) *$X(<r)$ is a F_σ linear subspace of X for each $r > 0$ and $X(\leqq r)$ is a $F_{\sigma\delta}$ linear subspace of X for each $r \geqq 0$;*

(2) *$X(<r) \subset X(<R) \subset X(\leqq R)$ for each $0 < r \leqq R$, and $X(\leqq r) \subset X(\leqq R)$ for each $0 \leqq r \leqq R$;*

(3) *if X is complete, then $X(<r)$ is of the 1st category in X for each $r \in (0, r(A)]$.*

Proof. (1) Let $X(n, r) = \{x \in X: \|A^n x\| \leqq r^n\}$ $(n \geqq 0,\ r \geqq 0)$. Each set $X(n, r)$ is closed in X. As $X(<r) = \bigcup_{k=1}^{\infty} \bigcup_{m=0}^{\infty} \bigcap_{n=m}^{\infty} X(n, r - 1/k)$ and $X(\leqq r) = \bigcap_{k=1}^{\infty} X(<r + 1/k)$, we have that $X(<r)$ is F_σ and $X(\leqq r)$ is $F_{\sigma\delta}$ in X. The linearity of both these sets follows from (1), (2) in Lemma 1.

(2) is trivial.

(3) Let $Y(k, m) = \bigcap_{n=m}^{\infty} X(n, r - 1/k)$. Since each $Y(k, m)$ is closed and $X(<r) = \bigcup_{k=1}^{\infty} \bigcup_{m=0}^{\infty} Y(k, m)$, it is sufficient to show that $\operatorn{int}(Y(k, m)) \neq \emptyset$ for some k and m leads to a contradiction. So assume that $\operatorname{int}(Y(k, m)) \neq \emptyset$ for some k, m. Then

$$\sup\{r(A, x): x \in Y(k, m)\} \leqq r - 1/k \leqq r(A) - 1/k.$$

As $\operatorname{int}(Y(k, m)) \neq \emptyset$, we have $X = \operatorname{sp}(Y(k, m))$ and, by (2) in Lemma 2,

$$r(A) = \max\{r(A, x): x \in Y(k, m)\}$$

which by the preceding inequality leads to $r(A) \leqq r(A) - 1/k$, a contradiction.

Theorem. *Let X be a Banach space and S a countable set of linear bounded operators in X. Then there exists a F_σ 1st category subset F of X such that*

$$r(A, x) = r(A) \quad \text{for each } x \text{ in } \quad X \setminus F \quad \text{and each } A \text{ in } S.$$

Proof. Set $F = \bigcup\{X(<r(A)): A \in S\}$ and use Lemma 3, part (3).

Using the technique of the local spectral theory for self-adjoint operators we have proved also that $r(A, x) = \lim_{n\to\infty} \|A^n x\|^{1/n}$ for any normal operator A on a Hilbert space X and each x in X. As this technique is closely related to iterative processes in Hilbert spaces, we have decided not to develop it here and the reader is referred to [1]. Since $r(A, x) = r(A)$ for "almost" all x, we may use the sequence $\{\|A^n x\|^{1/n}\}$ for computing $r(A)$ for some classes of operators A. But it should be pointed out that the (respective) convergence of this sequence is very bad.

Proposition 1. *Let X be a Banach space, $A: X \to X$ a linear bounded operator such that $\sigma(A) \cap S(0, r(A)) = \{\lambda_1, \dots, \lambda_m\}$ is a finite isolated subset of $\sigma(A)$ ($S(0, r(A))$ denotes the spectral circle of A) and the resolvent $R(A, \lambda)$ has a pole at λ_i of a finite order, for each $i = 1, 2, \dots, m$. Let $E_r = E(A, \{\lambda_1, \dots, \lambda_m\})$ and let x in X be such that $E_r x \neq 0$. Then the limit $\lim_{n\to\infty} \|A^n x\|^{1/n}$ exists and equals $r(A)$ $(= r(A, x))$.*

Proof. Let $\sigma = \sigma(A) \setminus \{\lambda_1, \dots, \lambda_m\}$, $E_i = E(A, \{\lambda_i\})$, $X_i = E_i(X)$ for $i = 1, \dots, m$, and $E_\sigma = E(A, \sigma)$, $X_\sigma = E_\sigma(X)$. Then $X = X_1 \oplus \dots \oplus X_m \oplus X_\sigma$ and each of $X_1, \dots, X_m, X_\sigma$ is invariant under A. Without loss of generality we may assume that $r(A) = 1$ and $\|x\|_X = \max\{\|E_1 x\|_{X_1}, \dots, \|E_m x\|_{X_m}, \|E_\sigma x\|_{X_\sigma}\}$ for each x in X. Let x in X be such that $E_r x \neq 0$, and set $x_i = E_i x$ $(i = 1, \dots, m)$ and $x_\sigma = E_\sigma x$. Then $x = x_1 + \dots + x_m + x_\sigma$ and the set $J = \{i \in \{1, \dots, m\}: x_i \neq 0\}$ is nonempty (because $E_r = E_1 + \dots + E_m$). Let $R(A, \lambda)$ have a pole at λ_i of order p_i $(i = 1, \dots, m)$. Take any $i \in J$ and let n_i be the largest integer such that $(\lambda_i - A)^{n_i} x_i \neq 0$; clearly $0 \leqq n_i < p_i$ (note that $X_i = \{x \in X: (\lambda_i - A)^{p_i} x = 0\}$). Then we have, for $n \geqq n_i$,

$$A^n x_i = (\lambda_i - (\lambda_i - A))^n x_i = \sum_{k=0}^{n_i} (-1)^k \binom{n}{k} \lambda_i^{n-k} (\lambda_i - A)^k x_i .$$

It is easy to see that there exist $a_i, b_i > 0$ such that

$$a_i \binom{n}{n_i} \leqq \|A^n x_i\| \leqq b_i \binom{n}{n_i} \quad \text{for large } n .$$

Therefore, for large n we have

$$\max\left\{a_i \binom{n}{n_i} : i \in J\right\} \leqq \|A^n x_r\| \leqq \max\left\{b_i \binom{n}{n_i} : i \in J\right\}.$$

This implies that $\lim_{n\to\infty} \|A^n x_r\|^{1/n} = r(A, x_r) = 1$. From this equality and from $r(A, x_\sigma) \leqq r(A_\sigma) < r(A) = 1$ (where A_σ denotes the restriction of A to X_σ) we conclude that the limit $\lim_{n\to\infty} \|A^n x\|^{1/n}$ exists and equals $r(A, x_r) = r(A, x) = r(A) = 1$ (see Lemma 1, (3) and its proof).

Corollary 1. *Let X be a Banach space and $A: X \to X$ a linear compact operator. Then $r(A, x) = \lim_{n\to\infty} \|A^n x\|^{1/n}$ for each x in X.*

In Proposition 1 and its corollary, the set $\{x \in X: r(A, x) < r(A)\}$ is a proper closed linear subspace of X. Corollary 1 extends a result of [3, § 9.1].

Proposition 2. *Let X be a normed linear space and $A: X \to X$ a linear operator. Assume that at least one of the following conditions is satisfied:*
(1) A is bounded;
(2) A is one-to-one and $A^{-1}: R(A) \to X$ is bounded.

Then, for each x in X, the set $L(A, x)$ of limits of all convergent subsequences of the sequence $\{\|A^n x\|^{1/n}\}_{n=1}^{\infty}$ is a closed segment in $[0, \infty]$.

Proof. Let x in X be given. It is clear that the set $L(A, x)$ is closed in $[0, \infty]$. Assume that $L(A, x)$ is not a segment, i.e. not connected. Then there exist nonnegative numbers u and v such that $u < v$ and $L(A, x) \cap [u, v] = \{u, v\}$. Take two numbers a and b such that $u < a < b < v$ and set $c = b/a$ (> 1). Define two sets R and S of positive integers by $R = \{n: \|A^n x\|^{1/n} \leqq a\}$ and $S = \{n: \|A^n x\|^{1/n} \geqq b\}$. It is clear that $R \cap S = \emptyset$ and the set of positive integers outside the set $R \cup S$ is finite, and hence there exists n_0 such that each integer $n \geqq n_0$ is either in R or in S. As both sets R and S are infinite, one can easily construct two sequences $n_1 < n_2 < \dots$ $\dots < n_k < \dots$ and $m_1 < m_2 < \dots < m_k < \dots$ such that $n_k \in R$, $n_k + 1 \in S$ and $m_k \in S$, $m_k + 1 \in R$ for all k. Then

$$\|A^{n_k} x\| \leqq a^{n_k}, \quad \|A^{n_k+1} x\| \geqq b^{n_k+1}, \quad \|A^{m_k} x\| \geqq b^{m_k}, \quad \|A^{m_k+1} x\| \leqq a^{m_k+1}$$

for all k. Set $x_k = A^{n_k} x$ and $y_k = A^{m_k} x$. Then

$$\|Ax_k\|/\|x_k\| \geqq bc^{n_k} \quad \text{and} \quad \|Ay_k\|/\|y_k\| \leqq ac^{-m_k} \quad \text{for all } k,$$

and hence neither (1) nor (2) is satisfied, a contradiction.

This proposition also shows that the claim in the proof of the second part of Lemma 2.2 in [2] is false.

Proposition 3. *Let H be a Hilbert space with an orthonormal basis $\{e_n\}_{n=1}^{\infty}$, and $0 \leqq a \leqq b \leqq r$. Then there exists a weighted shift operator A in H such that $r(A) = r$ and $L(A, x) = [a, b]$ for all nonzero x in $H_{\text{fin}} = \text{sp}\{e_1, e_2, \dots\}$.*

Proof. We may restrict ourselves to the case $r(A) = 1$ only. We shall consider five cases:

(i) $0 = a < b = 1$;
(ii) $0 < a < b = 1$;
(iii) $0 = a < b < 1$;
(iv) $0 < a < b < 1$; and
(v) $0 \leqq a = b \leqq 1$.

In cases (i)–(iv) we take $c \in (0, 1)$ and define (N denotes the set of nonnegative integers):

(a) a function $f: N \to N$ such that, for some $m_f \geqq 0$, $i > j \geqq m_f$ implies $f(i) > f(j)$ and $f(m + 1) - f(m) \to \infty$ as $m \to \infty$;

(b) a number $M(n) \in N$, for $n \geqq f(m_f)$, by the condition

$$f(M(n)) \leqq n < f(M(n) + 1);$$

(c) a nondecreasing function $s: N \cap [m_f, \infty) \to R^+$;

(d) a function $e: N \to R^+$ by

$$e(n) = \begin{cases} s(m) - s(m_f) & \text{if} \quad n = f(m), \quad m \geqq m_f + 1, \\ s(m_f) & \text{if} \quad n = f(m_f), \\ 0 & \text{otherwise}; \end{cases}$$

(e) a sequence $\{a_n\}_{n=1}^{\infty}$ by $a_n = c^{e(n)}$; and
(f) a weighted shift operator $A: H \to H$ by $Ae_n = a_n e_{n+1}$ $(n \geqq 1)$.

If $x = \sum_{k=1}^{u} x_k e_k \in H_{\text{fin}}$, then we have, for $n \geqq f(m_f)$,

$$A^n x = \sum_{k=1}^{u} \prod_{i=k}^{k+n-1} a_i e_{k+n} = \sum_{k=1}^{u} x_k c^{\Sigma_{i=k}^{k+n-1} e(i)} e_{k+n} =$$
$$= \sum_{k=1}^{u} x_k c^{s(M(k+n-1)) - \Sigma^k_{i=1} e(i)} e_{k+n}.$$

Assume $x \neq 0$ and define $g = \min\{|x_i|: x_i \neq 0\}$ and $h = \max\{|x_i|: i = 1, \ldots, u\}$. Then

$$\|A^n x\| \leqq h \max_{k=1,\ldots,u} c^{s(M(n)) - \Sigma^k_{i=1} e(i)} \leqq h c^{s(M(n)) - \Sigma^u_{i=1} e(i)}$$

and hence

(§) $$\|A^n x\| \leqq q c^{s(M(n))} \quad \text{for} \quad n \geqq f(m_f),$$

where $q = hc^{-\Sigma^u_{i=1} e(i)}$. Similarly one obtains

(§§) $$\|A^n x\| \geqq g c^{s(M(u+n-1))} \quad \text{for} \quad n \geqq f(m_f).$$

Case (i). Define $f(m) = m!$ (then $m_f = 1$) and $s(m) = (m+1)^{1/2} f(m)$. If $n(m) = f(m)$ $(m \geqq 1)$, then $M(n(m)) = m$ and, by (§), $\|A^{n(m)}x\| \leqq qc^{s(m)}$, so that

$$\|A^{n(m)}x\|^{1/n(m)} \leqq q^{1/n(m)} c^{(m+1)^{1/2}} \to 0 \quad \text{as} \quad m \to \infty.$$

If we set $n(m) = f(m) - u$ for large m, then $M(u + n(m) - 1) = M(f(m) - 1) = m - 1$ (for large m) and, by (§§), $\|A^{n(m)}x\| \geqq gc^{f(m-1)m^{1/2}}$, so that

$$\|A^{n(m)}x\|^{1/n(m)} \geqq g^{1/n(m)} c^{(m-1)!m^{1/2}/(m!-u)} \to 1 \quad \text{as} \quad m \to \infty.$$

We have just proved that both 0 and 1 lie in $L(A, x)$. But $L(A, x) \subset [0, 1]$ and hence, by Proposition 2, $L(A, x) = [0, 1]$. (One may show directly that $n(m) = [s(m)/d]$ satisfies $\|A^{n(m)}x\|^{1/n(m)} \to c^d$ as $m \to \infty$; similarly in the other cases.)

Case (ii). Let $t = \log a/\log c$ and define $f(m) = [m!\, t^m]$ for $m \in N$ and $s(m) = f(m+1)/m$ for $m \geqq m_f$. Take $n(m) = f(m+1) - u$. Then $M(n(m)) = m$ for large m and, by (§),

$$\|A^{n(m)}x\| \geqq gc^{s(M(n(m)))} = gc^{s(m)};$$

hence

$$\|A^{n(m)}x\|^{1/n(m)} \geqq g^{1/n(m)} c^{f(m+1)/(m(f(m+1)-u))} \to 1 \quad \text{as} \quad m \to \infty.$$

Since

$$\limsup_{n\to\infty} \frac{s(M(u+n-1))}{n} = \limsup_{n\to\infty} \frac{f(M(u+n-1)+1)}{M(u+n-1)\,n} \leqq$$

$$\leqq \limsup_{n\to\infty} \frac{f(M(u+n-1)+1)}{M(u+n-1)f(M(u+n-1))} = \lim_{m\to\infty} \frac{f(m+1)}{m\,f(m)} = t\,,$$

we conclude that $\liminf_{n\to\infty} \|A^n x\|^{1/n} \geqq c^t = a$ and hence $1 \in L(A, x) \subset [a, 1]$. But for $n(m) = f(m)$ $(m \geqq 1)$ one has $\|A^{n(m)}x\|^{1/n(m)} \leqq qc^{s(M(n(m)))} = qc^{s(m)}$ and hence $\|A^{n(m)}x\|^{1/n(m)} \leqq q^{1/n(m)}c^{s(m)/f(m)} = q^{1/n(m)}c^{f(m+1)/(mf(m))} \to a$ for $m \to \infty$. We have proved that both a and 1 lie in $L(A, x)$ and $L(A, x) \subset [a, 1]$. By Proposition 2, we have $L(A, x) = [a, 1]$.

Case (iii). Set $t = \log b/\log c$ and define $f(m) = [m!\,t^{-m}]$ $(m \in N)$ and $s(m) = m\,f(m)$ $(m \geqq m_f)$.

Case (iv). Set $t = \log a/\log b$ and define $f(m) = [t^m]$ $(m \in N)$ and $s(m) = f(m) \log a/\log c$ $(m \geqq m_f)$.

Both the cases (iii) and (iv) are treated similarly as the case (ii).

Case (v). Define $(a_1, a_2, \ldots) = (1, a^2, 1, 1, a^2, a^2, 1, 1, 1, a^2, a^2, a^2, 1, \ldots)$. One easily checks that $L(A, x) = \{a\}$ for each nonzero x in H_{fin}.

It remains to note that in all five cases the sequence $\{a_n\}_{n=1}^{\infty}$ lies in $(0, 1]$ and contains arbitrarily long segments of consecutive 1's, so that $\|A^n\| = 1$ for all n and hence $r(A) = 1$.

APPENDIX

Let us show the relation of this paper to the local spectral theory. Let X be a complex Banach space, A a bounded linear operator in X, and x in X. The local resolvent set of A at x, denoted by $\varrho(A, x)$, is the set of all complex numbers ζ for which there exists a neighbourhood U of ζ and an analytic X-valued function f on U such that $(\lambda - A)f(\lambda) = x$ for all λ in U; the local spectrum of A at x, denoted by $\sigma(A, x)$, is the complement of $\varrho(A, x)$ (to the whole complex plane). In [2], it is shown that for each $\zeta \in \partial\sigma(A)$, there is a set $X(\zeta)$ of the second category in X such that $\zeta \in \partial\sigma(A, x)$ for all $x \in X(\zeta)$. A more precise argument makes it possible to prove the following

Claim. *The set $X \setminus \{x \in X: \partial\sigma(A, x) \supset \partial\sigma(A)\}$ is of the first category in X.*

Proof. Let $\zeta \in \partial\sigma(A)$ be given. Then $\zeta_n \to \zeta$ for some sequence $\{\zeta_n\}_{n=1}^{\infty} \subset \varrho(A)$, the resolvent set of A. As $\|R(A, \zeta_n)\| \geqq 1/(\zeta_n - \zeta) \to \infty$ for $n \to \infty$, the Banach Theorem (see [6, Chap. II, § 4]) implies that the set $Z(\zeta) = \{x \in X:$

$\limsup_{n\to\infty} \|R(A, \zeta_n)\, x\| < \infty\}$ is of the first category in X and hence, by the definition of the local spectrum, $\zeta \in \partial\sigma(A, x)$ for all $x \in X \setminus Z(\zeta)$.

Now let $\{\zeta_n\}_{n=1}^{\infty}$ be a dense subset of $\partial\sigma(A)$. Then the set $Z = \bigcup_{n=1}^{\infty} Z(\zeta_n)$ is of the first category in X and, for each $x \in X \setminus Z$, $\partial\sigma(A, x)$ contains each ζ_n and hence the whole boundary $\partial\sigma(A)$.

In [5] it is proved (by a slightly different argument) that the set $M = \{x \in X: \sigma(A, x) \supset \partial\sigma(A)\}$ is not of the first category in X; in fact, the proof given there shows that the complement of M (to the whole space X) is of the first category. Since $\sigma(A, x) \subset \sigma(A)$, we have $\partial\sigma(A) \subset \partial\sigma(A, x)$ provided $\partial\sigma(A) \subset \sigma(A, x)$. Hence the above claim makes the assertion concerning M more precise. By the same argument as in [5] one can prove a more general result. Let $\sigma_s(A) = \{\lambda \in C: X \neq \operatorname{ran}(A - \lambda)\}$ (the surjective spectrum of A) and $\sigma_s(A, x) = \{\lambda \in C: x \notin \operatorname{ran}(A - \lambda)\}$ (this set may called the local surjective spectrum or the minimal local spectrum of A at x). It is clear that $\partial\sigma(A) \subset \sigma_s(A) \subset \sigma(A)$ and $\sigma_s(A, x) \subset \sigma(A, x)$. Note that $\sigma_s(A)$ is closed (this may be proved either directly or by using the fact that $A - \lambda$ is not surjective iff it is a right topological divisor of zero).

Theorem. *Let X be a Banach space and S a countable set of linear bounded operators in X. For each A in S let D_A be a countable subset of $\sigma_s(A)$. Then there exists a first category subset F of X such that, for each x in $X \setminus F$ and each A in S,* (1) $D_A \subset \sigma_s(A, s)$ *and* (2) $\sigma_s(A) \subset \operatorname{cl}(\sigma_s(A, x))$.

Proof. The sets $F_A = \bigcup \{\operatorname{ran}(A - \lambda): \lambda \in D_A\}$, $A \in S$, and $F = \bigcup \{F_A: A \in S\}$ are of the first category in X. If $x \in X \setminus F_A$, then $D_A \subset \sigma_s(A, x)$. If $x \in X \setminus F$, then $D_A \subset \sigma_s(A, x)$ for all A in S, i.e. (1) holds. Since we may assume that each D_A is dense in $\sigma_s(A)$, the assertion (2) is a consequence of (1) and of the equivalence of $D_A \subset \operatorname{cl}(\sigma_s(A, x))$ and $\sigma_s(A) \subset \operatorname{cl}(\sigma_s(A, x))$.

Since $r(A) = \max |\sigma(A)|$ and $r(A, x) \geqq \max |\sigma(A, x)|$, we have $\{x \in X: r(A, x) < r(A)\} \subset X \setminus \{x \in X: \partial\sigma(A, x) \supset \partial\sigma(A)\} = X \setminus \{x \in X: \sigma(A, x) \supset \partial\sigma(A)\} \subset X \setminus \{x \in X: \sigma_s(A, x) \supset \sigma_s(A)\}$, the theorem in the main text is a consequence of the claim and of the above theorem as well.

On the other hand, our theorem implies the above claim at least in the case when A possesses the single-valued extension property. Indeed, in this case $r(A, x) = \max |\sigma(A, x)|$. One easily checks that $\sigma((A - \lambda)^{-1}, x) = (\sigma(A, x) - \lambda)^{-1}$ and hence $r((A - \lambda)^{-1}, x) = \operatorname{dist}(\lambda, \sigma(A, x))^{-1}$ for all λ in $\varrho(A)$. Let D be a countable dense subset of $\varrho(A)$. Since also $r((A - \lambda)^{-1}) = \operatorname{dist}(\lambda, \sigma(A))^{-1}$ for all λ in $\varrho(A)$, our theorem ensures the existence of a first category subset F of X such that $\operatorname{dist}(\lambda, \sigma(A, x)) = \operatorname{dist}(\lambda, \sigma(A))$ for all x in $X \setminus F$ and λ in D. This immediately implies that $\partial\sigma(A, x) \supset \partial\sigma(A)$ for all x in $X \setminus F$.

In [2] the author conjectured that there exists an x with $\sigma(A, x) = \sigma(A)$. A simple example (see [4]) disproves the conjecture. Indeed, if S is the unilateral shift in a (complex infinite dimensional) separable Hilbert space H, then $\sigma(S^*, x) \subset \partial\sigma(S^*)$

for all $x \in H$, but $\sigma(S^*) = \sigma(S) = \{\lambda: |\lambda| \leqq 1\}$. (Given any nonzero x in H, set

$$f(\lambda) = -\sum_{n=0}^{\infty} \lambda^n S^{n+1} x .$$

Then f is analytic in the interior of $\sigma(S^*)$, because S is an isometry and hence

$$\liminf_{n\to\infty} \|S^{n+1}x\|^{1/n} = \lim_{n\to\infty} \|x\|^{1/n} = 1 ,$$

and $(\lambda - S^*) f(\lambda) = x$ for all λ with $|\lambda| < 1$.) The hitch is in the fact that S^* does not possess the single-valued extension property. If this obstruction is avoided by modifying the definition of the local spectrum (precisely, by incorporating the analytic residuum into it), then the above claim holds with $\partial\sigma(A, x) \supset \partial\sigma(A)$ replaced by $\sigma(A, x) = \sigma(A)$ (see [5]).

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Souhrn

O LOKÁLNÍM SPEKTRÁLNÍM POLOMĚRU

JOSEF DANEŠ

Pro omezený lineární operátor je definován lokální spektrální poloměr a je dokázáno, že lokální spektrální poloměr je roven spektrálním poloměru na množině, jejíž doplněk je 1. kategorie. Je také ukázána souvislost s lokální spektrální teorií.

Резюме

О ЛОКАЛЬНОМ СПЕКТРАЛЬНОМ РАДИУСЕ

JOSEF DANEŠ

Для ограниченного линейного оператора определяется локальный спектральный радиус и доказывается, что локальный спектральный радиус равен спектральному радиусу на множестве, дополнение которого является множеством первой категории. Рассматривается также связь с локальной спектральной теорией.

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